ON THE REPRODUCING KERNEL FOR THE SPACE OF SEMI-EXACT ANALYTIC DIFFERENTIALS

SACHIKO HAMANO

We discuss here some analytic invariants associated with a holomorphic family of Riemann surfaces.

Let $R$ be a bordered Riemann surface of genus $g(\geq 0)$ with a finite number of $C^\infty$-smooth contours $C_j$ ($j = 1, \ldots, \nu$) in a larger Riemann surface $\tilde{R}$. Let $S(R)$ be the space of semi-exact $L^2$-analytic differentials on $R$. Let $K(z, \zeta)$ denote the reproducing kernel function for $S(R)$.

**Definition.** Let $R$ be as above. Fix two points $a, b \in R$ with local coordinates $U_a : |z - \zeta| < r_a$ and $U_b : |z| < r_b$, where $a$ and $b$ correspond to $\zeta$ and $0$, respectively (where $U_a$ and $U_b$ have no relations). Among all harmonic functions $u$ on $R \setminus \{a, b\}$ with two logarithmic poles of $\log |z - \zeta|$ at $a$ and $-\log |z|$ at $b$ normalized so that $\lim_{z \to 0}(u(z) + \log |z|) = 0$, we have uniquely determined functions $h_i$ ($i = 1, 0$) with the $L_i$-boundary conditions ($i = 1, 0$): (L1) for each $C_j$, $h_1$ satisfies $h_1(z) = c_j$ (constant) on $C_j$ and $\int_{C_j} \partial h_i \partial\zeta = 0$; (L0) $h_0$ satisfies $\frac{\partial h_0}{\partial\zeta} = 0$ on $C_j$. We call $h_i(z)$ the $L_i$-principal function and $\mu_i := \lim_{z \to \zeta}(h_i(z) - \log |z - \zeta|)$ the $L_i$-constant for $(R, b, a)$ with respect to the local coordinates $U_a$ and $U_b$ (simply, for $(R, 0, \zeta)$).

**Theorem 1** ([3]). Let the notation be as above. We have

$$K(z, \zeta) = \frac{2}{\pi} \frac{\partial^2 h_1(z, \zeta)}{\partial z \partial\zeta}, \quad \tilde{K}(\zeta, \zeta) = \frac{1}{\pi} \frac{\partial^2 \mu_1(\zeta)}{\partial \zeta \partial\zeta}.$$

Let $\pi : \tilde{R} \to B$ be a holomorphic family such that $\tilde{R}$ is a complex 2-dimensional manifold, $\pi$ is a holomorphic projection from $\tilde{R}$ onto a disk $B$ in $C_t$, and each fiber $\tilde{R}(t) = \pi^{-1}(t)$, $t \in B$ is irreducible and non-singular in $\tilde{R}$. We set $\tilde{R} = \cup_{t \in B}(t, \tilde{R}(t))$. Let $R = \cup_{t \in B}(t, R(t))$ be a subdomain with $C^\infty$ smooth boundary $\partial R = \cup_{t \in B}(t, \partial R(t))$ in $\tilde{R}$ such that $\tilde{R}(t) \ni R(t) \neq \emptyset$ for $t \in B$, $R(t)$ is a bordered Riemann surface of genus $g(\geq 0)$ in $\tilde{R}(t)$, and $\partial R(t)$ in $R(t)$ consists of a finite number of $C^\infty$ smooth contours $C_j(t)$ ($j = 1, \ldots, \nu$).

**Theorem 2** ([3]). We assume that the total space $R = \cup_{t \in B}(t, R(t))$ is 2-dimensional pseudoconvex in $\tilde{R}$. Then $\log \tilde{K}(t, \zeta, \zeta)$ is a plurisubharmonic function on $R$.

This phenomenon is the same as the Bergman metrics (see [5]).

Here we recall the definition of Schiffer spans for planar Riemann surfaces. Let $R$ be a finite bordered planar Riemann surface. Let $\mathcal{P}(R)$ be the set of

Partly supported by JSPS Grant-in-Aid for Young Scientists (B), 23740098.
all univalent functions \( P \) on \( R \) with the expression
\[
P(z) = \frac{1}{z - \zeta} + 0 + A_1(z - \zeta) + A_2(z - \zeta)^2 + \cdots
\]

at a given point \( \zeta \in R \). The Schiffer span \( s \) for \((R, \zeta)\) is defined by
\[
s := \frac{2}{\pi} \sup_{P \in \mathcal{P}(R)} \{ \text{the Euclidean area} \ E_P \text{ of } E_P := \mathbb{C} \setminus P(R) \}.
\]
The Schiffer span \( s(\zeta) \) induces the metric \( s(\zeta)|d\zeta|^2 \) on \( R \).

**Theorem 3** ([3]). Let \( R \) be a finite bordered planar Riemann surface. Then we have the following:

(i) The metrics \( \tilde{K}(\zeta, \zeta)|d\zeta|^2, \frac{1}{\pi} \frac{\partial^2 \mu_1(\zeta)}{\partial \zeta \partial \bar{\zeta}}|d\zeta|^2, -\frac{1}{\pi} \frac{\partial^2 \rho_0(\zeta)}{\partial \zeta \partial \bar{\zeta}}|d\zeta|^2, \) and \( s(\zeta)|d\zeta|^2 \) are all identical on \( R \);

(ii) \( \tilde{K}(\zeta, \zeta)|d\zeta|^2 \) is of negative curvature at every point \( \zeta \in R \);

(iii) \( \tilde{K}(\zeta, \zeta)|d\zeta|^2 \) is complete on \( R \).

For the proofs of the above theorems, we use the following variational formulas for principal functions and the plurisubharmonic variation of the Schiffer span under pseudoconvexity ([2]).

**Lemma 4** ([1], [4]). Let \( R(t) \) is of genus \( g(\geq 0) \), and let \( \{A_i(t), B_i(t)\}_{i=1}^g \) be a canonical homology basis on \( R(t) \) such that each \( A_i(t) \) and \( B_i(t) \) varies continuously with \( t \in B \). Then we have
\[
\frac{\partial^2 \mu_1(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \left| \frac{\partial h_1(t, z)}{\partial z} \right|^2 ds + \frac{4}{\pi} \int_{R(t)} \left| \frac{\partial^2 h_1(t, z)}{\partial t \partial \bar{z}} \right|^2 dx dy;
\]
\[
\frac{\partial^2 \mu_0(t)}{\partial t \partial \bar{t}} = -\left( \frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \left| \frac{\partial h_0(t, z)}{\partial z} \right|^2 ds + \frac{4}{\pi} \int_{R(t)} \left| \frac{\partial^2 h_0(t, z)}{\partial t \partial \bar{z}} \right|^2 dx dy \right)
\]
\[
-2 \operatorname{Im} \left\{ \sum_{l=1}^g \frac{\partial}{\partial t} \left( \int_{A_l(t)} *d h_0(t, z) \right) \frac{\partial}{\partial t} \left( \int_{B_l(t)} *d h_0(t, z) \right) \right\}.
\]

Here, for the defining function \( \varphi(t, z) \) of \( \partial R \),
\[
k_2(t, z) = \left( \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \right)^2 - 2 \operatorname{Re} \left\{ \frac{\partial^2 \varphi}{\partial t \partial \bar{z}} \frac{\partial^2 \varphi}{\partial t \partial \bar{z}} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \left| \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \right|^3.
\]

**References**


Department of Mathematics, Fukushima University, 960-1296 JAPAN
E-mail address: hamano@educ.fukushima-u.ac.jp