Energy-minimal diff. between doubly conn. Riemann surfaces

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Harmonic mappings between Riemann surfaces

Let (M, σ) and (N, ρ) be Riemann surfaces with metrics σ and ρ, respectively. If a mapping $f : (M, \sigma) \to (N, \rho)$ is C^2, then f is said to be harmonic (to avoid the confusion we will sometimes say ρ-harmonic) if

$$f_{z\bar{z}} + (\log \rho^2)_w \circ f f_z f_{\bar{z}} = 0, \quad (1)$$

where z and w are the local parameters on M and N respectively.

Also f satisfies (1) if and only if its Hopf differential $\Psi = \rho^2 \circ f f_z \overline{f_{\bar{z}}}$ is a holomorphic quadratic differential on M. For $g : M \mapsto N$ the energy integral is defined by

$$E_\rho[g] = \int_M (|\partial g|^2 + |\bar{\partial} g|^2) dV_\sigma.$$

Then f is harmonic if and only if f is a critical point of the corresponding functional where the homotopy class of f is the range of this functional.

Mappings of finite distortion

A homeomorphism $w = f(z)$ between planar domains Ω and D has finite distortion if a) f lies in the Sobolev space $W^{1,1}_{loc}(\Omega, D)$ of functions whose first derivatives are locally integrable, and b) f satisfies the distortion inequality $|f_{\bar{z}}| \leq \mu(z)|f_z|$, $0 \leq \mu(z) < 1$ almost everywhere in Ω. Such mappings are generalizations of quasiconformal homeomorphisms.

The Nitsche conjecture

The conjecture in question concerns the existence of a har. homeo. between circular annuli $A(r, 1)$ and $A(\tau, \sigma)$, and is motivated in part by the existence problem for doub.-conn. min. surf. with prescribed boundary. In 1962 J. C. C. Nitsche observed that the image annulus cannot be too thin, but it can be arbitrarily thick (even a punctured disk).

Indeed Nitsche observed that a radial harmonic mapping $f : A(r, 1) \to A(\tau, \sigma)$, i.e. satisfy. the condition $f(se^{it}) = f(s)e^{it}$ ($f(z) = az + b/\bar{z}$), is a homeomorphism if and only if $\frac{\sigma}{\tau} \geq \frac{1}{2}\left(\frac{1}{r} + r\right)$ (Nitsche bound).

Then he conjectured that for arbitrary harmonic homeomorphism between annuli we have Nitsche bound. Some partial solutions are presented by Weitsman, Lyzzaik, Kalaj (2001,2003).

The Nitsche was right

The Nitsche conjecture for Euclidean harmonic mappings is settled recently by Iwaniec, Kovalev and Onninen (2010, JAMS), showing that, only radial harmonic mappings

$h(\zeta) = C\left(\zeta - \frac{\omega}{\overline{\zeta}}\right)$, $C \in \mathbb{C}$, $\omega \in \mathbb{R}$,

$|C|(1-\omega) = \sigma$, **which inspired the Nitsche conjecture, make the extremal distortion of rounded annuli.**

Radial $\rho-$harmonic mappings

We state a similar conjecture (see Kalaj, arXiv:1005.5269) with respect to $\rho-$ harmonic mappings. In order to do this, we find all examples of radial ρ-harmonic maps between annuli. We put $w(z) = g(s)e^{it}, \quad z = se^{it}$ in harm. eq. $h_{z\bar{z}} + (\log \rho^2)_w \circ h h_z h_{\bar{z}} = 0$ where g is an increasing or a decreasing function.

The resulting functions are $w(z) = g(s)e^{it}$, $z = se^{it}$, **where** g **is the inverse of**

$$h(\gamma) = \exp\left(\int_{\sigma}^{\gamma} \frac{dy}{\sqrt{y^2 + c\varrho^2}}\right), \ \tau \leq \gamma \leq \sigma,$$

$\varrho = 1/\rho$. **Moreover they are homeomorphisms iff we have** $\rho-$**Nitsche bound:**

$$r \geq \exp\left(\int_{\sigma}^{\tau} \frac{\rho(y)dy}{\sqrt{y^2\rho^2(y) - \tau^2\rho^2(\tau)}}\right).$$

Generalization of Nitsche conjecture

Let ρ be a radial metric. If $r < 1$, and there exists a $\rho-$ harmonic mapping of the annulus $A' = A(r, 1)$ onto the annulus $A = A(\tau, \sigma)$, then there hold the $\rho-$Nitsche bound. Notice that if $\rho = 1$, then this conjecture coincides with standard Nitsche conjecture. For some partial solution see Kalaj (2011, Israel J. of Math)

Another justification of the previous conjecture

Assume that a homeomorphism $f : A(1, r) \to A(\tau, \sigma)$ has a finite distortion and minimize the integral means $\mathcal{K}_\rho[f] = \int_\Omega \mathbb{K}(z, f)\rho^2(z)dxdy$. Here $\mathbb{K}(z, f) = (|f_z|^2 + |f_{\bar{z}}|^2)/(|f_z|^2 - |f_{\bar{z}}|^2)$ is Distortion function. If the annuli $A(1, r)$ and $A(\tau, \sigma)$ are conformaly equivalent then the absolute minimum is achieved by a conformal mapping.

If the annuli $A(1, r)$ and $A(\tau, \sigma)$ are not conformaly equivalent then the absolute minimum is achieved by a homeomorphism f whose inverse h is a ρ harmonic mapping between $A(\tau, \sigma)$ and $A(1, r)$ if and only if the annuli satisfy the $\rho-$Nitsche bound.

The previous result is generalization of a result of Astala, Iwaniec and Martin (2010, ARMA) and has been obtained by Kalaj (2010, arXiv:1005.5269).

If the annuli $A(1, r)$ and $A(\tau, \sigma)$ do not satisfies the $\rho-$Nitsche condition, then no such homeomorphisms of finite distortion exists between them.

Energy minimizers, radial metric, (Kalaj, arXiv:1005.5269)

Let ρ be a regular metric. Within the Nitsche rang, for the annuli A and A', the absolute minimum of the energy integral $h \to E_\rho[h], \quad h \in W^{1,2}(A, A')$ is attained by a $\rho-$Nitsche map

$$h^c(z) = q^{-1}(s)e^{i(t+\beta)}, \ z = se^{it}, \ \beta \in [0, 2\pi),$$

where

$$q(s) = \exp\left(\int_\sigma^s \frac{dy}{\sqrt{y^2+c\varrho^2}}\right), \tau < s < \sigma.$$

Energy minimizers, Euclidean metric

Let A and A' be doub. conn. domains in compl. plane such that $\mathrm{Mod}(A) \leq \mathrm{Mod}(A')$. The absolute minimum of the energy integral $h \to E[h], \quad h \in W^{1,2}(A, A')$ is attained by an Euclidean harmonic homeomorphism between A and A'. (Iwaniec, Kovalev, Onninen, Inventiones, (2011))

Energy minimizers, Arbitrary metric with bounded Gauss curvature

Let A and A' be doub. conn. domains in Riemann surfaces (M, σ) and (N, ρ), such that $\mathrm{Mod}(A) \leq \mathrm{Mod}(A')$. The absolute minimum of the energy integral $h \to E_\rho[h], \quad h \in W^{1,2}(A, A')$ is attained by an $\rho-$ harmonic homeomorphism between A and A'. (See Kalaj, arXiv:1108.0773)

The proof uses

- **The so called deformations $D^{\rho}(\Omega, \Omega^*)$ which make a compact family.**
- **A modification of Choquet-Rado-Kneser theorem for Riemann surfaces with bounded Gauss curvature**
- **The fact that energy integral is weak lower semicontinuous.**

And the key lemma

Let $\Omega = A(r, R)$ be a circular annulus, $0 < r < R < \infty$, and Ω^* a doubly connected domain. If $h \in D^\rho(\Omega, \Omega^*)$ is a stationary deformation, then the Hopf differential of the mapping h: $\rho^2(h(z))h_z\overline{h_{\bar{z}}} \equiv \frac{c}{z^2}$ in Ω where $c \in \mathbb{R}$ is a constant. This lemma follows by a result of Jost.

Conjecture a)

If f is a minimizer of the energy E^ρ between two doubly connected domains Ω, ($\tau = \mathrm{Mod}(\Omega)$) and Ω^*, then f is harmonic and $K(\tau)$-quasiconformal if and only if τ is smaller that the modulus $\tau_\diamond$ of critical Nitsche domain $A(\tau_\diamond)$.

Conjecture b)

b) Under condition of a) we conjecture that $c_\diamond < 0$ and

$$K(\tau) = \max\{\sqrt{\frac{c_\diamond - c}{c_\diamond}}, \sqrt{\frac{c_\diamond}{c_\diamond - c}}\}. \quad (1)$$

Notice that, this is true if the image domain is a circular annulus with a radial metric.

Quasiconformal harmonic mappings (HQC)

The first author who studied the class HQC is Olli Martio on 1968 (AASF). The class of HQC has been studied later by: Hengartner, Schober, Mateljević, Pavlović, Partyka, Sakan, Kalaj, Vuorinen, Manojlović, Nesi, Alessandrini, Božin, Marković, Wan, Onninen, Iwaniec, Kovalev etc.

Lipschitz and co-Lipschitz mappings

Let $w : \Omega \to D$ be a mapping between two domains Ω and D. Then w is called Lipschitz (co-Lipschitz) continuous if there exists a constant $C > 1$ $(c > 0)$ such that

$$|w(z) - w(z')| \leq C|z - z'|,$$

$$(c|z - z'| \leq |w(z) - w(z')|) \quad z, z' \in \Omega.$$

Kellogg theorem

Let $w : \Omega \to D$ be a conformal map between two Jordan d. $\Omega, D \in C^{1,\alpha}$. Then $w \in C^{1,\alpha}(\overline{\Omega})$, $w^{-1} \in C^{1,\alpha}(\overline{D})$.

Corollary

Under the conditions of Kellogg theorem w is bi-Lipschitz continuous.

There exists a conf. mapp. of the unit disk onto a C^1 Jordan domain which is not Lips. cont. (Lesley & Warschawski, 1978, Math Z).

Motivation

Let $f(e^{it}) = e^{i(t+\alpha \sin t)}$, $0 \leq \alpha \leq 1$ and let $w = P[f](z)$ be harmonic extension of f in the unit disk. In my master thesis (1997), I posed the following question, is w q.c. At that time I was not aware of the paper of Martio (1969, AASF). It follows by results of Martio that, w is q.c. if and only if $\alpha < 1$, or what is the same iff f is bi-Lipschitz.

Kellogg type results, Lipschitz continuity

Let $w : \Omega \to D$ be a q.c and $\mathbb{U}$ be the unit disk. Then w is Lipschitz provided that

- $\Omega = D = \mathbb{U}$ **and** $\Delta u = 0$**: (Pavlovic-AASF 2002)**
- $\Omega = D = \mathbb{U}$ **and** $\Delta u = 0$**: (Partyka & Sakan quant. estim. - AASF, 2005, 2007)**
- $\Omega = D =$**Half-plane and** $\Delta u = 0$**: (Pavlovic & Kalaj, 2005, AASF)**

Kellogg type results, Lipschitz continuity

- $\Omega = D =$**Half-plane and** $\Delta u = 0$**, explicit constants (Mateljević& Knežević, 2007, JMAA)**
- **If** $\partial\Omega, \partial D \in C^{2,\alpha}$ **and** $|\Delta w| \leq A|\nabla w|^2 + B$**, (Mateljević & Kalaj, J. D. Analyse, 2006 & Potential Analysis 2010)**
- **If** $\partial\Omega, \partial D \in C^{1,\alpha}$**, and** $\Delta w = 0$ **(Kalaj, Math Z, 2008)**

Kellogg type results, bi-Lipschitz continuity with respect to quasihyperbolic metric

- **If w is hyp. harm. q.c. mapping of the unit disk onto itself, then w is bi-Lipschitz w. r. to hyp. metric (Wan, J. Dif. Geom. 1992).**
- **If $f : \Omega \to D$ is a quasiconformal and harmonic mapping, then it is bi-Lipschitz with respect to quasihyperbolic metric on D and D', (Manojlović, 2009, Filomat)**

Higher dimensional, Lipschitz continuity

- **If $\partial\Omega, \partial D \in C^{2,\alpha}$ and $|\Delta w| \leq A|\nabla w|^2 + B$, (Kalaj, to appean in J. d' Analyse)**
- **Some other High. dim. gen. (Mateljevic & Vuorinen, JIA-2010)**

- **Hyperbolic harmonic (Tam & Wan, Pac. J. math, 1998)**
- **If $\Omega = D =$ Unit ball and $\Delta w = g$, then u is $C(K)$ Lipschitz with**

Kellogg type results, co-Lipschitz continuity

Let $w : \Omega \to D$ be a q.c. Then w is co-Lipschitz provided that

- $\Omega = D = \mathbb{U}$ **and** $\Delta u = 0$**: (Follows by Heinz theorem)**
- **If** $\partial\Omega \in C^{1,\alpha}$**,** D **is convex and** $\Delta w = 0$ **(Kalaj-2002,)**
- **If** $\partial\Omega, \partial D \in C^{2,\alpha}$ **and** $\Delta w = 0$**, (Kalaj, AASF-2009)**
- **If** $\partial\Omega, \partial D \in C^2$ **and** $\Delta w = 0$**, (Kalaj, Annali SNSP, 2011.)**

Main tools of the proofs of results for HQC

- **Poisson integral formula**
- **Lewy's theorem**
- **Choquet-Rado-Kneser theorem**
- **Mori's theorem**
- **Kellogg theorem**
- **Isoperimetric inequality**
- **Hopf's Boundary Point lemma**
- **Heinz inequality**
- **Max. principle for (sub)harmonic**

- **Distance function from the boundary of the domain Ω:** $d(x) = \operatorname{dist}(x, \partial\Omega)$.
- **Möbius transformations of the unit disk**

$$w = e^{it}\frac{z-a}{1-z\bar{a}}, \quad |z| < 1, |a| < 1$$

or of the unit ball B^n.

- **The Carleman-Hartman-Wintner lemma**

Composing by conformal mappings

If w is a harmonic and g a conformal mapping, then the function $w \circ g$ is harmonic, but the function $g \circ w$ need not be harmonic. Thus the results, for example for the half plane and for the unit disk are not equivalent.

The Poisson integral formula

Let $P(r,x) = \frac{1-r^2}{2\pi(1-2r\cos x+r^2)}$ **denote the Poisson kernel. Then every bounded harm. func.** w **defined on the unit disk** $\mathbf{U} := \{z : |z| < 1\}$ **has the foll. rep.** $w(z) = P[f](z) = \int_0^{2\pi} P(r, x-\varphi) f(e^{ix}) dx$, **where** $z = re^{i\varphi}$ **and** f **is a bounded integr. func. def. on the unit circle T. There exist Poisson integral formula for the halp-plane (space) and for the unit ball.**

The Mori's theorem (Fehlmann & Vuorinen (AASF))

If u is a K quasi-conformal self-mapping of the unit ball B^n with $u(0) = 0$, then there exists a constant $M_1(n, K)$, satisfying the condition $M_1(n, K) \to 1$ as $K \to 1$, such that

$$|u(x) - u(y)| \leq M_1(n, K)|x - y|^{K^{1/(1-n)}}. \quad (2)$$

The Lewy theorem

If $f = g + \overline{h}$ is a univalent harmonic mapping between two plane domain, then $J_f := |g'|^2 - |h'|^2 > 0$. Some extensions of this theorem have been done by: Schulz and Berg.

The Choquet-Rado-Kneser theorem

If $f = g + \overline{h}$ is a harmonic mapping between two Jordan domains Ω and D such that D is convex, and $f|_{\partial\Omega} : \partial\Omega \to \partial D$ is a homeomorphism (or more general if $f|_{\partial\Omega}$ is a pointwise limit of homeomorphisms) then f is univalent.

Some extensions are given by: Alessandrini and Nessi, Kalaj, Duren, Schober, Jost, Yau and Schoen.

The Heinz theorem, Pac. J. Math, 1959

If $f = g + \overline{h}$ is a univalent harmonic mapping of the unit disk onto itself with $f(0) = 0$, then there holds the inequality

$$|g'(z)|^2 + |h'(z)|^2 \geq \frac{2}{\pi^2}, \quad z \in \mathbf{U}.$$

This theorem has been generalized by Kalaj on 2002 (Comp. Variabl.)

The Hilbert transformation

of a function χ is defined by the formula

$$H(\chi)(\varphi) = -\frac{1}{\pi}\int_{0+}^{\pi} \frac{\chi(\varphi+t)-\chi(\varphi-t)}{2\tan(t/2)}\mathrm{d}t$$

for a.e. φ and $\chi \in L^1(S^1)$. Assume that $w = P[F](z)$, $z = re^{it} \in \mathbb{U}$, $F' \in L^1(S^1)$. Then, if w_t and rw_r are bounded, there hold

$$w_t = P[F'] \text{ and } rw_r = P[H(F')]. \qquad (3)$$

Characterization theorem for HQC

Let $F : S^1 \to \gamma$ be a sense pr. homeo. of the unit circle onto the Jordan curve $\gamma = \partial D \in C^2$. Then $w = P[F]$ is a q.c. mapping of the unit disk onto D if and only if F is abs. cont. and

$$0 < I(F) := \operatorname{ess\,inf} I(\nabla w(e^{i\tau})), \quad (4)$$

$$||F'||_\infty := \operatorname{ess\,sup} |F'(\tau)| < \infty \quad (5)$$

$$||H(F')||_\infty := \operatorname{ess\,sup} |H(F')(\tau)| < \infty. \quad (6)$$

If F satisfies the conditions (4), (5) and (6), then $w = P[F]$ is K quasiconformal, where

$$K := \frac{\sqrt{||F'||_\infty^2 + ||H(F')||_\infty^2 - I(F)^2}}{I(F)}. \quad (7)$$

The constant K is the best possible in the following sense, if w is the identity or it is a mapping close to the identity, then $K = 1$ or $K$is close to 1 (respectively).

Remark

a) This theorem has been proved by Pavlović in (AASF, 2002) for $D =$ U.

b) If the image domain D is convex, then the condition (4) is equivalent (and can be replaced) with

$$0 < l(F) := \operatorname{ess\,inf} |F'(\tau)|, \qquad (8)$$

(Kalaj, Math z. 2008)

c) In this form by Kalaj (Arxiv). It is a general. of a theorem of Choquet.

Elliptic operator

Let $A(z) = \{a^{ij}(z)\}_{i,j=1}^{2}$ **be a symm. matrix function,** $z \in \Omega \subset \mathbf{C}$**. Assume that**

- $\Lambda^{-1} \leq \langle A(z)h, h\rangle \leq \Lambda$ **for** $|h| = 1$, $\Lambda \geq 1$
- $|A(z) - A(\zeta)| \leq L|\zeta - z|$ $z, \zeta \in D$**.**

For $L[u] := \sum_{i,j=1}^{2} a^{ij}(z) D_{ij} u(z)$ **under previous cond. consider the diff. ineq.** $|L[u]| \leq \mathcal{B}|\nabla u|^2 + \Gamma$**, (***ell. part. diff. ineq.***),** $\mathcal{B} > 0$ **and** $\Gamma > 0$**. If** $A = Id$**, then** $L = \Delta$

Theorem 1

(To appear in Proceedings A of The Royal Society of Edinburgh)
If $w : \mathbf{U} \to \mathbf{U}$ is a q.c. solution of the elliptic partial differential inequality

$$|L[w]| \leq a|\nabla w|^2 + b,$$

then ∇w is bounded and w is Lipschitz continuous.

The main steps of the proofs

The proofs are differs from the proof of corresponding results for the class HQC. Some methods of the proof are borrowed from the paper of Nagumo. Two main steps in the proof are the following two lemmas

Global behaviour

Let D be a complex domain with diameter d satisfying exterior sphere cond. Let $u(z)$ be a twice diff. mapping satisfying the ell. diff. ineq. in D, $u = 0$ $(z \in \partial D)$. Assume also that $|u(z)| \leq M$, $z \in D$, $64\mathcal{B}\Gamma M < \pi$ and $u \in C(\overline{D})$. Then $|\nabla u| \leq \gamma, \quad z \in D,$ where γ is a constant depending only on M, $\mathcal{B}$, Γ, $\mathfrak{L}$, Λ and d.

Interior estimate of gradient

Let D be a bounded domain, whose diameter is d. Let $u(z)$ be any C^2 solution of elliptic partial differential inequality such that $|u(z)| \leq M$. Then there exist constants $C^{(0)}$ and $C^{(1)}$, depending on modulus of continuity of u, Λ, L, B, Γ, M and d such that

$$|\nabla u(z)| < C^{(0)}\rho(z)^{-1} \max_{|\zeta - z| \leq \rho(z)}\{|u(\zeta)|\} + C^{(1)}$$

where $\rho(z) = \operatorname{dist}(z, \partial D)$.

The idea of the proof of Theorem 1

By using the interior estimates of gradient and estimates near the boundary we show global and a priory bondedness of gradient of a q.c. mapping w. It is previously showed that the function $u = |w|$ satisfies a certain elliptic differential inequality near the boundary of the unit disk. By using the quasiconformality, we prove that ∇w is bounded by a constant not depending on w.

Some open problems

- **Is every q.c. harmonic between smooth domains on the space bi-Lipschitz continuous?**
- **Is every q.c. solution of elliptic PDE between smooth domains on plane or on the space bi-Lipschitz continuous?**
- **Do some q.c. harmonic mappings on the space have critical points?**

Thanks for attention