On the length spectrum metric in infinite-dimensional Teichmüller spaces

Erina Kinjo

Tokyo Institute of Technology

Riemann surfaces and Discontinuous groups 2013
Abstract.
We study when the length spectrum metric and Teichmüller metric define the same topology on Teichmüller space.
1. Introduction

$R_0$: a Riemann surface ($\neq \mathbb{C}, \mathbb{C}, \mathbb{C} - \{0\}$, torus)

$T(R_0)$: the Teichmüller space of $R_0$

$$:= \{(R, f) \mid f : R_0 \to R \text{ a quasiconformal map }\}/_{\text{Teich.}}$$

$$(R, f) \sim_{\text{Teich.}} (S, g) \iff \exists h : R \to S \text{ conformal s.t. } h \cong g \circ f^{-1}.$$

We denote the equivalence class of $(R, f)$ by $[R, f]$. 
We define Teichmüller metric and the length spectrum metric on $T(R_0)$.

- Teichmüller metric $d_T$

$$d_T([R_1, f_1], [R_2, f_2]) = \inf_{f \approx f_2 \circ f_1^{-1}} \log K(f),$$

where the infimum is taken over all quasiconformal maps which are homotopic to $f_2 \circ f_1^{-1}$ and $K(f)$ is the maximal dilatation of $f$. 

$$d_T([R_1, f_1], [R_2, f_2]) = 0 \iff f_2 \circ f_1^{-1} \text{ is homotopic to some conformal map.}$$
the length spectrum metric $d_L$

$$d_L([R_1, f_1], [R_2, f_2]) = \sup_{\alpha \in \Sigma_{R_0}} \left| \log \frac{\ell_{R_2}(f_2(\alpha))}{\ell_{R_1}(f_1(\alpha))} \right|,$$

where $\Sigma_{R_0} := \{\alpha \mid \alpha \text{ is a closed curve in } R_0\}$ and $\ell_{R_i}(\alpha)$ is the hyperbolic length of a geodesic which is freely homotopic to $\alpha$.

$$d_L([R_1, f_1], [R_2, f_2]) = 0 \iff \ell_{R_1}(f_1(\alpha)) = \ell_{R_2}(f_2(\alpha)) \text{ for any } \alpha \in \Sigma_{R_0}.$$
Lemma (Sorvali, 1972)

For any $p_1, p_2 \in T(R_0)$,

$$d_L(p_1, p_2) \leq d_T(p_1, p_2)$$

holds.
Question:
When do $d_T$ and $d_L$ define the same topology on Teichmüller space?

$d_T$ and $d_L$ define the same topology on $T(R_0)$.

$\iff$ For sequence $\{p_n\}_{n=0}^\infty \subset T(R_0)$,

$$\lim_{n \to \infty} d_L(p_n, p_0) = 0 \iff \lim_{n \to \infty} d_T(p_n, p_0) = 0$$

Lemma

$\iff$ For sequence $\{p_n\}_{n=0}^\infty \subset T(R_0)$,

$$\lim_{n \to \infty} d_L(p_n, p_0) = 0 \Rightarrow \lim_{n \to \infty} d_T(p_n, p_0) = 0$$

We write $d_T \sim d_L$ on $T(R_0)$ if both metrics define the same topology.
★ History

1972: T. Sorvali conjectures that $d_T \sim d_L$ on $T(R_0)$ if $R_0$ is a topologically finite Riemann surface (i.e., a compact surface from which at most finitely many points have been removed).

1999: Z. Li and L. Liu prove that Sorvali’s conjecture is true.

We consider Teichmüller spaces of topologically infinite Riemann surfaces below.
2003: H. Shiga shows that there exists a topologically infinite Riemann surface $R_0$ s.t. $d_T \sim d_L$ on $T(R_0)$. Also he gives a sufficient condition for $d_T$ and $d_L$ to define the same topology.

2008: Liu-Sun-Wei give a sufficient condition for $d_T$ and $d_L$ to define different topologies.

Our results

- We extend a theorem of Liu-Sun-Wei.
- We show that the converse of Shiga's theorem is not true.
- We extend Shiga's theorem.
2. Extension of a theorem of Liu-Sun-Wei

Liu-Sun-Wei gave a sufficient condition for \(d_T\) and \(d_L\) to define different topologies on the Teichmüller space.

**Theorem (Liu-Sun-Wei)**

\(R_0: a \text{ Riemann surface,}\)

\[\exists \{\alpha_n\}_n^{\infty} \subset \Sigma_{R_0} \text{ s.t. } \ell_{R_0}(\alpha_n) \to 0 \ (n \to \infty).\]

\[\Rightarrow d_T \sim d_L \text{ on } T(R_0).\]
2. Extension of a theorem of Liu-Sun-Wei

Liu-Sun-Wei gave a sufficient condition for $d_T$ and $d_L$ to define different topologies on the Teichmüller space.

**Theorem (Liu-Sun-Wei)**

$R_0$: a Riemann surface,

\[ \exists \{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0} \text{ s.t. } \ell_{R_0}(\alpha_n) \to 0 \ (n \to \infty). \]

\[ \Rightarrow d_T \not\sim d_L \text{ on } T(R_0). \]

**Example.**
We extend a theorem of Liu-Sun-Wei as follows.

**Theorem 1**

\( R_0: \) a Riemann surface.
\[ \exists \{ \alpha_n \}_{n=1}^{\infty} \subset \Sigma R_0 \text{ s.t. for } \forall \{ \beta_n \}_{n=1}^{\infty} \subset \Sigma R_0 \text{ with } \alpha_n \cap \beta_n \neq \emptyset (n = 1, 2, \ldots), \]
\[ \frac{\#(\alpha_n \cap \beta_n) \ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty). \]

\[ \Rightarrow d_T \sim d_L \text{ on } T(R_0). \]
We extend a theorem of Liu-Sun-Wei as follows.

**Theorem 1**

* $R_0$: a Riemann surface.

$\exists \{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ s.t. for $\forall \{\beta_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ with $\alpha_n \cap \beta_n \neq \emptyset$ ($n = 1, 2, \ldots$),

$$\frac{\#(\alpha_n \cap \beta_n) \ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty).$$

$\Rightarrow d_T \asymp d_L$ on $T(R_0)$.

**Example 1.**

Any Riemann surface $R_0$ satisfying the assumption of Theorem of Liu-Sun-Wei satisfies the assumption of Theorem 1 by the collar lemma.
Example 2.
The Riemann surface $R_0$ constructed by Shiga (the first example of $R_0$ s.t. $d_T \sim d_L$ on $T(R_0)$) satisfies the assumption of Theorem 1. (This does not satisfy the assumption of Theorem of Liu-Sun-Wei.)

Example 3.
Except for Examples 1 and 2, we can construct a Riemann surface $R_0$ satisfying the assumption of Theorem 1 as follows:
$P_0$: a pair of pants with boundary lengths $(a_0, b_0, b_0)$.

Make countable copies of $P_0$ and glue them as in the below Figure.

$\{a_n\}_{n=1}^{\infty}$: a monotone divergent sequence of positive numbers.

$P_n$: a pair of pants with boundary lengths $(a_0, a_n, a_n)$.

Make two copies of $P_n$ and glue each copy with the union of the copies of $P_0$ as in the below Figure.

Let $R'_0$ denote a Riemann surface with boundary we have obtained.

Take some pants $\{P'_m\}_{m=1}^{\infty}$ and define $R_0 := R'_0 \cup \bigcup_{m=1}^{\infty} P'_m$. 
We prove Theorem 1. (We write it again.)

**Theorem 1**

*R₀*: a Riemann surface.

∃{αₙ}₀⁻→ₙ ⊂ Σ₉₀ s.t. for ∀{βₙ}₀⁻→ₙ ⊂ Σ₉₀ with αₙ ∩ βₙ ≠ ∅ (n = 1, 2, ...),

\[
\frac{\#(αₙ ∩ βₙ)ℓ₉₀(αₙ)}{ℓ₉₀(βₙ)} \to 0 \quad (ₙ \to ∞).
\]

⇒ dₜ ∼ dₗ on T(R₀).

The proof is short.
We prove Theorem 1. (We write it again.)

**Theorem 1**

\( R_0: \) a Riemann surface.

\[ \exists \{ \alpha_n \}_{n=1}^{\infty} \subset \Sigma_{R_0} \text{ s.t. for } \forall \{ \beta_n \}_{n=1}^{\infty} \subset \Sigma_{R_0} \text{ with } \alpha_n \cap \beta_n \neq \emptyset \ (n = 1, 2, \ldots), \]

\[ \frac{\#(\alpha_n \cap \beta_n) \ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty). \]

\[ \Rightarrow d_T \sim d_L \text{ on } T(R_0). \]

The proof is short. We use the following lemma.

**Matsuzaki’s Lemma**

\( \alpha: \) a simple closed geodesic on a Riemann surface \( R_0, \)

\( f: R_0 \to R_0: \) the \( n \)-times Dehn twist along \( \alpha. \)

\[ \Rightarrow \] The maximal dilatation \( K(f) \) of an extremal quasiconformal map of \( f \) satisfies

\[ K(f) \geq \left\{ \left( \frac{2|n| - 1}{\pi} \ell_{R_0}(\alpha) \right)^2 + 1 \right\}^{1/2}. \]
Shiga gave a sufficient condition for $d_T$ and $d_L$ to define the same topology on the Teichmüller space.

**Theorem (Shiga)**

$R_0$: a Riemann surface

\[ \exists \text{ a pants decomposition } R_0 = \bigcup_{k=1}^{\infty} P_k \text{ satisfying following conditions:} \]

1. Each connected component of $\partial P_k$ is either a puncture or a simple closed geodesic of $R_0$. ($k = 1, 2, \ldots$)

2. \( \exists M > 0 \text{ s.t. if } \alpha \text{ is a boundary curve of some } P_k \text{ then} \)

\[ 0 < M^{-1} < \ell_{R_0} (\alpha) < M. \]

\( \Rightarrow d_T \sim d_L \text{ on } T(R_0). \)
3. A counterexample to the converse of Shiga’s theorem

Shiga gave a sufficient condition for $d_T$ and $d_L$ to define the same topology on the Teichmüller space.

**Theorem (Shiga)**

$R_0$: a Riemann surface

$\exists$ a pants decomposition $R_0 = \bigcup_{k=1}^{\infty} P_k$ satisfying following conditions:

1. Each connected component of $\partial P_k$ is either a puncture or a simple closed geodesic of $R_0. (k = 1, 2, \ldots)$
2. $\exists M > 0$ s.t. if $\alpha$ is a boundary curve of some $P_k$ then
   \[0 < M^{-1} < \ell_{R_0}(\alpha) < M.\]

$\Rightarrow d_T \sim d_L$ on $T(R_0)$.

**Example 1.**

All Riemann surfaces of finite topological type satisfy Shiga’s condition.
Example 2.
Some Riemann surfaces of infinite topological type satisfy Shiga’s condition.
Example 2.
Some Riemann surfaces of infinite topological type satisfy Shiga's condition.

Now, we show that the converse of Shiga's theorem is not true by giving a counterexample.
Counterexample (to the converse of Shiga’s theorem)

\( \Gamma \): a hyperbolic triangle group of signature \((2,4,8)\) acting on \( \mathbb{D} \).

\( P \): a fundamental domain for \( \Gamma \) with angles \((\pi, \pi/4, \pi/4, \pi/4)\).

\( O, a, b, c \): the vertices of \( P \), where the angle at \( O \) is \( \pi \).

\( \varepsilon > 0 \): a sufficiently small number.

\( b' \): the point on the segment \([Ob]\) whose hyperbolic distance from \( b \) is \( \varepsilon \).

Similarly, we take \( a' \) and \( c' \) in \( P \).

We define a Riemann surface \( R_0 \) by removing the \( \Gamma \)-orbits of \( a', b', c' \) from the unit disk \( \mathbb{D} \); \( R_0 := \mathbb{D} - \{ \gamma(a'), \gamma(b'), \gamma(c') \mid \gamma \in \Gamma \} \).
It is not difficult to show that $R_0$ does not satisfy the assumption of Shiga’s Theorem.

It is difficult to show that $d_T \sim d_L$ on $T(R_0)$.

**Outline of the proof** (that $d_T \sim d_L$ on $T(R_0)$).

We show that for the sequence $\{p_n\}_{n=0}^\infty \subset T(R_0)$ s.t. $d_L(p_n, p_0) \to 0$ ($n \to \infty$), $d_T(p_n, p_0)$ converges to 0 as $n \to \infty$.

We may assume that $p_0 = [R_0, id]$. Put $p_n = [R_n, f_n]$.

**Step 1**: We divide $R_0$ into punctured-disks and hyperbolic right-hexagons. Also, we divide $R_n$ for a sufficiently large $n$ similarly.
In Step 1, we note the following lemma.

**Lemma**

\( R_0: \) a hyperbolic Riemann surface.
\( \alpha_1, \alpha_2: \) disjoint simple closed geodesics in \( R_0. \)
\( \beta_{12}: \) a simple arc connecting \( \alpha_1 \) and \( \alpha_2. \)

Assume that a closed curve \( \alpha_{12} := \alpha_1 \cdot \beta_{12} \cdot \alpha_2 \cdot \beta_{12}^{-1} \) is non-peripheral.
\[ \Rightarrow \exists \ \beta_{12}^*: \text{a geodesic connecting } \alpha_1 \text{ and } \alpha_2 \text{ s.t.} \]

1. \( \beta_{12} \) and \( \beta_{12}^* \) are homotopic, where the homotopy map moves each endpoint on each closed geodesic;
2. \( \beta_{12}^* \) is orthogonal to \( \alpha_1 \) and \( \alpha_2; \)
3. the length of \( \beta_{12}^* \) is determined by \( \ell_{R_0}(\alpha_1), \ell_{R_0}(\alpha_2) \) and \( \ell_{R_0}(\alpha_{12}). \)

**Proof.** This follows from properties of pants and right-hexagons. □
Step 2: We construct \((1 + C_n)\)-qc maps from right-hexagons in \(R_0\) to right-hexagons in \(R_n\), where \(C_n \to 0\) \((n \to \infty)\).

![Diagram](image)

Step 3: We construct \((1 + C_n)\)-qc maps from punctured-disks in \(R_0\) to punctured-disks in \(R_n\), where \(C_n \to 0\) \((n \to \infty)\). Consequently, we obtain a quasiconformal map \(g_n\) of the whole of \(R_0\) such that \(g_n\) is homotopic to \(f_n\) and \(K(g_n) \to 1\) \((n \to \infty)\). Thus \(d_T(p_n, p_0) \to 0\) \((n \to \infty)\).
In Steps 2 and 3, we use Bishop’s Lemma.

**Bishop’s Lemma**

$T_1, T_2 \subset \mathbb{D}$: two hyperbolic triangles with sides $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$. Suppose all their angles are bounded below by $\theta > 0$ and

$$\varepsilon := \max(|\log \frac{a_1}{a_2}|, |\log \frac{b_1}{b_2}|, |\log \frac{c_1}{c_2}|) \leq A.$$

$\Rightarrow \exists \ C = C(\theta, A) > 0 \text{ and } \exists \ a \ (1 + C\varepsilon)\text{-quasiconformal map } \varphi : T_1 \to T_2 \text{ such that } \varphi \text{ maps each vertex to the corresponding vertex and } \varphi \text{ is affine on the edge of } T_1.$
4. Extension of Shiga’s theorem

We extend Shiga’s theorem as follows.

**Theorem 2.**

\( R_0: \) a Riemann surface.

\[ \exists \ M > 0 \text{ and } \exists \text{ a decomposition} \]

\[ R_0 = S \cup (R_0 - S) \]

s.t.

1. \( S \) is an open subset of \( R_0 \) whose relative boundary consists of simple closed geodesics and each connected component of \( S \) has a pants decomposition satisfying the same condition as that of Shiga’s theorem for \( M \).

2. \( R_0 - S \) is of genus 0 and \( d_{R_0}(x, S) < M \) for any \( x \in R_0 - S \).

\[ \Rightarrow d_T \sim d_L \text{ on } T(R_0). \]
Examples 1.
The above counterexample to the converse of Shiga’s theorem satisfies Theorem 2. Let $S_i$ ($i = 1, 2, ...$) be a punctured disk, then $S = \bigcup_{i=1}^{\infty} S_i$.

Also, we can construct a Riemann surface $R_0$ satisfying Theorem 2 by replacing a hyperbolic triangle group $\Gamma$ with an arbitrary Fuchsian group with a compact fundamental region.
Examples 2.
In a Riemann surface of Example 1, we replace a punctured disk $S_i$ with a Riemann surface satisfying Shiga's condition. We regard it as a block and construct a Riemann surface $R_0$ with two or more holes. (See figure.) Then $R_0$ satisfies Theorem 2.
Corollary 3.

Let $R_0$ be a Riemann surface with bounded geometry. Also, assume that $R_0$ has finite genus.

$\Rightarrow d_T \sim d_L$ on $T(R_0)$.

We say that a Riemann surface $R_0$ has **bounded geometry** if it satisfies the following condition:

There exists a constant $M > 0$ such that any closed geodesic has the length greater than $1/M$ and for any $x \in R_0$, there exists a closed curve based on $x$ with the length less than $M$. 


