On the length spectrum metric in infinite-dimensional Teichmüller spaces

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Abstract.
We study when the length spectrum metric and Teichmüller metric define the same topology on Teichmüller space.

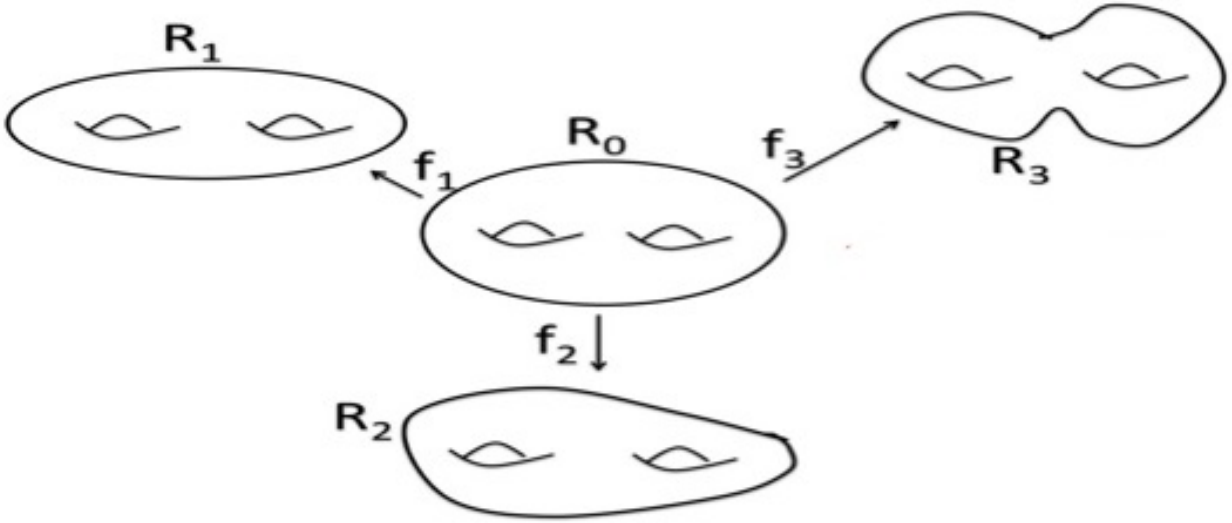
1. Introduction

R_0: a Riemann surface ($\neq \hat{\mathbb{C}}, \mathbb{C}, \mathbb{C} - \{0\}$, torus)
$T(R_0)$: the Teichmüller space of R_0

$$:= \{(R, f) \mid f : R_0 \to R \text{ a quasiconformal map } \}/ \overset{\text{Teich.}}{\sim}$$

$$(R, f) \overset{\text{Teich.}}{\sim} (S, g) \overset{\text{def.}}{\Leftrightarrow} \exists\, h : R \to S \text{ conformal s.t. } h \cong g \circ f^{-1}.$$

We denote the equivalence class of (R, f) by $[R, f]$.

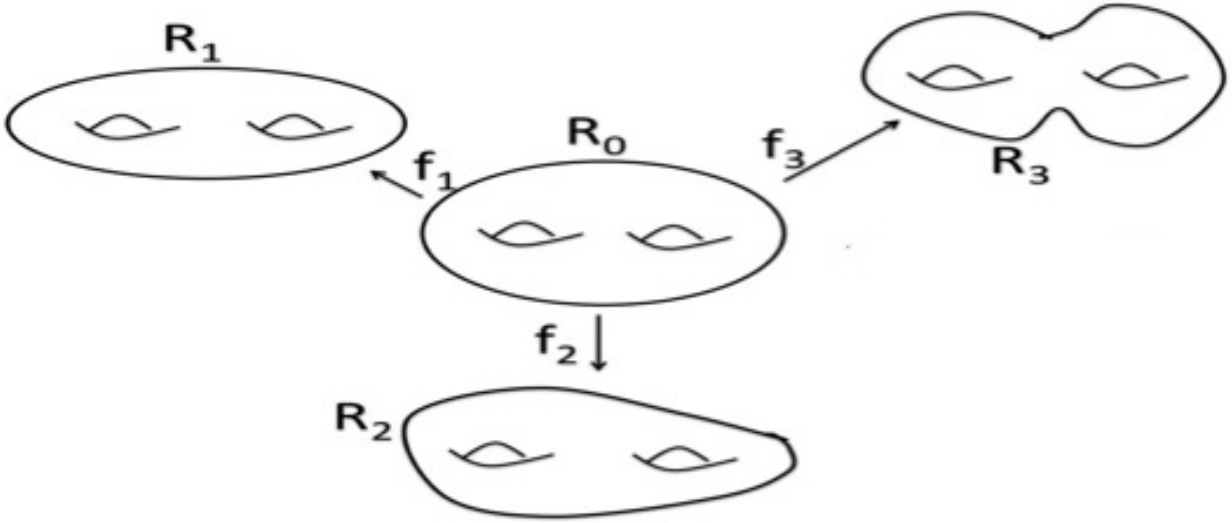


We define Teichmüller metric and the length spectrum metric on $T(R_0)$.

- Teichmüller metric d_T

$$d_T([R_1, f_1], [R_2, f_2]) = \inf_{f \cong f_2 \circ {f_1}^{-1}} \log K(f),$$

where the infimum is taken over all quasiconformal maps which are homotopic to $f_2 \circ {f_1}^{-1}$ and $K(f)$ is the maximal dilatation of f.

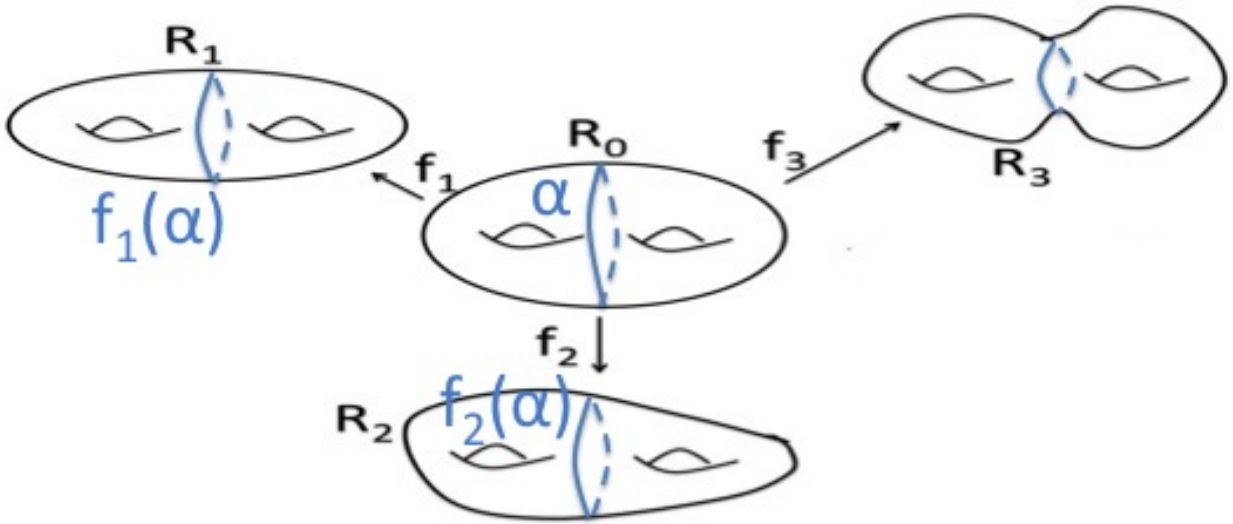


$d_T([R_1, f_1], [R_2, f_2]) = 0 \Leftrightarrow f_2 \circ {f_1}^{-1}$ is homotopic to some conformal map.

- the length spectrum metric d_L

$$d_L([R_1, f_1], [R_2, f_2]) = \sup_{\alpha \in \Sigma_{R_0}} \left| \log \frac{\ell_{R_2}(f_2(\alpha))}{\ell_{R_1}(f_1(\alpha))} \right|,$$

where $\Sigma_{R_0} := \{\alpha \mid \alpha$ is a closed curve in $R_0\}$ and $\ell_{R_i}(\alpha)$ is the hyperbolic length of a geodesic which is freely homotopic to α.



$d_L([R_1, f_1], [R_2, f_2]) = 0 \Leftrightarrow \ell_{R_1}(f_1(\alpha)) = \ell_{R_2}(f_2(\alpha))$ for any $\alpha \in \Sigma_{R_0}$.

Lemma (Sorvali, 1972)

For any $p_1, p_2 \in T(R_0)$,

$$d_L(p_1, p_2) \leq d_T(p_1, p_2)$$

holds.

Question:

When do d_T and d_L define the same topology on Teichmüller space?

d_T and d_L define the same topology on $T(R_0)$.
$\Leftrightarrow$ For sequence $\{p_n\}_{n=0}^{\infty} \subset T(R_0)$,
$$\lim_{n\to\infty} d_L(p_n, p_0) = 0 \Leftrightarrow \lim_{n\to\infty} d_T(p_n, p_0) = 0$$

$\overset{\text{Lemma}}{\Leftrightarrow}$ For sequence $\{p_n\}_{n=0}^{\infty} \subset T(R_0)$,
$$\lim_{n\to\infty} d_L(p_n, p_0) = 0 \Rightarrow \lim_{n\to\infty} d_T(p_n, p_0) = 0$$

We write $d_T \sim d_L$ on $T(R_0)$ if both metrics define the same topology.

⋆ History

1972: T. Sorvali conjectures that $d_T \sim d_L$ on $T(R_0)$ if R_0 is a topologically finite Riemann surface (i.e., a compact surface from which at most finitely many points have been removed).

1999: Z. Li and L. Liu prove that Sorvali's conjecture is true.

We consider Teichmüller spaces of topologically infinite Riemann surfaces below.

2003: H. Shiga shows that there exists a topologically infinite Riemann surface R_0 s.t. $d_T \nsim d_L$ on $T(R_0)$. Also he gives a sufficient condition for d_T and d_L to define the same topology.

2008: Liu-Sun-Wei give a sufficient condition for d_T and d_L to define different topologies.

Our results

- We extend a theorem of Liu-Sun-Wei.
- We show that the converse of Shiga's theorem is not true.
- We extend Shiga's theorem.

2. Extension of a theorem of Liu-Sun-Wei

Liu-Sun-Wei gave a sufficient condition for d_T and d_L to define different topologies on the Teichmüller space.

Theorem (Liu-Sun-Wei)

R_0: a Riemann surface,
$\exists \{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *s.t.* $\ell_{R_0}(\alpha_n) \to 0$ $(n \to \infty)$.
$\Rightarrow$ $d_T \not\sim d_L$ *on* $T(R_0)$.

2. Extension of a theorem of Liu-Sun-Wei

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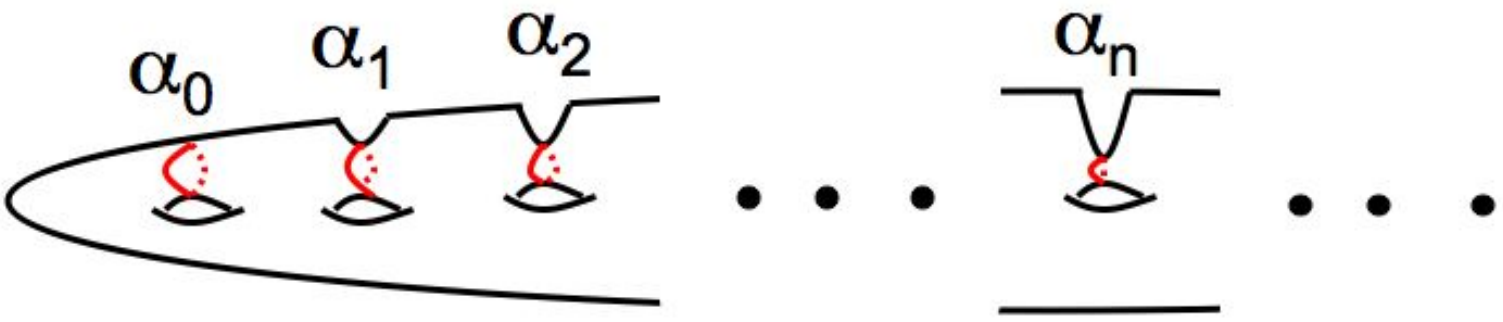
Example.



We extend a theorem of Liu-Sun-Wei as follows.

Theorem 1

R_0: a Riemann surface.
$\exists\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *s.t. for* $\forall\{\beta_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *with* $\alpha_n \cap \beta_n \neq \emptyset$ $(n = 1, 2, \dots)$,

$$\frac{\sharp(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty).$$

$\Rightarrow$ $d_T \nsim d_L$ *on* $T(R_0)$.

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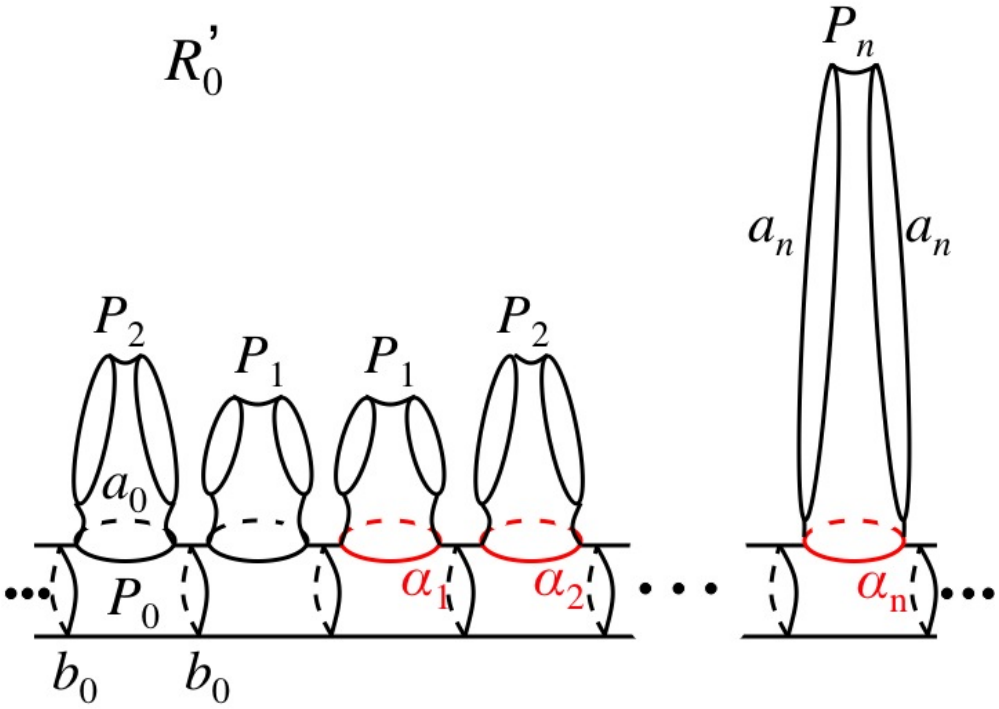
$\Rightarrow$ $d_T \nsim d_L$ *on* $T(R_0)$.

Example 1.
Any Riemann surface R_0 satisfying the assumption of Theorem of Liu-Sun-Wei satisfies the assumption of Theorem 1 by the collar lemma.

Example 2.
The Riemann surface R_0 constructed by Shiga (the first example of R_0 s.t. $d_T \nsim d_L$ on $T(R_0)$) satisfies the assumption of Theorem 1. (This does not satisfy the assumption of Theorem of Liu-Sun-Wei.)

Example 3.
Except for Examples 1 and 2, we can construct a Riemann surface R_0 satisfying the assumption of Theorem 1 as follows:

P_0: a pair of pants with boundary lengths (a_0, b_0, b_0).
Make countable copies of P_0 and glue them as in the below Figure.
$\{a_n\}_{n=1}^{\infty}$: a monotone divergent sequence of positive numbers.
P_n: a pair of pants with boundary lengths (a_0, a_n, a_n).
Make two copies of P_n and glue each copy with the union of the copies of P_0 as in the below Figure.
Let R_0' denote a Riemann surface with boundary we have obtained.



Take some pants $\{P_m'\}_{m=1}^{\infty}$ and define $R_0 := R_0' \cup \bigcup_{m=1}^{\infty} P_m'$.

We prove Theorem 1. (We write it again.)

Theorem 1

R_0: a Riemann surface.
$\exists\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *s.t. for* $\forall\{\beta_n\}_{n=1}^{\infty} \subset \Sigma_{R_0}$ *with* $\alpha_n \cap \beta_n \neq \emptyset$ $(n = 1, 2, \dots)$,

$$\frac{\sharp(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty).$$

$\Rightarrow$ $d_T \nsim d_L$ *on* $T(R_0)$.

The proof is short.

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Theorem 1

R_0: a Riemann surface.
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$$\frac{\sharp(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 0 \ (n \to \infty).$$

$\Rightarrow$ $d_T \nsim d_L$ on $T(R_0)$.

The proof is short. We use the following lemma.

Matsuzaki's Lemma

α: a simple closed geodesic on a Riemann surface R_0,
$f : R_0 \to R_0$: the n-times Dehn twist along α.
$\Rightarrow$ The maximal dilatation $K(f)$ of an extremal quasiconformal map of f satisfies

$$K(f) \geq \left\{ \left(\frac{(2|n| - 1)\ell_{R_0}(\alpha)}{\pi} \right)^2 + 1 \right\}^{1/2}.$$

3. A counterexample to the converse of Shiga's theorem

Shiga gave a sufficient condition for d_T and d_L to define the same topology on the Teichmüller space.

Theorem (Shiga)

R_0: a Riemann surface
$\exists$ a pants decomposition $R_0 = \cup_{k=1}^{\infty} P_k$ satisfying following conditions:
(1)Each connected component of ∂P_k is either a puncture or a simple closed geodesic of R_0.$(k = 1, 2, \dots)$
(2)$\exists M > 0$ s.t. if α is a boundary curve of some P_k then

$$0 < M^{-1} < \ell_{R_0}(\alpha) < M.$$

$\Rightarrow d_T \sim d_L$ on $T(R_0)$.

3. A counterexample to the converse of Shiga's theorem

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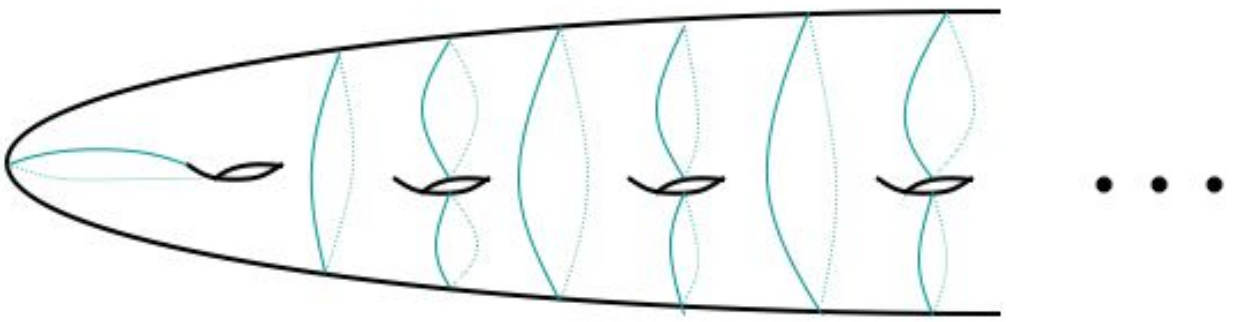
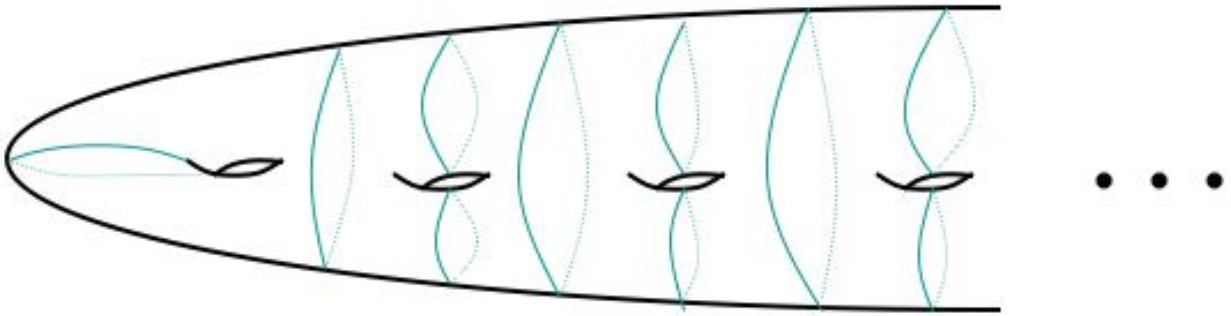
Example 1.
All Riemann surfaces of finite topological type satisfy Shiga's condition.

Example 2.
Some Riemann surfaces of infinite topological type satisfy Shiga's condition.



Example 2.
Some Riemann surfaces of infinite topological type satisfy Shiga's condition.



Now, we show that the converse of Shiga's theorem is not true by giving a counterexample.

Counterexample (to the converse of Shiga's theorem)

Γ: a hyperbolic triangle group of signature (2,4,8) acting on $\mathbb{D}$

P: a fundamental domain for Γ with angles $(\pi, \pi/4, \pi/4, \pi/4)$.

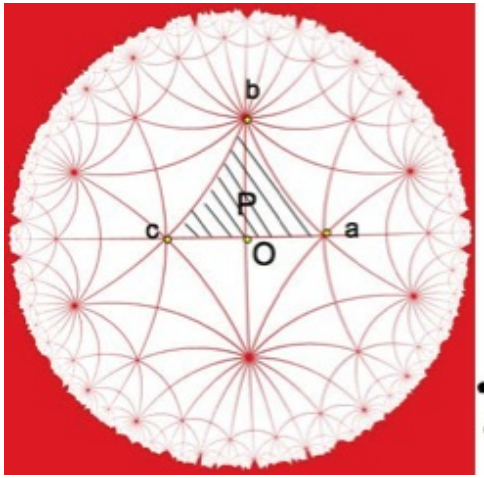
O, a, b, c: the vertices of P, where the angle at O is π.

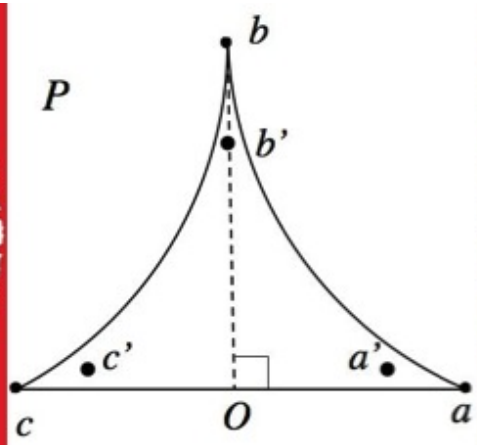
$\varepsilon > 0$: a sufficiently small number

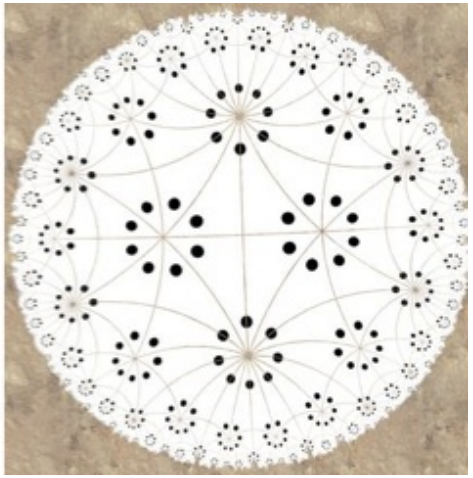
b': the point on the segment $[Ob]$ whose hyperbolic distance from b is ε.

Similarly, we take a' and c' in P.

We define a Riemann surface R_0 by removing the Γ-orbits of a', b', c' from the unit disk $\mathbb{D}$; $R_0 := \mathbb{D} - \{\gamma(a'), \gamma(b'), \gamma(c') \mid \gamma \in \Gamma\}$.

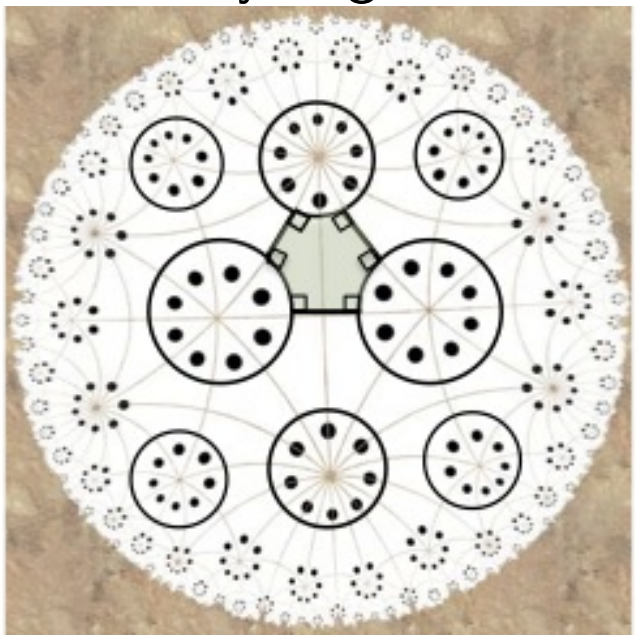






It is not difficult to show that R_0 does not satisfy the assumption of Shiga's Theorem.
It is difficult to show that $d_T \sim d_L$ on $T(R_0)$.

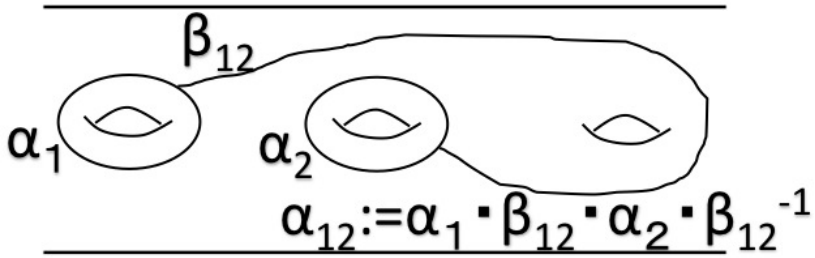
Outline of the proof (that $d_T \sim d_L$ on $T(R_0)$).
We show that for the sequence $\{p_n\}_{n=o}^{\infty} \subset T(R_0)$ s.t. $d_L(p_n, p_0) \to 0$ $(n \to \infty)$, $d_T(p_n, p_0)$ converges to 0 as $n \to \infty$.
We may assume that $p_0 = [R_0, id]$. Put $p_n = [R_n, f_n]$.
Step 1: We divide R_0 into punctured-disks and hyperbolic right-hexagons.
Also, we divide R_n for a sufficiently large n similarly.



In Step 1, we note the following lemma.

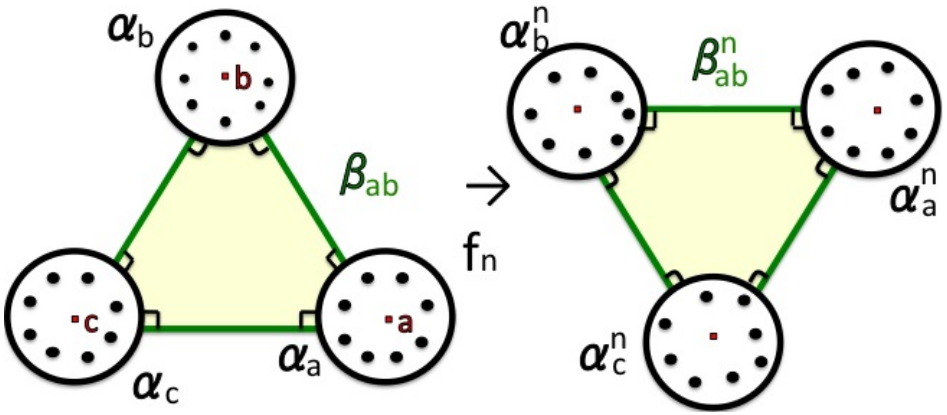
Lemma

R_0: a hyperbolic Riemann surface.
α_1, α_2: disjoint simple closed geodesics in R_0.
β_{12}: a simple arc connecting α_1 and α_2.
Assume that a closed curve $\alpha_{12} := \alpha_1 \cdot \beta_{12} \cdot \alpha_2 \cdot \beta_{12}^{-1}$ is non-peripheral.
$\Rightarrow \exists\ \beta_{12}^{\star}$: a geodesic connecting α_1 and α_2 s.t.
(1) β_{12} and $\beta_{12}^{\star}$ are homotopic, where the homotopy map moves each endpoint on each closed geodesic;
(2) $\beta_{12}^{\star}$ is orthogonal to α_1 and α_2;
(3) the length of $\beta_{12}^{\star}$ is determined by $\ell_{R_0}(\alpha_1)$, $\ell_{R_0}(\alpha_2)$ and $\ell_{R_0}(\alpha_{12})$.



Proof. This follows from properties of pants and right-hexagons. □

Step 2: We construct $(1 + C_n)$-qc maps from right-hexagons in R_0 to right-hexagons in R_n, where $C_n \to 0$ $(n \to \infty)$.



Step 3: We construct $(1 + C_n)$-qc maps from punctured-disks in R_0 to punctured-disks in R_n, where $C_n \to 0$ $(n \to \infty)$. Consequently, we obtain a quasiconformal map g_n of the whole of R_0 such that g_n is homotopic to f_n and $K(g_n) \to 1$ $(n \to \infty)$. Thus $d_T(p_n, p_0) \to 0$ $(n \to \infty)$.

In Steps 2 and 3, we use Bishop's Lemma.

Bishop's Lemma

$T_1, T_2 \subset \mathbb{D}$: two hyperbolic triangles with sides (a_1, b_1, c_1) and (a_2, b_2, c_2). Suppose all their angles are bounded below by $\theta > 0$ and

$$\varepsilon := \max(|\log \frac{a_1}{a_2}|, |\log \frac{b_1}{b_2}|, |\log \frac{c_1}{c_2}|) \leq A.$$

$\Rightarrow \exists\ C = C(\theta, A) > 0$ and $\exists$ a $(1 + C\varepsilon)$-quasiconformal map $\varphi : T_1 \to T_2$ such that φ maps each vertex to the corresponding vertex and φ is affine on the edge of T_1.

$\square$

4. Extension of Shiga's theorem

We extend Shiga's theorem as follows.

Theorem 2.

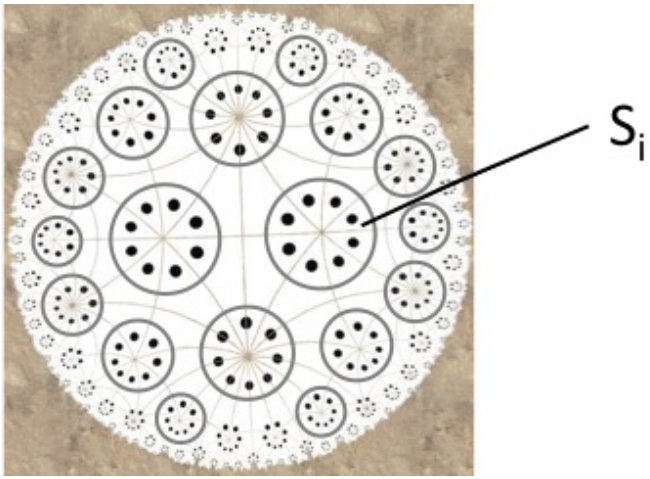
R_0: a Riemann surface.
$\exists\ M > 0$ and $\exists$ a decomposition

$$R_0 = S \cup (R_0 - S)$$

s.t.
(1) S is an open subset of R_0 whose relative boundary consists of simple closed geodesics and each connected component of S has a pants decomposition satisfying the same condition as that of Shiga's theorem for M
(2) $R_0 - S$ is of genus 0 and $d_{R_0}(x, S) < M$ for any $x \in R_0 - S$.
$\Rightarrow d_T \sim d_L$ on $T(R_0)$.

Examples 1.

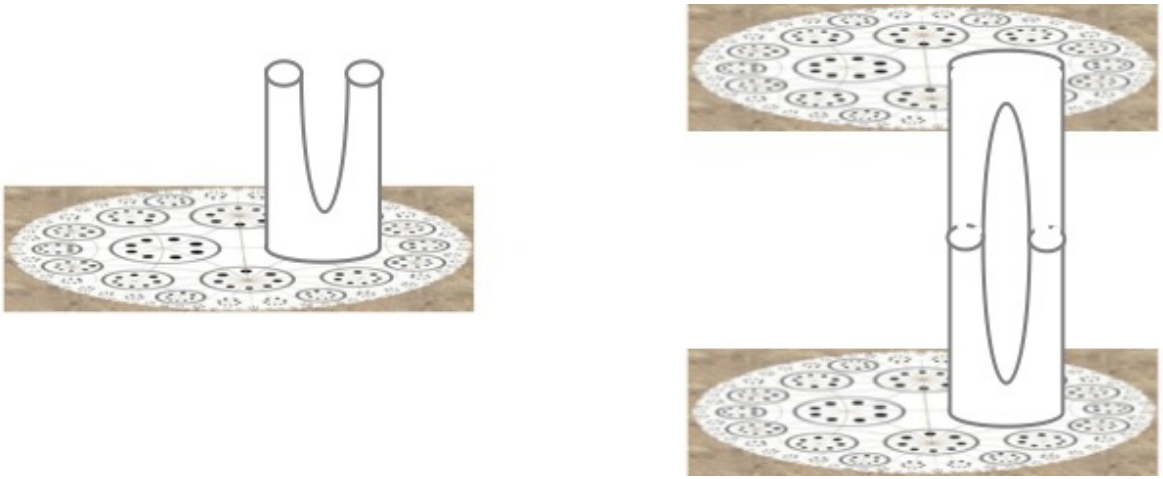
The above counterexample to the converse of Shiga's theorem satisfies Theorem 2. Let S_i $(i = 1, 2, ...)$ be a punctured disk, then $S = \cup_{i=1}^{\infty} S_i$.



Also, we can construct a Riemann surface R_0 satisfying Theorem 2 by replacing a hyperbolic triangle group Γ with an arbitrary Fuchsian group with a compact fundamental region.

Examples 2.

In a Riemann surface of Example 1, we replace a punctured disk S_i with a Riemann surface satisfying Shiga's condition. We regard it as a block and construct a Riemann surface R_0 with two or more holes. (See figure.) Then R_0 satisfies Theorem 2.



Corollary 3.

Let R_0 be a Riemann surface with bounded geometry. Also, assume that R_0 has finite genus.
$\Rightarrow d_T \sim d_L$ *on* $T(R_0)$.

We say that a Riemann surface R_0 has *bounded geometry* if it satisfies the following condition:
There exists a constant $M > 0$ such that any closed geodesic has the length greater than $1/M$ and for any $x \in R_0$, there exists a closed curve based on x with the length less than M.

Reference

- C. J. Bishop, *Quasiconformal mapping of Y-pieces*; Rev. Mat. Iberoamericana **18** (2002), 627-652.
- E. Kinjo, *On Teichmülcer metric and the length spectrums of topologically infinite Riemann surfaces*; Kodai Math. J. **34** (2011), 179-190.
- L. Liu, Z. Sun and H. Wei, *Topological equivalence of metrics in Teichmüller space*; Ann. Acad. Sci. Fenn. Math. Vol. **33** (2008), 159-170.
- K. Matsuzaki, *The infinite direct product of Dehn twists acting on infinite dimensional Teichmüller spaces*; Kodai Math. J. **26** (2003), 279-287.
- H. Shiga, *On a distance by the length spectrum on Teichmüller space*; Ann. Acad. Sci. Fenn. Series A, I Math. **28** (2003), 315-326.
- T. Sorvali, *The boundary mapping induced by an isomorphism of covering groups*; Ann. Acad. Sci. Fenn. Series A, I Math. **526** (1972), 1-31.