# STABILITY AND COERCIVITY FOR TORIC POLARIZATIONS

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ABSTRACT. We prove that a toric polarized manifold is uniformly K-stable in the toric sense if and only if the K-energy functional is coercive modulo the maximal torus action.

## CONTENTS



## INTRODUCTION

The idea of *uniform K-stability* was first introduced by the thesis [Szé06] and developed as it can be seen in [Der14a], [Der14b], [BBJ15], [BHJ15], [DR15], [BHJ16], and [BDL16]. Especially it was shown by [BDL16] that if the automorphism group is discrete, any polarized manifold $(X, L)$ admitting a constant scalar curvature Kähler metric is uniformly K-stable.

In analytic point of view the counterpart of the uniform K-stability should be the coercivity property of the K-energy. Let $\mathcal{H}$ be the collection of positively curved fiber metrics on $L$ and denote by $\varphi_g$ the pull-back of $\varphi \in \mathcal{H}$ by the bundle automorphism $g \in \mathrm{Aut}(X, L)$.

**Definition.** Let $G$ be a closed algebraic subgroup of the identity component $\mathrm{Aut}^0(X, L)$. We say that K-energy functional $M : \mathcal{H} \to \mathbb{R}$ is $G$-coercive (with respect to Aubin's $J$-functional) if there exists a constant $\delta, C \in \mathbb{R}_{>0}$ such that for any $\varphi \in \mathcal{H}$

$$M(\varphi) \geqslant \delta \inf_{g \in G} J(\varphi_g) - C$$

holds.

This growth condition for the K-energy originates from Aubin's strong Moser-Trudinger inequality on the two-sphere. The relation with the existence of a Kähler-Einstein metric was first discussed by [Tian97]. It assures the critical point in a certain completion of $\mathcal{H}$ and in fact in the Fano case the obtained a priori singular metric defines a smooth

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Kähler-Einstein metric (see [BBGZ13]). As well, the uniform estimate could help seeking a constant scalar curvature Kähler metric. See also [BBJ15].

This paper is devoted to establish one expected correspondence between the stability and the coercivity, for toric polarized manifolds. Instead we restrict ourselves to the space of torus-invariant metrics $\mathcal{H}^{\mathbb{S}}$ and toric test configurations, according to the symmetry of this class of manifolds. A toric test configuration is represented by a convex, rational piecewise-linear function $f$ on the moment polytope $P \subset M_{\mathbb{R}}$. Following [Don02] we define

$$L(f) := \int_{\partial P} f - \frac{\operatorname{area}(\partial P)}{\operatorname{vol}(P)} \int_P f. \tag{0.1}$$

In addition, let us introduce a new invariant *J-norm* as

$$\|f\|_J := \inf_{\ell} \left\{ \frac{1}{\operatorname{vol}(P)} \int_P (f+\ell) - \min_P \{f+\ell\} \right\}, \tag{0.2}$$

where $\ell$ runs through all the affine functions. This gives the toric correspondence of the non-Archimedean J-functional introduced by [BHJ15]. See also Proposition 7.8 of [BHJ15].

**Main Theorem.** For any toric polarized manifold with the maximal torus $\mathbb{T}$, K-energy functional is $\mathbb{T}$-coercive on $\mathcal{H}^{\mathbb{S}}$ if and only if there exists a constant $\delta > 0$ such that

$$L(f) \geqslant \delta \|f\|_J \tag{0.3}$$

holds for any convex, rational piecewise-linear function $f : P \to \mathbb{R}$.

We call the algebraic condition *uniform K-stability in the toric sense*. Whenever $f$ is affine, the condition yields $L(f) = 0$ hence it includes classical Futaki's obstruction. The proof of the coercivity is essentially due to [ZZ08b] where they adopt the larger "boundary norm" to measure the uniformity of stability. At the same time [ZZ08b], Theorem 0.2 assures a lot of example of uniformly K-stable toric polarizations. The converse implication needs rather new argument, however, as the both proofs show our main declaration is that $J$-norm more naturally fits into the coercivity concept and it could work for general polarizations. In fact we derive the stability from $\mathbb{T}$-coercivity for general polarized manifolds. One key ingredient is the slope formula for the energies which originates from the seminal work of G. Tian. The other is the generalization of Hilbert-Mumford type argument to the stability of pairs, developed by [Paul12]. Lastly we refer the paper [CLS14] where they claim that a toric polarization with a constant scalar curvature Kähler metric satisfies our uniform K-stabiliy. It combined with our result yields that $\mathbb{T}$-coercivity follows from the existence of a constant scalar curvature Kähler metric. In one conclusion the problem of finding a constant scalar curvature Kähler metric on a toric polarized manifold is reduced to the regularity of a weak solution.

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## 1. Preliminary toric materials

We start from quickly reviewing the toric setting, as convenience for the readers. See [Gui94], [Abr98], [Don02], and [SZ12] for the detail. Let $M$ be a integral lattice of rank $n$. A toric polarized manifold is defined by a Delzant polytope $P$ in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. In our notation $P$ contains its boundary and we denote the interior by $P^{\circ}$. Let us denote the dual lattice by $N$. Then the complex torus $\mathbb{T} := N \otimes \mathbb{C}^*$ naturally acts on $X$, having an open dense orbit. Notice that in the setting there exists a natural real form $\mathbb{S} := N \otimes_{\mathbb{Z}} \mathbb{S}^1$.

1.1. **Torus invariant metrics.** An arbitrary moment map $\mu : X \to M_{\mathbb{R}}$ gives a homeomorphism $\mu^{-1}(P^{\circ}) \simeq \mathbb{S} \times P^{\circ}$ so that $N_{\mathbb{R}} \times P^{\circ}$ gives the universal cover of $X_0 := \mu^{-1}(P^{\circ})$. In a fixed coordinate $(x_1, \dots x_n)$ of $M_{\mathbb{R}}$ we have the complex coordinate $(z_1, \dots z_n)$ of $X_0 \simeq (\mathbb{C}^*)^n$ such that $\log z_i =: \xi_i + \sqrt{-1}\eta_i$ defines the dual coordinate $(\eta_1, \dots \eta_n)$ of $N_{\mathbb{R}}$. Then the Kähler metric defining $\mu$ is written in $X_0$ as

$$\omega = \sum dx_i \wedge d\eta_i.$$

Any $\mathbb{S}$-invariant Kähler metric $\omega_{\varphi} = dd^c\varphi$ is represented by the local potential $\varphi$ on $X_0$, which is a function in $\xi_i$. If one denotes it by $\psi(\xi_1, \dots \xi_n)$ the gradient

$$(z_1, \dots z_n) \mapsto \big(\frac{\partial \psi}{\partial \xi_1}, \dots, \frac{\partial \psi}{\partial \xi_n}\big)$$

gives the moment map for $\omega_{\varphi}$. Notice that the image $P$ is independent of the choice of moment maps.

The Legendre transform

$$\begin{aligned} u(x_1, \dots x_n) &:= \sup \big\{ \sum x_i \xi_i - \psi(\xi) \big\} \\ &= \sum x_i \xi_i - \psi(\xi) \;\; \text{when } x_i = \frac{\partial \psi}{\partial \xi_i} \end{aligned}$$

is called the symplectic potential. By the standard Delzant construction, one can prove that $u$ defines a convex function on $P^{\circ}$ such that if $P$ is written in the form

$$P = \big\{ x \in M_{\mathbb{R}} \mid \ell_k(x) := \langle x, \alpha_k \rangle - \beta_k \geqslant 0 \text{ for any } 1 \leqslant k \leqslant d \big\}, \tag{1.1}$$

then $u - \frac{1}{2}\sum_{k=1}^{d} \ell_k \log \ell_k$ is smooth up to the boundary. Conversely, such a convex function defines an $\mathbb{S}$-invariant Kähler metric.

If one rescales $u$ to $u - \sum a_i x_i - b$ by an affine function, $\psi$ changes into $\psi(\xi_i + a_i) + b$. In particular we can rescale $u$ to have a point $x^0 \in P^{\circ}$ as a minimizer. Then rescaling again by

$$\sum a_i x_i + b = \sum \frac{\partial u}{\partial x_i}(x^0)(x_i - x_i^0) + u(x^0)$$

one obtains the symplectic potential $u$ satisfying

$$\begin{aligned} &(i) \quad \inf_P u = u(x^0) = 0 \quad \text{ for some } x^0 \in P^{\circ} \text{ and} \\ &(ii) \quad \frac{\partial u}{\partial x_i}(x^0) = 0 \quad \text{for all } 1 \leqslant i \leqslant n. \end{aligned}$$

We say $u$ is normalized at this end. Note that the change of variables $\xi_i \mapsto \xi_i + a_i$ just corresponds to the $\mathbb{T}$-action.

The scalar curvature is given by a fourth-order differential of $u$.

**Theorem 1.1** ([Abr98]). *(i) Hessian is transformed as*

$$(u_{ij}) := \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big) = \Big(\frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j}\Big)^{-1} \quad when\ x_i = \frac{\partial \psi}{\partial \xi_i}.$$

*(ii) Denoting $(u^{ij}) := (u_{ij})^{-1}$, (twice of) the scalar curvature is given by*

$$\begin{aligned} S_{\omega_\varphi} &:= \frac{n \operatorname{Ric} \omega_\varphi \wedge \omega_\varphi^{n-1}}{\omega_\varphi^n} \\ &= -\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} \quad when\ x_i = \frac{\partial \psi}{\partial \xi_i}. \end{aligned}$$

□

1.2. **Toric test configurations.** Let us continuously fix the coordinate of $M_{\mathbb{R}}$. By the definition of moment map, Hamilton functions associated with the vector fields in $N_{\mathbb{R}}$ are translated to affine functions on $P$. A general toric test configuration correspond to a convex, rational piecewise-linear function $f : P \to \mathbb{R}$. This is a torus-equivariant morphism $(\mathcal{X}, \mathcal{L}) \to (\mathbb{P}^1, \mathcal{O}(1))$ constructed by the big polytope

$$\mathfrak{P} := \big\{(u, \lambda) \in M_{\mathbb{R}} \times \mathbb{R} \mid f(u) \leqslant \lambda \leqslant \max_P f\big\}. \tag{1.2}$$

The corresponding toric polarization $(\mathcal{X}, \mathcal{L})$ in fact forms a $\mathbb{C}^*$-equivariant flat family of $\mathbb{Q}$-polarized schemes with $(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}) \simeq (X \times \mathbb{C}, p_1^* L)$ so that it give a test configuration in the terminology of [Don02]. Note that by the construction $\mathcal{X}$ is necessary normal. Toric version of the Donaldson-Futaki invariant is given by the form

$$\cdot\, L(f) = \int_{\partial P} f - \hat{S} \int_P f \tag{1.3}$$

with the volume $V = \mathrm{vol}(P)$ and with the mean of the scalar curvature

$$\hat{S} = \frac{\mathrm{area}(\partial P)}{\mathrm{vol}(P)}. \tag{1.4}$$

The integration on $P$ is for the Lebesgue measure of $M_{\mathbb{R}} \simeq \mathbb{R}^n$ and we adopt the area measure $d\sigma$ on $\partial P$ determined by

$$dx_1 \cdots dx_n = \pm d\sigma \wedge d\ell_k.$$

**Remark 1.2.** *The above $L(f)$ is not precisely equivalent to the Donaldson-Futaki invariant introduced by* [Don02] *for general polarized manifolds. Indeed $L(f)$ is homogeneous with respect to the* normalized *finite base change but the Donaldson-Futaki invariant is not. It actually corresponds to the* non-Archimedean K-energy $M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ *which can be defined by (*3.3*). All these invariants are equivalent when the central fiber $\mathcal{X}_0$ is reduced. See* [BHJ15] *for the detail.*

1.3. **Energy functionals.** Recall that classical K-energy $M$ and the Aubin-Mabuchi energy $E$ on $\mathcal{H}$ are characterized by their differential:

$$\delta M(\varphi) = -\frac{1}{V}\int_X (\delta\varphi)(S_\varphi - \hat{S})\omega_\varphi^n/n! \quad \text{and} \tag{1.5}$$

$$\delta E(\varphi) = \frac{1}{V}\int_X (\delta\varphi)\omega_\varphi^n/n!. \tag{1.6}$$

As a scale-free version of $E$, we use Aubin's J-functional given by

$$J(\varphi) = \frac{1}{V}\int_X \varphi\omega^n/n! - E(\varphi), \tag{1.7}$$

where $\omega$ is a reference metric to define $E$.

For a general polarized manifold $V$ denotes the self-intersection number $L^n$ and $\hat{S}$ denotes the mean of the scalar curvature $-\frac{n}{2}V^{-1}K_X L^{n-1}$. Restricted to $\mathbb{S}$-invariant metrics these functionals have the following description on $P$.

**Theorem 1.3.** *Let $u$ be the symplectic potential of an $\mathbb{S}$-invariant Kähler metric $\varphi$ on toric $(X, L)$. Then the energies are written in the forms*

(*i*)

$$VE(\varphi) = -\int_P u \quad \textit{and} \tag{1.8}$$

(*ii*)

$$VM(\varphi) = -\int_P \log\det(u_{ij}) + \int_{\partial P} u - \hat{S}\int_P u. \tag{1.9}$$

It is remarkable in the toric case that the linear part is given by

$$L(u) = \int_{\partial P} u - \hat{S}\int_P u. \tag{1.10}$$

Abuse of notation we will write the energies like $M(u)$ or $J(u)$ as functions in $u$. For the $J$-functional we recall:

**Lemma 1.4** ([ZZ08b], Lemma 2.1)**.** *There exists a constant $C > 0$ such that for any normalized symplectic potential $u$*

$$\left| \ J(\varphi) - \frac{1}{V}\int_P u \ \right| \leqslant C.$$

□

## 2. Stability to Coercivity

In this section we prove the half of the main theorem, from stability to coercivity direction. Let $\mathcal{S}$ be the collection of convex functions with the growth condition:

$$u - \frac{1}{2}\sum_k \ell_k \log \ell_k \ \in C^\infty(P).$$

By a standard approximation result (Proposition 5.2.8 and Corollary 5.2.5 of [Don02]), we may assume that $L(u) \geqslant \delta \|u\|_J$ holds for any $u \in \mathcal{S}$. We also need the space $\mathcal{C}_\infty$ which consists of continuous convex function smooth on $P^\circ$.

Fix $u_0 \in \mathcal{S}$ to set

$$L_0(u) := \int_{\partial P} u - \int_P \Big( -\frac{1}{2} \sum_{i,j} \frac{\partial^2 u_0^{ij}}{\partial x_i \partial x_j} \Big) u \quad \text{and} \tag{2.1}$$

$$VM_0(u) := -\int_P \log\det(u_{ij}) + L_0(u). \tag{2.2}$$

The idea of the proof originates from the following lemma.

**Proposition 2.1** ([Don02], Proposition 3.3.4)**.** *For any $u \in \mathcal{C}_\infty$, $M_0(u) \geqslant M_0(u_0)$ holds.*

If $u \geqslant 0$, it follows from Hölder's inequality that

$$|L(u) - L_0(u)| = \left| \int_P \Big( \hat{S} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 u_0^{ij}}{\partial x_i \partial x_j} \Big) u \right| \leqslant \frac{C}{V} \int_P u.$$

For an arbitrary $u \in \mathcal{S}$ one can take any affine function $\ell$ for which $(u+\ell) - \min_P\{u+\ell\}$ is applied to the above inequality so that

$$|L(u) - L_0(u)| \leqslant C \|u\|_J$$

holds. Let us decompose like

$$\|u\|_J = (1+k)\|u\|_J - k\|u\|_J$$

and bound the first term by $L(u) \geqslant \delta \|u\|_J$. This leads us

$$|L(u) - L_0(u)| \leqslant \delta^{-1} C(1+k) L(u) - Ck\|u\|_J$$

to conclude

$$\begin{aligned} VM(u) &\geqslant -\int_P \log\det(u_{ij}) + \frac{1}{1+\delta^{-1}C(1+k)} L_0(u) + \frac{Ck}{1+\delta^{-1}C(1+k)} \|u\|_J \\ &= VM_0\Big(\frac{u}{1+\delta^{-1}C(1+k)}\Big) - n\log(1+\delta^{-1}C(1+k)) + \frac{Ck}{1+\delta^{-1}C(1+k)} \|u\|_J . \end{aligned}$$

Then the value of $M_0$ in the first term is bounded from below by Proposition 2.1. Notice that as $k \to \infty$ the coefficient of $J$-norm approaches to $\delta$. In particular for normalized $u \in \mathcal{S}$ we obtain

$$M(u) \geqslant \delta' J(u) - C'$$

with effective $\delta'$ and $C'$.

For any affine function $\ell$, the non-linear term of $M(u+\ell)$ is the same as that of $M(u)$. It is equivalently to say

$$M(u+\ell) = M(u) + L(\ell). \tag{2.3}$$

The stability assumption yields $L(\ell) \geqslant \|\ell\|_J = 0$ and $L(-\ell) \geqslant 0$ since $-\ell$ is also an affine function. Thus we conclude

$$M(u) \geqslant \delta' \inf_{\ell} J(u+\ell) - C'.$$

As we noted, adding an affine function corresponds to the $\mathbb{T}$-action so that we get $\mathbb{T}$-coercivity.

□

To show the converse, it is natural to attach the "ray" emanating from $u \in \mathcal{S}$, to a convex $f$. For example this is given in [SZ12] by the form $u + t\tilde{f}$ with $\tilde{f} := f - \frac{1}{V}\int_P f$ and $t \geqslant 0$. Since the non-linear term of $M(u + t\tilde{f})$ does not depend on $t$ we expect the stability dividing the estimate

$$M(u + t\tilde{f}) \geqslant \delta \inf_{\ell} J(u + t\tilde{f} + \ell) - C \tag{2.4}$$

by $t \to \infty$. In fact the affine function $\ell$ which attains the infimum depends on $t$. In the next section we will fix the problem by taking a natural filtration of $\mathcal{H}$.

## 3. Argument of Hilbert-Mumford criterion

Let us now treat with a general complex polarized manifold $(X, L)$. In this section we would rather like to represent a test configuration by a one-parameter subgroup (1-PS for short). For an exponent $r \in \mathbb{N}$ global sections $H^0(X, L^{\otimes r})$ define the Kodaira embedding $X \to \mathbb{P}^{N_r - 1}$. Any 1-PS $\lambda : \mathbb{C}^* \to \mathrm{GL}(N_r; \mathbb{C})$ defines a test configuration as the Zariski closure

$$\mathcal{X}_\lambda := \overline{\bigcup_{\tau \in \mathbb{C}^*} (\lambda(\tau)X \times \{\tau\})} \subseteq \mathbb{P}^{N_r - 1} \times \mathbb{C}^* \tag{3.1}$$

with the $\mathbb{Q}$-line bundle $\mathcal{L}_\lambda := \mathcal{O}(1/r)|_{\mathcal{X}}$. By the equivariant version of Kodaira embedding one can prove that any test configuration is obtained in this way from some (but not unique) 1-PS.

Let $\mathbb{T} \subseteq \mathrm{Aut}^0(X, L)$ be a maximal torus. We say that a test configuration is $\mathbb{T}$-equivariant if it is endowed with a $\mathbb{T}$-action which preserves fiber, commutes the $\mathbb{C}^*$-action, and is compatible with the identification $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L)$. For any representative 1-PS $\lambda$ it is equivalent to say that $\lambda$ is commutative with $\mathbb{T}$. In other words there exists a maximal torus $D \subset \mathrm{GL}(N_r; \mathbb{C})$ satisfying $\lambda(\mathbb{C}^*) \subseteq D$ and $\mathbb{T} \subseteq D$. In case $(X, L)$ is toric this is equivalent to give a toric test configuration. In fact $\mathbb{T} \times \mathbb{C}^*$ acts on $\mathcal{X}$ with the open dense orbit. Let us denote the space of 1-PS to $D$ by $N(D)$.

A guideline for the stability from the coercivity should be the fact that the Donaldson-Futaki invariant of a test configuration is given by the slope of the K-energy along the associated ray $\varphi^t$ on $\mathcal{H}$ ($t \in \mathbb{R}_{>0}$). The idea originates from the introduction of K-stability (for the Fano manifolds) in [DT92], [Tian97] and it has been further developed as one can see in [PRS08], [PT09], and [BHJ15]. Actually in [BHJ15] the *non-Archimedean energy* $M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$, $J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ of a test configuration, which are characterized by the property

$$M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} M(\varphi^t)/t \quad \text{and} \quad J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} J(\varphi^t)/t \tag{3.2}$$

were introduced, as the canonical energy functionals for the metrics on the Berkovich analytification. Here $\varphi^t$ is a smooth ray on $\mathcal{H}$ with appropriate compatibility with $(\mathcal{X}, \mathcal{L})$. For example one can take $\varphi^t(x) := \Phi_{\mathrm{FS}}(\lambda(e^{-t})x, e^{-t})$; pull-back of the restriction of some Fubini-Study metric in (3.1). The relation of $M^{\mathrm{NA}}$ and the Donaldson-Futaki invariant is given by the explicit form

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) + ((\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}})\overline{\mathcal{L}}^n), \tag{3.3}$$

where the second term is the intersection of the non-reduced part of the central fiber and the extension of $\mathcal{L}$ to the trivial compactification of $\mathcal{X}$ (which is obtained by (1.2 for a toric test configuration). What we need in the sequel is a consequence that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geqslant M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ and the equality holds if $\mathcal{X}_0$ is reduced. Notice that in the toric case normality of $\mathcal{X}_0$ implies $\mathcal{X} \simeq X \times \mathbb{C}$ (see the proof of [DS15], Theorem 17).

At this point we assume the coercivity

$$M(\varphi^t) \geqslant \delta \inf_{\sigma \in \mathbb{T}} J(\varphi^t_\sigma) - C$$

to show the stability. The trouble here is of course that the element $\sigma \in \mathbb{T}$ attaining the infimum of $J(\varphi^t_\sigma)$ depends on the time parameter $t$. We go through the collection of all Fubini-Study fiber metrics for $X \to \mathbb{P}^{N_r - 1}$ denoted by $\mathcal{H}_r$ to avoid the point. Tian-Zeldich-Catlin's Bergman kernel asymptotic indicates us to exploit the space, as it shows $\mathcal{H} = \overline{\bigcup_r \mathcal{H}_r}$.

3.1. **Function of log-norm singularities.** We basically quote the formalism of [BHJ16]. In the sequel we fix a sufficiently large exponent $r$ and some $\varphi \in \mathcal{H}_r$ to consider the functions in $g \in \mathrm{GL}(N_r; \mathbb{C})$:

$$m(g) := M(\varphi_g), \quad j(g) := J(\varphi_g), \tag{3.4}$$

and $e(g) := E(\varphi_g)$ for the Aubin-Mabuchi energy $E$.

Let us starts from the following observation which I learned from the discussion with Professor S. Boucksom.

**Proposition 3.1.** *The energies $e$ and $m$ are pluriharmonic and $j$ is plurisubharmonic along* $\mathrm{Aut}^0(X, L)$.

*Proof.* Let us take an arbitrary holomorphic map from the one-dimensional open disk $\Delta \to \mathrm{Aut}^0(X, L)$ which sends $z \in \Delta$ to $g(z)$. We have the well-known formula which gives the curvature by fiber integration:

$$dd^c_z E(\varphi_{g(z)}) = (n+1)^{-1} V^{-1} \int_X (dd^c_{z,x} \varphi_{g(z)}(x))^{n+1}.$$

Defining the holomorphic map $F : \Delta \times X \to X$ by $F(z, x) := g(z) \cdot x$ we may proceed as

$$\begin{aligned} (dd^c_{z,x} \varphi_{g(z)}(x))^{n+1} &= (dd^c_{z,x} F^* \varphi)^{n+1} \\ &= F^*(dd^c_x \varphi)^{n+1} = 0, \end{aligned}$$

where the last line vanishes for the big degree. This completes the proof for $e$. Plurisubharmonicity of $j$ is immediate. Let us focus on $M$. We regard $M$ as a metric on the

Deligne pairing $\langle K_{\Delta\times X/\Delta}\mathcal{L}^n\rangle + (n+1)^{-1}\hat{S}\langle\mathcal{L}^{n+1}\rangle$ with $\mathcal{L} := F^*L$ (see *e.g.* [BHJ16]). such that the proof reduces to the equality

$$dd^c_{z,x}\langle\log(dd^c\varphi_{g(z)})^n, \varphi_{g(z)}, \dots, \varphi_{g(z)}\rangle = 0,$$

where the Monge-Ampére operator is taken over $X$. Set $\Psi := \log(dd^c\varphi_{g(z)})^n$ as a metric on $K_{\Delta\times X/\Delta}$ and $\psi := \log(dd^c\varphi)^n$ as a metric on $K_X$. Since the natural automorphism $G : (z,x) \mapsto (z, g(z)\cdot x)$ of $\Delta \times X$ induces $F^*K_X \simeq p_2^*K_X = K_{\Delta\times X/\Delta}$ we are led to show

$$F^*\psi = \Psi.$$

This can be checked fiberwisely. Indeed for each fixed $z$ we have the inclusion $i_z : X \to \Delta \times X$ such that

$$\begin{aligned} e^{\Psi_z} &= (dd^c i_z^* F^*\varphi)^n = V^{-1} i_z^*(dd^c_{z,x}F^*\varphi)^n \\ &= i_z^*F^*(dd^c\varphi)^n = g(z)^*(dd^c\varphi)^n. \end{aligned}$$

□

Plurisubharmonicity on $\mathrm{GL}(N_r, \mathbb{C})$ is not the case. A fundamental result of [Paul12] (see also [Li12]) tells us that asymptotic of these functions are controlled by the difference of two plurisubharmic functions. More specifically it is described by the actions to two homogeneous polynomials; $X$-resultant $R$ and $X$-hyperdiscriminant $\Delta$ associated with the Kodaira embedding. To be precise, taking norms $\|\cdot\|$ of $\mathrm{GL}(N_r;\mathbb{C})$-vector space where $R$ and $\Delta$ lives and the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ of $\mathrm{GL}(N_r;\mathbb{C})$ we have

$$m(g) = V^{-1}\log\|g\cdot\Delta\| - V^{-1}\frac{\deg\Delta}{\deg R}\log\|g\cdot R\| + O(1) \tag{3.5}$$

and

$$j(g) = V^{-1}\log\|g\|_{\mathrm{HS}} - V^{-1}\frac{1}{\deg R}\log\|g\cdot R\| + O(1). \tag{3.6}$$

Moreover the second term for $j(g)$ just corresponds to the Aubin-Mabuchi functional. The point here is that we can not expect to have a single log-norm term as we had in the classical GIT setting. In the terminology of [BHJ16], for a general reductive group $G$ a function $f : G \to \mathbb{R}$ of the form

$$f(g) = a\log\|g\cdot v\| - b\log\|g\cdot w\| + O(1)$$

is said to have log-norm singularities. Even in this generalized "pair of log-norm terms " setting, we have the correspondence of the Hilbert-Mumford weight for a given 1-PS.

**Theorem 3.2** (Theorem 4.4 of [BHJ16])**.** *Let $f$ be a function on $G$ with log norm singularities. Then,*

(*i*) *For each 1-PS $\lambda : \mathbb{C}^* \to G$ there exists $f^{\mathrm{NA}} \in \mathbb{Q}$ such that*

$$f(\lambda(\tau)) = f^{\mathrm{NA}}(\lambda)\log|\tau|^{-1} + O(1)$$

*for $|\tau| \leqslant 1$.*

(*ii*) *$f$ is bounded below on $G$ iff $f^{\mathrm{NA}}(\lambda) \geqslant 0$ for all 1-PS $\lambda$.*

Applying the theorem to $f(g) = M(\varphi_g)$ (*resp.* $J(\varphi_g)$) with $\varphi$ a Fubini-Study weight, one obtains $f^{\mathrm{NA}}(\lambda) = M^{\mathrm{NA}}(\lambda)$ (*resp.* $J^{\mathrm{NA}}(\lambda)$). In the proof of the main theorem we like to study how the above $O(1)$-term is determined. For the reason let us go back to the construction of $f^{\mathrm{NA}}(\lambda)$. As a standard fact of $\mathbb{C}^*$-representation any $v \in V$ in the $GL(N_r;\mathbb{C})$-vector space has the decomposition

$$\lambda(\tau)v = \sum \tau^{\lambda_i} v_i \tag{3.7}$$

so that

$$\begin{aligned} \log\|\lambda(\tau)v\| &= \max\big\{\lambda_i \log|\tau| + \log\|v_i\|\big\} + O(1) \\ &= -\min \lambda_i \cdot \log|\tau|^{-1} + O(1). \end{aligned}$$

In the first line

$$\log\|\lambda(\tau)v\| \geqslant \max\big\{\lambda_i \log|\tau| + \log\|v_i\|\big\}$$

is trivial and $O(1)$ only depends on $N_r$. To investigate the second $O(1)$ we take a maximal complex torus $D \subseteq \mathrm{GL}(N_r;\mathbb{C})$ which contains the image of $\lambda$. We denote the character space by $M(D) := \mathrm{Hom}(D, \mathbb{C}^*) \simeq \mathbb{Z}^{N_r}$. The dual space $N(D) := \mathrm{Hom}(\mathbb{C}^*, D)$ is naturally identified with the collection of one-parameter subgroups. Then any $\mathrm{GL}(N_r;\mathbb{C})$-module $V$ has the decomposition $V = \oplus_\chi V_\chi$ such that $d\cdot v = \chi(d)v$ for $v \in V_\chi$. Set the weight space of a given $v \in V$ as

$$M_v := \big\{\chi \in M \;\big|\; V_\chi\text{-component of } v \text{ is non-zero}\big\} \tag{3.8}$$

and denote its convex hull in $M_{\mathbb{R}} := M \otimes \mathbb{R}$ by $P_v$. Define the characteristic function $p_v(\chi)$ whose value is identically zero if $\chi \in P_v$ and $\infty$ otherwise. We set the Legendre transform of the convex function $p_v$ as

$$h_v(\lambda) = \max_{\chi \in M_v} \langle \chi, \lambda\rangle \tag{3.9}$$

and call it the support function.

Next we fix any maximal compact group $U \subset \mathrm{GL}(N_r;\mathbb{C})$. Notice that $U$ ensures the following property.

($i$) $D \cap U \subseteq U$ is a maximal real torus.
($ii$) Any two maximal tori $D_1$ and $D_2$ are conjugate by an element of $U$.
($iii$) The Cartan(polar) decomposition $\mathrm{GL}(N_r;\mathbb{C}) = UDU$ holds.

From the property ($i$) we have $\mathrm{Log}\,|\cdot| : D \to N_{\mathbb{R}}$ which yields $D/(D\cap U) \simeq N_{\mathbb{R}}$. One observes for any $\chi \in M$ $\log|\chi(d)| = \langle \chi, \mathrm{Log}\,|d|\rangle$ and for any $\lambda \in N$ $\mathrm{Log}\,|\lambda(\tau)| = \log|\tau|\cdot\lambda$ holds. In this setting we repeat the formula in [BHJ16] that for any $u, u' \in U$ and $d \in D$

$$f(u'du) = h_{u\cdot v_+}(\mathrm{Log}\,|d|) - h_{u\cdot v_-}(\mathrm{Log}\,|d|) + O(1) \tag{3.10}$$

holds and claim that $O(1)$-term is determined by $U$ hence independent of $D$. We may assume that the norm is $U$-invariant so

$$\log\|(u'du)v\| = \log\|(du)v\| = \log\left\|\sum_{\chi\in M_{uv}} \chi(d)(uv)_\chi\right\|.$$

The last term is written to be

$$\max_{\chi\in M_{uv}} \log\left\{\langle\chi, \operatorname{Log}|d|\rangle + \log\|(uv)_\chi\|\right\} + O(1)$$

with $O(1)$ determined by the cardinality of $M_{uv}$. On the other hand compactness of $U$ yields a constant $C$ such that

$$-C \leqslant \log\|(uv)_\chi\| \leqslant C \tag{3.11}$$

for any $u \in U$ and $\chi \in M_{uv}$. If we take another maximal torus $D' = uDu^{-1}$ characters are naturally related in the manner

$$\chi \in M(D) \longleftrightarrow \chi' := \chi(u^{-1}\cdot u) \in M(D').$$

For $v_\chi \in V_\chi$ and $d' = udu^{-1}$ we have

$$d'(uv_\chi) = udv_\chi = u\chi(d)v_\chi.$$

The last one equals to $\chi(d)(uv_\chi) = \chi'(d')(uv_\chi)$ by linearity, hence $uv_\chi \in V_{\chi'}$. It concludes that the multiplication $v \mapsto uv$ is compatible to the weight decomposition in the sense that

$$(u\cdot v)_{\chi'} = u\cdot v_\chi$$

holds for $v \in V$. Therefore the bound in (3.11) is determined only by $U$. By the definition of $h_{u\cdot v}$ we then obtain the formula (3.10) with the desired $O(1)$-term.

For a fixed maximal torus $D$ any 1-PS $\lambda : \mathbb{C}^* \to \mathrm{GL}(N_r;\mathbb{C})$ has its image in some $D'$ which is conjugate to $D$. Finally we agree that $O(1)$ of Theorem 3.2 is determined by a fixed maximal compact subgroup $U$ of $G = \mathrm{GL}(N_r;\mathbb{C})$.

3.2. **Completion of the proof of the main theorem.** Assume $\mathbb{T}$-coercivity, namely that

$$M(\varphi) \geqslant \delta \inf_{\sigma\in\mathbb{T}} J(\varphi_\sigma) - C$$

holds for any $\varphi \in \mathcal{H}^{\mathbb{S}}$. Then

$$m(g) \geqslant \delta \inf_{\sigma\in\mathbb{T}} j(\sigma g) - C \tag{3.12}$$

for any $g \in \mathrm{GL}(N_r;\mathbb{C})$. Note that $\varphi_g(x) = \varphi(g\cdot x)$ hence $(\varphi_g)_\sigma = \varphi_{\sigma g}$.

Let us take any $\mathbb{T}$-equivariant test configuration and its representative 1-PS $\lambda \in N(D)$. For $k \in \mathbb{N}$ set $d_k := \lambda(e^{-k}) \in D$. In a fixed coordinate, $d_k$ can be written as a diagonal matrix $(e^{-k\lambda_1}, \dots, e^{-k\lambda_{N_r}})$ and $\mathbb{T}$ may be regarded as the collection of diagonal matrices the last $n$ components of which are just $1 \in \mathbb{C}^*$. Then each infimum of $j(\sigma d_k)$ is approximated by some $\sigma_k := (e^{a_{1k}}, \dots, e^{a_{nk}}, 1, \dots, 1)$ with $a_{ik} \in \mathbb{Q}$ so that

$$\inf_{\sigma\in\mathbb{T}} j(\sigma d_k) \geqslant j(\sigma_k d_k) - 1/k \tag{3.13}$$

holds. Set $\mu_{ik} := -k^{-1}a_{ik} \in \mathbb{Q}$ and $q_k \in \mathbb{N}$ such that $q_k\mu_{ik} \in \mathbb{Z}$ holds for any $1 \leqslant i \leqslant n$. Then $\mu_{ik}$ defines $\mu_k \in N_{\mathbb{Q}}(\mathbb{T})$ for which $q_k\mu_k$ is a 1-PS. As a direct consequence of (3.2), $M^{\mathrm{NA}}$ and $J^{\mathrm{NA}}$ are homogeneous in $\lambda \in N(D)$. Therefore one can extend them to $N_{\mathbb{Q}}(D)$ so that $f^{\mathrm{NA}}(\lambda) = q^{-1}f^{\mathrm{NA}}(q\lambda)$ holds for $q \in \mathbb{N}$. In addition, $\mu_k$ itself is not a 1-PS but some value *e.g.*

$$(\mu_k + \lambda)(e^{-k}) := (e^{-k(\mu_{1k}+\lambda_1)}, \dots, e^{-k(\mu_{nk}+\lambda_n)}, e^{-k\lambda_{n+1}}, \dots, e^{-k\lambda_N})$$

makes sense in $\mathrm{GL}(N_r, \mathbb{C})$. Noting these points the argument of subsection 3.1 can be applied to $q_k(\mu_k + \lambda) \in N(D)$ and $\tau := e^{-k/q_k}$ so that

$$j((\mu_k + \lambda)(e^{-k})) \geqslant J^{\mathrm{NA}}(\mu_k + \lambda) \cdot k + O(1) \tag{3.14}$$

holds with $O(1)$ which is not only for $k$ but also for $\mu_k$ and $\lambda$. The right-hand side is obviously bounded from below by

$$\inf_{\mu \in N_{\mathbb{Q}}(\mathbb{T})} J^{\mathrm{NA}}(\mu + \lambda) \cdot k + O(1).$$

It follows that

$$m(\lambda(e^{-k}))/k \geqslant \delta \inf_{\mu \in N_{\mathbb{Q}}(\mathbb{T})} J^{\mathrm{NA}}(\mu + \lambda) + O(1/k). \tag{3.15}$$

Letting $k \to \infty$ we obtain the stability:

$$M^{\mathrm{NA}}(\lambda) \geqslant \delta \inf_{\mu \in N_{\mathbb{Q}}(\mathbb{T})} J^{\mathrm{NA}}(\mu + \lambda). \tag{3.16}$$

In the toric case $N_{\mathbb{Q}}(\mathbb{T})$ consists of rational linear functions on $M_{\mathbb{R}}$ so that we conclude the main theorem.

□

**Remark 3.3.** *Note that $\mathbb{T}$-coercivity in particular implies $m(g)$ bounded from below. Since Proposition 3.1 shows that $m(g)$ is pluriharmonic on the quasi-projective variety $\mathrm{Aut}^0(X, L)$, it is constant along the automorphism group. It follows from the slope formula (3.2) that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = 0$ if $(\mathcal{X}, \mathcal{L})$ is product.*

More generally one can consider a reductive subgroup $G = K_{\mathbb{C}} \subseteq \mathrm{Aut}^0(X, L)$ and the coercivity condition

$$M(\varphi) \geqslant \delta \inf_{g \in C(G)} J(\varphi_g) - C, \tag{3.17}$$

restricted to any $K$-invariant positively curved metrics. Note that $\varphi_g$ is $K$-invariant for any $g \in C(G)$. From this coercivity, the above argument actually derives uniform K-stability relative to $G$ in the sense of [His16].

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