MABUCHI'S SOLITON METRIC AND RELATIVE D-STABILITY

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Abstract. For Fano manifolds T. Mabuchi introduced a generalization of the Kähler-Einstein metric, which is characterized as the critical point of the Ricci-Calabi functional. We show that a Fano manifold admits Mabuchi's metric if and only if it is uniformly relatively D-stable.

Contents

1. Introduction

Let X be a Fano manifold. In a central problem of complex geometry we are guided to look for a standard Kähler metric in the first Chern class $c_1(X) = c_1(-K_X)$. The fundamental result established in [CDS15] states that there exists a Kähler-Einstein metric if and only if X is K-polystable (see also [Tia15]). Not all the Fano manifold satisfy the stability; for example one-point blow up of $\mathbb{P}^2$ is never Kähler-Einstein. On the other hand, for an *arbitrary* Fano manifold X we may consider a canonical geometric flow which should optimally destabilize X. The self-similar solution of the flow coincides with T. Mabuchi's generalization of Kähler-Einstein metric. The purpose of this paper is to clarify which Fano manifold admits such a metric.

For the definition, let us denote a Kähler metric by ω and the normalized Ricci potential function by ρ which is the unique function satisfies

$$\operatorname{Ric}\omega - \omega = dd^c\rho, \ \int_X (e^\rho - 1)\omega^n = 0. \tag{1.1}$$

We also write $\omega = dd^c\varphi$ locally so as to identify the metric with a collection of smooth functions φ patching together to define the fiber metric of $-K_X$. Our standard metric first introduced by [M01] is the critical point of the Ricci-Calabi functional

$$R(\omega) = R(\varphi) := \frac{1}{V}\int_X (e^\rho - 1)^2\omega^n. \tag{1.2}$$

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Here the volume $V = \int_X \omega^n$ is independent of ω. From the straightforward variational computation one can see that the metric ω is a critical point iff $e^\rho - 1$ is the Hamilton function for some one-parameter subgroup $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(X, -K_X)$. It is clear from the definition that the condition gives the Ricci-analogue of the extremal Kähler metric defined in terms of the classical Calabi functional. These two metrics are not the same but in fact the above η is generated by the extremal vector field. There as well exists the infinite-dimensional GIT picture [D15] so that the Ricci-Calabi functional can be seen as the square norm of a certain moment map. Then the role of the Kemp-Ness functional in GIT is played by the famous D-energy

$$D(\varphi) = -\log \frac{1}{V}\int_X e^{-\varphi} - \frac{1}{(n+1)V}\sum_{i=0}^{n}\int_X (\varphi - \varphi_0)\omega^i \wedge \omega_0^{n-i}.$$

It first appeared in [BM86] and was written down to this form by [D88]. The gradient flow

$$\frac{\partial}{\partial t}\varphi = 1 - e^\rho$$

was initially studied in our previous work [CHT17]. In [H19], [X19] it was shown that the flow indeed minimizes $R(\varphi)$ and is naturally related with the optimal degeneration of the Fano manifold. From now on we call the pair of the critical point of $R(\varphi)$ and the one-parameter subgroup *Mabuchi soliton*, since it is characterized as the self-similar solution of the flow.

Our main result claims that the existence of Mabuchi soliton is equivalent to certain algebraic stability condition. It extends the result of [Y17], [N17] for the toric case to general Fano manifolds. Our approach precisely follows [BBJ18] where they give a new variational proof of [CDS15] for a Fano manifold with finite automorphism group.

Theorem A. A Fano manifold X admits a Mabuchi soliton if and only if it is uniformly relatively D-stable, with respect to the equivariant test configurations.

If the extremal vector field is zero i.e. $\eta = 0$, we obtain the existence result of Kähler-Einstein metric, with no restriction for the automorphism group. For this we adopt the equivariant formulation which was suggested in [DS16], [H18].

See Definition 3.20 for the stability. The concept of D-stability originates from [B16]. As the K-stability introduced by [D02] naturally arises from the Calabi functional and the K-energy, D-stability arises from the above Ricci-Calabi functional and the D-energy. The uniform stability was introduced in [BHJ15] and [Der16a] independently. In regard to the torus containing the soliton vector field one can formulate the relative version of the D-stability. Putting these together we formulate the *uniform relative stability* which reflects the coercivity of the modified D-energy. In fact it was shown by [LZ17] that the relevant coercivity is equivalent to the existence of Mabuchi soliton in a slight different formulation.

If we derive the coercivity from the stability, the uniformity is critical in controlling the sequence of test configurations. The relative consideration of the energies relies on [BWN14]. Although they were mainly concerned with the Kähler-Ricci soliton the techniques are valid for the general situations including the present case. Our task is to show these ideas naturally extend and fit into the situation of Mabuchi soliton.

Unlike K-stability, D-stability works only for Fano manifolds, however, as Theorem A and its proof show, the treatment is much easier. Existence of extremal Kähler metric is still open problem, even for the anti-canonical polarizations. A simple argument shows that the Mabuchi soliton assures the extremal Kähler metric. A new circumstance in the relative setting is that the two metrics are in fact not equivalent. The first counterexample is raised in the latest version of [NSY17].

Compared with Kähler-Ricci soliton, Mabuchi soliton has in some sense more algebraic nature. For example the soliton vector field is periodic and actually generates η. Moreover we may expect a generalization of the CM line bundle. On the other hand, the gradient flow is not so flexible as the Kähler-Ricci flow and there exists a toric Fano 3-fold which does not admits a Mabuchi soliton. This is in contrast to the result of [WZ04].

Along the variational approach we may naturally understand the uniqueness of Mabuchi soliton.

Theorem B ([M03], Theorem C)**.** Let (ω_0, η_0) and (ω_1, η_1) be smooth Mabuchi solitons. Then there exists an automorphism $f \in \mathrm{Aut}^0(X)$ in the identity component such that $f^*\omega_1 = \omega_0$, $f^*\eta_0 = \eta_1$.

Our argument also gives a new proof of the Matsushima-type theorem in [M03], [N19]. Namely, if a Fano manifold admits the Mabuchi soliton, the identity component of the group of automorphism preserving the extremal vector field is reductive. The uniqueness and the reductivity are key materials for the derivation of the coercivity from existence of the metric.

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2. Mabuchi soliton and modified D-energy

2.1. **Notation.** Throughout the paper X denotes an n-dimensional Fano manifold and a Kähler metric ω is taken in the first Chern class $c_1(X)$. We adopt the additive notation writing the anti-canonical bundle as $-K_X$ and the fiber metric as φ. While we do not fix a specific covering $\{U_\alpha\}_\alpha$ of local coordinate patches U_α, the symbol φ is interpreted to a function φ_α on each $U_\alpha \subset X$. In a local frame any section s of $-K_X$ is identified with a function s_α and it is evaluated by the multiplication of $e^{-\varphi_\alpha}$ to $|s_\alpha|^2$. On the intersection $U_\alpha \cap U_\beta$ for two indicies α, β and coordinates z^i_α, z^j_β $1 \leqslant i, j \leqslant n$ the transition function is written to $g_{\alpha\beta} = \det \left[\frac{\partial z^i_\alpha}{\partial z^j_\beta}\right]_{ij}$ and the compatibility $\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|^2$ holds. If we put $d^c = \frac{\partial - \bar\partial}{4\pi\sqrt{-1}}$, it follows that the Chern curvature $\omega_\varphi = dd^c\varphi$ is globally well-defined. We set $\mathcal{H}(X, -K_X)$ as the collection of smooth fiber metric φ on $-K_X$ such that ω_φ is positive. By dd^c-Lemma, any metric ω in $c_1(X)$ equals to ω_φ for some $\varphi \in \mathcal{H}(X, -K_X)$ which is unique up to addition of a constant. For this reason $\mathcal{H}(X, -K_X)$ is called *space* *of Kähler metrics.* We essentially need φ in place of $\omega = \omega_\varphi$ to look at the action of the Hamilton diffeomorphism group.

2.2. **Ricci curvature formulation.** We briefly review an energy formulation to the Kähler-Einstein problem, which make use of the Ricci potential. There has been another (and probably major) scalar curvature formulation which works for a general polarized manifold. In terms of the scalar curvature one may introduce the Calabi functional and notion of K-stability observing the behavior of the K-energy along the degeneration of the manifold. See the milestone works [C82], [C85], [F83], [M86], [T97], and [D02]. D-energy which we will explain is as well classical but the determination of the corresponding D-stability [B16] and the momentum map picture [D15] were rather recent.

Let us start from defining two probability measures associated to a Kähler metric $\omega = \omega_\varphi$, or equivalently $\varphi \in \mathcal{H}(X, -K_X)$. One is the Monge-Ampère measure $V^{-1}\omega_\varphi^n$. The other one which we call the *canonical measure* is special for the Fano case and defined to be

$$\mu_\varphi := \frac{e^{-\varphi}}{\int_X e^{-\varphi}}, \tag{2.1}$$

where $e^{-\varphi}$ denotes the global volume form described as $e^{-\varphi_\alpha} \bigwedge_{i=1}^n dz_\alpha^i \wedge d\bar{z}_\alpha^i$ on a coordinate patch U_α. Note that the metric is Kähler-Einstein iff it satisfies the Monge-Ampère equation $V^{-1}\omega_\varphi^n = \mu_\varphi$. Therefore we should focus on the difference of these two measures. In fact it precisely gives the infinite-dimensional moment map. Namely, once we regard a fixed Kähler metric ω as a symplectic form and instead collect all the complex structures J compatible with ω, one may attach to each J the measure

$$J \mapsto \mu_\varphi - V^{-1}\omega^n. \tag{2.2}$$

The group of Hamilton diffeomorphisms naturally acts on the complex structures. The Lie algebra of this group is naturally identified with smooth function space $C^\infty(X;\mathbb{R})$ with Poisson bracket and hence the above defines a map to the dual Lie algebra. It indeed satisfies the moment map condition. More precisely, we should impose to J the compatibility condition with the fiber metric φ, but see [D15] for the detail explanation.

The square norm of the moment map is written down to

$$R(\varphi) = \frac{1}{V}\int_X (e^\rho - 1)^2 \omega^n, \tag{2.3}$$

which we call the Ricci-Calabi functional. Our interest is the critical point of the Ricci-Calabi functional which gives a generalization of Kähler-Einstein metric. The first variation of $R \colon \mathcal{H}(X, -K_X) \to \mathbb{R}$ is given as follows. See also [N19] for calculating the second variation.

Proposition 2.1 ([CHT17], Proposition 2.3)**.** *Set the twisted Laplacian on functions $f \in C^\infty(X, \mathbb{C})$ by*

$$L_\rho f = \Delta_\omega f + (\bar\partial\rho, \bar\partial f)_\omega. \tag{2.4}$$

Then the first variation of the Ricci-Calabi functional is given as

$$\delta R(\varphi) = -\frac{2}{V}\int_X \delta\varphi \left(L_\rho \tilde{f} + \tilde{f}\right) d\mu_\varphi, \tag{2.5}$$

where

$$\tilde{f} = (e^{\rho} - 1) - \frac{1}{V}\int_X (e^{\rho} - 1) d\mu_{\varphi}.$$

As a consequence, φ is the critical point of the Ricci-Calabi functional iff $e^{\rho} - 1$ is a Hamilton function. One can check this by a simple application of the Bochner-Kodaira formula. Since X is Fano any holomorphic vector field defines a function h unique up to addition of a constant such that

$$\sqrt{-1}\bar{\partial} h = i_v \omega. \tag{2.6}$$

We call h a Hamilton function.

Definition 2.2. *A Kähler metric $\omega \in c_1(X)$ is called a Mabuchi soliton if $e^{\rho} - 1$ is a Hamilton function for some holomorphic vector field.*

The vector field is zero iff $\rho = 0$ and in this case Mabuchi soliton is nothing but Kähler-Einstein.

Going back to the moment map picture, we also have the canonical energy functional $D\colon \mathcal{H}(X, -K_X) \to \mathbb{R}$ with the outer derivative $(dD)_{\varphi} = \mu_{\varphi} - V^{-1}\omega^n$ at φ. We call it D-energy. It is in fact separated into two terms $D = L - E$ and each term is specifically defined as

$$L(\varphi) := -\log \frac{1}{V}\int_X e^{-\varphi}, \ E(\varphi) := \frac{1}{(n+1)V}\sum_{i=0}^{n}\int_X (\varphi - \varphi_0)\omega^i \wedge \omega_0^{n-i}. \tag{2.7}$$

We here take a reference φ_0 and $\omega_0 = dd^c\varphi_0$. Note that φ does not define a global function but the difference $\varphi - \varphi_0$ does. One can easily compute to check the differential

$$(dL)_{\varphi} = \mu_{\varphi}, \ (dE)_{\varphi} = V^{-1}\omega^n. \tag{2.8}$$

The definition of the Monge-Ampère energy E chose φ_0 but it is characterized by (2.8), up to addition of a constant.

2.3. **Space of finite energy metrics.** The fundamental property of D-energy is that it is convex along any geodesic in the space of Kähler metrics. Since the difference $\varphi - \psi$ of any two $\varphi, \psi \in \mathcal{H}(X, -K_X)$ defines a global function, tangent space at any point of $\mathcal{H}(X, -K_X)$ is identified with $C^{\infty}(X; \mathbb{R})$. Mabuchi's inner product [M87] for any tangents $u, v \in C^{\infty}(X; \mathbb{R})$ at φ is

$$\langle u, v \rangle = \frac{1}{V}\int_X uv\omega^n. \tag{2.9}$$

Any curve φ^t ($t \in [a, b]$) in $\mathcal{H}(X, -K_X)$ defines a function $\Phi(\tau, x) := \varphi^{-\log|\tau|}(x)$ of complex variables $e^{-b} \leqslant |\tau| \leqslant e^{-a}$ and $x \in X$. It is well-known (from [S92]) that the geodesity for (2.9) is equivalent to the degenerate Monge-Ampère equation

$$(dd^c_{\tau,x}\Phi)^{n+1} = 0. \tag{2.10}$$

The left-hand side at the same time describes the Monge-Ampère energy by the fiber integration

$$dd^c_{\tau}E(\varphi) = \int_X (dd^c_{\tau,x}\Phi)^{n+1}. \tag{2.11}$$

It follows that E is affine along any geodesics. In fact for given smooth endpoints the bounded geodesic Φ connecting them uniquely exists, but it is not C^2 in general.

Variational approach even requires the appropriate completion of the space of smooth metrics. These facts strongly motivate to consider a singular fiber metric φ which is only locally integrable and satisfies $dd^c\varphi \geqslant 0$ in the sense of current. We denote the collection of all such singular φ by $\mathrm{PSH}(X, -K_X)$. It equivalent to say that in a coordinate patch U_α, φ_α is plurisubharmonic (psh for short) function. For the bounded psh function the wedge product of the current $\omega_\varphi^n = (dd^c\varphi)^n$ is safely defined thanks to the celebrated work of [BT76]. In particular we may define the Monge-Ampère energy E for locally bounded φ. For a smooth boundary data we have the bounded solution of (2.10). From the recent result [CTW17], the solution is actually of $C^{1,1}$.

The Monge-Ampère operator $\varphi \mapsto V^{-1}\omega_\varphi^n$ can not be continuously extended to $\mathrm{PSH}(X, -K_X)$. Following [BEGZ10] and [BBGZ13], one can however take the reference φ_0 smooth and bounded approximation $\varphi^{(j)} := \max\{\varphi, \varphi_0 - j\}$ of $\varphi \in \mathrm{PSH}(X, -K_X)$, to define the *non-pluripolar Monge-Ampère measure*

$$\mathrm{MA}(\varphi) := \lim_{j\to\infty} 1_{\{\varphi > \varphi_0 - j\}} V^{-1}\omega_{\varphi^{(j)}}. \tag{2.12}$$

By the construction $\mathrm{MA}(\varphi)$ drops the mass of the unbounded locus so it is no longer a probability measure. It can be further shown that $\mathrm{MA}(\varphi)$ is local in plurifine topology and has no mass on any pluripolar set. In a similar idea taking bounded ψ such that $\psi \geqslant \varphi$ locally we define the Monge-Ampère energy as

$$E(\varphi) := \inf_{\psi} E(\psi) \in \mathbb{R} \cup \{-\infty\}. \tag{2.13}$$

The extended Monge-Ampère energy is upper-semicontinuous in the L^1-topology of $\mathrm{PSH}(X, -K_X)$. Moreover, the level set $\{E \geqslant C\}$ is compact in this weak topology. One can regard this fact as an analogue of Banach-Alaoglu theorem.

Let us now consider $p \geqslant 1$ and the L^p-Finsler distance d_p of $\mathcal{H}(X, -K_X)$, defined by the norm of tangents

$$\|u\|_p := \left[\frac{1}{V}\int_X |u|^p\, \omega_\varphi^n\right]^{\frac{1}{p}}. \tag{2.14}$$

As we shall see, $p = 1$ plays the special role in the variational approach.

Theorem 2.3 ([D15], Theorem 2. See also [BBJ18], Theorem 1.7)**.** *Take a smooth non-increasing sequence of $\varphi_j \in \mathrm{PSH}(X, -K_X)$ converges to φ. Endow the space of metrics with finite Monge-Ampère energy*

$$\mathcal{E}^1(X, -K_X) := \left\{\varphi \in \mathrm{PSH}(X, -K_X) : E(\varphi) > -\infty\right\} \tag{2.15}$$

with the metric

$$d_1(\varphi, \psi) := \lim_{j\to\infty} d_1(\varphi_j, \psi_j). \tag{2.16}$$

It then gives the coarsest refinement of the L^1-topology so that E is continuous. The Monge-Ampère energy is affine along every geodesic on $(\mathcal{E}^1(X, -K_X), d_1)$. Moreover, $(\mathcal{E}^1(X, -K_X), d_1)$ realizes the completion of $(\mathcal{H}(X, -K_X), d_1)$.

We usually refer to the L^1-topology as the "weak" topology and the d_1-topology as the "strong" topology. The space $\mathcal{E}^1(X,-K_X)$ is contained in the *finite energy class*

$$\mathcal{E}(X,-K_X) := \left\{\varphi \in \mathrm{PSH}(X,-K_X) : \int_X \mathrm{MA}(\varphi) = 1\right\}. \tag{2.17}$$

Restricted to $\mathcal{E}(X,-K_X)$ the non-pluripolar Monge-Ampère operator is continuous along any monotone sequence. The determination of the domain of Monge-Ampère operator owes to the pioneering work [C98]. The compact setting is treated in [GZ07], [BEGZ10]. See also the comprehensive textbook [GZ17].

As it was shown in [D17a], geodesics connecting two points are not unique in $\mathcal{E}^1$, however, for any $\varphi^0, \varphi^1 \in \mathcal{E}^1(X,-K_X)$ the solution of (2.10) has the unique solution $\varphi^t \in \mathcal{E}^1(X,-K_X)$ and provides a weak geodesic for d_1. These are called *psh geodesics* in [BBJ18]. The convexity of the D-energy functional along such a weak geodesic is established by the fundamental work [B09], [BP08], and [B11]. Since it is not scale free, i.e. $E(\varphi+c) = E(\varphi)+c$ for constants c, it is convenient to introduce the Aubin's J-functional:

$$J(\varphi) = L_0(\varphi) - E(\varphi) := \sup_X(\varphi - \varphi_0) - E(\varphi). \tag{2.18}$$

It follows that $J(\varphi) - d_1(\varphi,\varphi_0)$ is uniformly bounded. As we have the uniform estimate

$$\sup_X(\varphi - \varphi_0) \leqslant \frac{1}{V}\int_X (\varphi - \varphi_0)\omega_0^n + C \tag{2.19}$$

for $\varphi \in \mathcal{H}(X,-K_X)$, sometimes $V^{-1}\int_X(\varphi-\varphi_0)\omega_0^n$ is adopted for the definition of J.

We say that D-energy is coercive if

$$D(\varphi) \geqslant \varepsilon J(\varphi) - C$$

for any smooth φ. From the weak compactness of the level set $\{E \geqslant -C\}$ the coercivity guarantees a minimizer.

Remark 2.4. *If one considers coercivity for d_2 there is no example of Fano manifolds satisfy the condition. This is confirmed by* [BHJ15], *Proposition 8.5 for the K-energy. We may check the same for the D-energy, using Definition 3.2.*

2.4. **Modified D-energy.** Using the inner product (2.9) we may also modify the D-energy such that the critical point gives the Mabuchi soliton.

It is consistent to consider the group of bundle automorphism $\mathrm{Aut}(X,-K_X)$, indeed any $g \in \mathrm{Aut}(X,-K_X)$ pulls-back $\varphi \in \mathrm{PSH}(X,-K_X)$ to $g^*\varphi$. More precisely, for any $x \in X$ a vector $v \in (-K_X)_x$ is evaluated as

$$|v|^2 e^{-(g^*\varphi)(x)} = |g\cdot v|^2 e^{-\varphi(gx)}. \tag{2.20}$$

Note that the local frame identifying the function $\varphi(gx)$ with the pull-backed fiber metric depends on g. Indeed we see from (2.20) that the function $\varphi(gx)$ is unbounded in g. Since the line bundle is anti-canonical, any automorphism of X can be lifted to $-K_X$, hence the group splits into $\mathrm{Aut}(X,-K_X) = \mathrm{Aut}(X)\times\mathbb{G}_m$. We denote the identity component by $\mathrm{Aut}^0(X,-K_X)$. In particular, constant multiplication on each fiber defines the identical one-parameter subgroup $1\colon \mathbb{G}_m \to \mathrm{Aut}^0(X,-K_X)$.

Our first step is to specify the soliton vector field of the Mabuchi soliton. For the purpose we fix an algebraic subtorus $T \subset \mathrm{Aut}(X, -K_X)$. The compact part $S = \mathrm{Hom}(\mathbb{S}^1, T)$ is canonically defined. Henceforth we take a compact subgroup $K \subset \mathrm{Aut}^0(X, -K_X)$ which contains S and commutes with T. We define the space of K-invariant metrics

$$\mathcal{H}(X, -K_X)^K := \Big\{ \varphi \in \mathcal{H}(X, -K_X) : g^*\varphi = \varphi \text{ for any } g \in K \Big\}. \tag{2.21}$$

The case $K = S$ is possible. From the assumption that K commutes with the torus, T acts on $\mathcal{H}(X, -K_X)^K$. Tangents are identified with smooth K-invariant functions. We take K-invariant functions in 2.14 to define the distance d_1. Similarly the space of finite energy K-invariant metrics $\mathcal{E}^1(X, -K_X)^K$ can be defined and has the same property as Theorem 2.3. Note that in the definition

$$d_1(\varphi_0, \varphi_1) = \inf_{\varphi_t} \int_0^1 \frac{1}{V} \int_X |\dot{\varphi}_s| \, \omega_{\varphi_t}^n ds$$

a path φ_t, connecting φ_0 to φ_1, is taken as K-invariant. We may however take a weak geodesic of (2.10), which is K-invariant by the uniqueness if $\varphi_0, \varphi_1 \in \mathcal{H}(X, -K_X)^K$. It follows that d_1 for $\mathcal{H}(X, -K_X)^K$ equals to the previous one for $\mathcal{H}(X, -K_X)$.

Let us denote by $N := \mathrm{Hom}(\mathbb{G}_m, T)$ the lattice of all one-parameter subgroups $\mu \colon \mathbb{G}_m \to T$. The dual lattice $M := \mathrm{Hom}(T, \mathbb{G}_m)$ is identified with the set of characters. Observe that the vector space $N_{\mathbb{R}} := N \otimes \mathbb{R}$ is identified with the Lie algebera $\mathfrak{s}$ of S. From the basic symplectic geometry S defines the moment polytope $P \subset M_{\mathbb{R}}$ as the image of the moment map

$$m_\varphi \colon X \to M_{\mathbb{R}}. \tag{2.22}$$

Actually for any smooth K-invariant ω_φ and $\mu \in N_{\mathbb{R}}$ we have the unique map satisfying

$$\langle \mu, m_\varphi(x) \rangle = \frac{d}{dt}\bigg|_{t=0} \varphi(\mu(e^t)x). \tag{2.23}$$

It is easy to show that m_φ is independent of the metric. Once $\mu \in N_{\mathbb{R}}$ is fixed $h_\mu := \langle \mu, m_\varphi(x) \rangle$ gives the (unnormalized) Hamilton function. Notice that when $\mu \in N$ is generated by a vector field $v \in \mathfrak{t}$ we have the relation (2.6). The S-invariance of ω guarantees that h_μ is real. For the identical one-parameter subgroup we observe $h_1 = 1$.

In this convention following [FM95] we introduce the inner product

$$\langle \mu, \nu \rangle := \int_X h_\mu h_\nu \omega^n \tag{2.24}$$

for $\mu, \nu \in N_{\mathbb{R}}$. Of course h_μ depends on the choice of metric ω but as we will see in the next section the above inner product is determined only by μ, ν. The Hamilton function can be regarded as the tangent vector of the associated (smooth) geodesic ray

$$\varphi^t = \mu(e^{-t})^* \varphi^0 \tag{2.25}$$

for a given initial $\varphi^0 \in \mathcal{H}(X, -K_X)^K$. Note again that we assumed K commutes with T, hence φ^t is K-invariant for each t.

The slope of D-energy along this ray is independent of $t \in [0, \infty)$ and explicitly computed as

$$F(\mu) := \frac{1}{V}\int_X h_\mu(e^\rho - 1)\omega^n. \tag{2.26}$$

It is precisely the classical Futaki invariant [F83] for the vector field generating μ. The extremal vector field naturally arises from the optimization of $F(\mu)$ normalized by $\|\mu\| = \langle \mu, \mu\rangle^{\frac{1}{2}}$. Actually a simple variational computation

$$\delta\bigg(\frac{F(\mu)}{\|\mu\|}\bigg) = \frac{1}{\|\mu\|}\bigg(F(\delta\mu) - \frac{\langle \delta\mu, \mu\rangle}{\langle \mu, \mu\rangle}F(\mu)\bigg)$$

suggest us to introduce the extremal one-parameter subgroup $\eta \in N_{\mathbb{R}}$, which satisfies

$$F(\mu) - \langle \mu, \eta\rangle = 0 \tag{2.27}$$

for any $\mu \in N_{\mathbb{R}}$. Since (2.27) is a system of linear equations, one easily see that η is uniquely characterized by the above relation. It is also easy to check $\eta \in N_{\mathbb{Q}}$ and automatically

$$\langle 1, \eta\rangle = \int_X h_\eta \omega^n = 0. \tag{2.28}$$

On the other hand the Mabuchi soliton should minimize $R(\varphi)$. In fact if there exists a Mabuchi soliton ω_φ with $e^\rho - 1 = h_\mu$ for some $\mu \in N_{\mathbb{R}}$ we have $\mu = \eta$ and

$$R(\varphi) = \frac{F(\eta)}{\|\eta\|}.$$

That is, the both optimizer φ and η attain the same value. In general the lower bound of the Ricci-Calabi functional is attained by the normalized non-Archimedean D-energies which we introduce in the next section. See the recent work [X19] and [H19] for this topic. The following is a consequence of Theorem 2.8 in the next subsection.

Proposition 2.5. *There exists the modified Monge-Ampère energy $E_\eta \colon \mathcal{H}(X, -K_X)^K \to \mathbb{R}$ satisfying*

$$(dE_\eta)_\varphi = (1 + h_\eta)V^{-1}\omega_\varphi^n$$

at each point $\varphi \in \mathcal{H}(X, -K_X)^K$. Moreover, E_η is geodesically affine.

We define modified D-energy as $D_\eta := L - E_\eta$. It follows from the proposition that a smooth metric ω is Mabuchi soliton iff it is a critical point of the modified D-energy.

2.5. **Modified Monge-Ampère measure.** For the variational approach it is necessary to handle with $E_\eta(\varphi)$ for singular φ. In this part following [BWN14] we discuss basic properties of the modified Monge-Ampère measure. In [BWN14] the case $K = S$ is considered but the same argument works for general K which contains S and commuts with T. Let continuously $m_\varphi \colon X \to P$ be the moment map.

Definition 2.6 ([BWN14])**.** *Let $\varphi \in \mathcal{H}(X, -K_X)^K$. For a non-negative continuous function $g \colon P \to \mathbb{R}$ define the modified Monge-Ampère measure*

$$\mathrm{MA}_g(\varphi) := g(m_\varphi(x))\,\mathrm{MA}(\varphi).$$

The definition further extends to general $\varphi \in \mathrm{PSH}(X, -K_X)^K$ *so that the measure* $\mathrm{MA}_g(\varphi)$ *is local in plurifine topology and non-pluripolar.*

Theorem 2.7 ([BWN14], Theorem 2.7)**.** *The Duistermatt-Heckmann measure*

$$\mathrm{DH}_T := (m_\varphi)_* \mathrm{MA}(\varphi)$$

is independent of smooth φ *and defines a positive measure on* $M_{\mathbb{R}}$*. For any* $\varphi \in \mathrm{PSH}(X, -K_X)^K$ *we have*

$$\int_X \mathrm{MA}_g(\varphi) \leqslant \int_P g \mathrm{DH}_T.$$

The equality holds if $\varphi \in \mathcal{E}(X, -K_X)^K$*, namely when* $\mathrm{MA}(\varphi)$ *is a probability measure.*

Let $\langle , \rangle$ be the canonical paring of the lattices N and M. We are interested in the case

$$g(x) := 1 + \langle \eta, x \rangle - \int_P \langle \eta, x \rangle \mathrm{DH}_T. \tag{2.29}$$

Note that g of this form is not necessarily non-negative. At least when $g \geqslant 0$ and φ smooth we observe $\mathrm{MA}_g(\varphi) = (1 + h_\eta) \mathrm{MA}(\varphi)$ and

$$(\inf_P g) \mathrm{MA}(\varphi) \leqslant \mathrm{MA}_g(\varphi) \leqslant (\sup_P g) \mathrm{MA}(\varphi). \tag{2.30}$$

Notice that $1 + h_\eta > 0$ holds if X admits a Mabuchi soliton. As in the next section we shall see that the condition $1 + h_\eta > 0$ is numerical, from now on we assume that the above g is positive. Then the equation of Mabuchi soliton may be interpreted into the Monge-Ampère type equation

$$\mathrm{MA}_g(\varphi) = \mu_\varphi. \tag{2.31}$$

We call $\varphi \in \mathrm{PSH}(X, -K_X)^K$ satisfying this condition a *weak Mabuchi soliton*.

On the other hand, if we choose

$$g(x) = \frac{e^{\langle \mu, x \rangle}}{\int_P e^{\langle \mu, x \rangle} \mathrm{DH}_T} \tag{2.32}$$

with certain μ, equation (2.31) gives the weak Kähler-Ricci soliton. In this case g is always positive but $\mu \notin N_{\mathbb{Q}}$.

Theorem 2.8 ([BWN14], Lemma 2.14, Proposition 2.15)**.** *We have the canonical energy* $E_g \colon \mathcal{H}(X, -K_X)^K \to \mathbb{R}$ *such that* $(dE_g)_\varphi = \mathrm{MA}_g(\varphi)$*. For general* $\varphi \in \mathrm{PSH}(X, -K_X)^K$ *we have*

$$E_g(\varphi) := \inf_{\psi \geqslant \varphi} E_g(\psi),$$

where ψ *runs through bounded ones, or* $\mathcal{H}(X, -K_X)^K$*. The functional* E_g *is monotone, upper-semicontinuous in* L^1*-topology, and continuous for any non-increasing sequence in* $\mathrm{PSH}(X, -K_X)^K$*.*

The description of E_g is easily specified so we briefly sketch it. For the path $\varphi_t = (1-t)\varphi + t\varphi_0$ the demanded E_g is computed as

$$\begin{aligned} E_g(\varphi) &= \int_0^1 \frac{d}{dt} E_g(\varphi_t) dt = \frac{1}{V}\int_0^1 dt \int_X (\varphi - \varphi_0) g(m_{\varphi_t}) \omega_{\varphi_t}^n \\ &= \frac{1}{V}\sum_{i=0}^n \binom{n}{i} \int_0^1 t^i (1-t)^{n-i} dt \int_X (\varphi-\varphi_0) g(m_{\varphi_t}) \omega_\varphi^i \wedge \omega_0^{n-i}. \end{aligned}$$

Note $m_{\varphi_t} = t m_\varphi + (1-t) m_{\varphi_0}$ and that the last integrant is just a variant of modified Monge-Ampère measure. Therefore we may exploit Definition 2.6 to derive the required property of E_g. If $\inf_P g$ is positive it follows

$$(\sup_P g) E(\varphi) \leqslant E_g(\varphi) \leqslant (\inf_P g) E(\varphi) \tag{2.33}$$

provided $\sup_X(\varphi - \varphi_0) = 0$. It implies that $E_g(\varphi) > -\infty$ if φ has finite Monge-Ampère energy. At any case we define the g-modified J-energy by

$$J_g(\varphi) := L_0(\varphi) - E_g(\varphi). \tag{2.34}$$

This is after all equivalent to the J-functional.

Lemma 2.9. *When $g > 0$, we have*

$$(\inf_P g) J_g(\varphi) \leqslant J(\varphi) \leqslant (\sup_P g) J_g(\varphi)$$

for all $\varphi \in \mathcal{H}(X, -K_X)^K$.

Proof. Since $J_g(\varphi + c) = J_g(\varphi)$ for any constant $c \in \mathbb{R}$, we may assume $\sup_X(\varphi - \varphi_0) = 0$. The claim is then a consequence of (2.33). □

For a given probability measure μ one can consider the Monge-Ampère type equation $\mathrm{MA}_g(\varphi) = \mu$. It was also shown in [BWN14] Theorem 2.18 that there exists the unique solution $\varphi \in \mathcal{E}^1(X, -K_X)^K$ iff the Legendre dual of the Monge-Ampère energy

$$E_g^*(\mu) := \sup_{\varphi \in \mathcal{E}^1(X, -K_X)} \left[E_g(\varphi) - \int_X (\varphi - \varphi_0) d\mu \right] \in \mathbb{R} \cup \{\infty\} \tag{2.35}$$

is finite. Moreover, the above supremum is attained by the solution. We denote the dual of E_η by E_η^*.

What we will study is the g-modified D-energy $D_g(\varphi) := L(\varphi) - E_g(\varphi)$ and the equation (2.31).

Lemma 2.10. *Let $D_\eta \colon \mathcal{E}^1(X, \omega_0)^K \to \mathbb{R}$ be the modified D-energy. For each $\varphi \in \mathcal{E}^1(X, -K_X)^K$, the map $\mathbf{d}_\eta \colon K_{\mathbb{C}} \to \mathbb{C}$ defined by $\mathbf{d}_\eta(g) := D_\eta(g^*\varphi)$ is pluriharmonic. In particular if D_η is bounded from below, then $\mathbf{d}_\eta$ is constant on the center.*

Proof. This is similar to [H18], Theorem 1.6 and Remark 2.6 which are for the case $\eta = 0$. The statement simply interprets geodecically affineness of D_η into the complex variables.

The log part is obviously pluriharmonic. We show that $\mathbf{e}_\eta(g) := E_\eta(g^*\varphi)$ is pluriharmonic. Let us take an arbitrary holomorphic map $g \colon \Delta \to \mathrm{Aut}^0(X, L)$ which sends

$z \in \Delta$ in the one dimensional disk to the automorphism $g(z)$. By a direct computation we have the fiber integration formula (compare with (2.11) of [BWN14]):

$$dd^c E_\eta(\varphi_{g(z)}) = \frac{1}{(n+1)V} \int_X (1 + h_{\varphi_g,\eta})(dd^c_{z,x}\varphi_{g(z)}(x))^{n+1}.$$

For the holomorphic map $F\colon \Delta \times X \to X$ by $F(z,x) := g(z)\cdot x$ we have

$$(dd^c_{z,x}\varphi_{g(z)}(x))^{n+1} = (dd^c_{z,x}F^*\varphi)^{n+1} = F^*(dd^c_x\varphi)^{n+1} = 0.$$

It implies that $\mathbf{e}$ is pluriharmonic. □

Proposition 2.11. *Assume that T contains the center of the complexified Lie group $K_{\mathbb{C}}$. If there exists a minimizer of modified D-energy $D_\eta\colon \mathcal{E}^1(X,\omega_0)^K \to \mathbb{R}$ it defines a weak solution of (2.31).*

Proof. For a function v we define the point-wise upper envelope

$$Pv := \sup\Big\{\psi \in \mathrm{PSH}(X, -K_X)^K, \psi \leqslant v\Big\}.$$

The proof is due to the highly non-trivial derivation formula ([BWN14], Proposition 2.16):

$$\frac{d}{dt}\Big|_{t=0} E_g(P(\varphi + tu)) = \int_X u\,\mathrm{MA}_g(\varphi)$$

for $\varphi \in \mathcal{E}^1(X, -K_X)^K$, $u \in C^0(X;\mathbb{R})^K$. This was first established in [BB10] for $g = 1$, $K = \{\mathrm{id}\}$ case. If φ is a minimizer of D_η, we observe

$$\begin{aligned} f(t) &:= L(\varphi + tu) - E_\eta(P(\varphi + tu)) \\ &\geqslant L(P(\varphi + tu)) - E_\eta(P(\varphi + tu)) \\ &\geqslant L(\varphi) - E_\eta(\varphi) = f(0). \end{aligned}$$

The derivation formula yields $f'(0) = 0$ and hence

$$\int_X u\,\mathrm{MA}(\varphi) = \int_X u\mu_\varphi. \tag{2.36}$$

for every $u \in C^0(X;\mathbb{R})^K$.

We should show that the same holds for any $u \in C^0(X;\mathbb{R})$. By Lemma 2.10, $\mathbf{d}_\eta$ is constant on the center. We observe that for any one-parameter subgroup $\mu \in N$ the slope of $\mathbf{d}(\mu(e^{-t}))$ is equivalent to the classical Futaki character. Since the character is defined on the reductive Lie algebra $\mathfrak{k}_{\mathbb{C}}$ which can be written as the direct sum of the center and the derived algebra, the slopes are nontrivial only on the center. Therefore $\mathbf{d}_\eta$ is actually constant on whole $K_{\mathbb{C}}$. Thus the measure $\mu := (dD)_\varphi$ is K-invariant. It then follows that for any smooth function v and $g \in K$

$$\int_X v\mu = \int_X g_*(v\mu) = \int_X ((g^{-1})^*v)g_*(\mu) = \int_X ((g^{-1})^*v)\mu.$$

Integrating against the Haar measure we have

$$\int_X v\mu = \int_X u\mu = 0$$

so that $\mu = 0$ as desired.

Conversely, if $\varphi \in \mathcal{E}^1(X, -K_X)^K$ is a weak solution, convexity of D_g implies that φ is a minimizer. $\square$

In the next subsection a refinement of the latter argument will show the uniqueness of the weak Mabuchi soliton.

From now on we rather start from the extremal one-parameter subgroup η. Let

$$\mathrm{Aut}(X, \eta) := \left\{ g \in \mathrm{Aut}(X, -K_X) : \eta(\tau) g = g \eta(\tau) \text{ for all } \tau \in \mathbb{G}_m. \right\} \tag{2.37}$$

The identity component is denoted by $\mathrm{Aut}^0(X, \eta)$. We take afresh $T = C(\mathrm{Aut}^0(X, \eta))$ as the center of the automorphisms commuting with η. Moreover, we entirely consider a maximal compact subgroup K containing S. It clearly commutes with the center. We set

$$J_T(\varphi) := \inf_{\sigma \in T} J(\sigma^* \varphi). \tag{2.38}$$

Definition 2.12. *Let $T = C(\mathrm{Aut}^0(X, \eta))$ and K be a maximal compact subgroup of $\mathrm{Aut}^0(X, \eta)$, which contains the compact part of T. We say that the modified D-energy is coercive if there exists a positive constants ε, C such that*

$$D_\eta(\varphi) \geqslant \varepsilon J_T(\varphi) - C$$

holds for every invariant metric $\varphi \in \mathcal{H}(X, -K_X)^K$.

By the standard argument we may obtain the weak solution from the coercivity. Actually for a minimizing sequence φ_j, we have $\sigma_j \in T$ by the coercivity such that $\sigma_j^* \varphi_j$ is contained in the sublevel set $\{J_\eta \leqslant C\}$. Since $\{E \geqslant -C\}$ is weakly compact, we obtain a weakly convergent subsequence $\sigma_j^* \varphi_j \to \varphi$ in $\mathcal{E}^1(X, -K_X)^K$. From Lemma 2.10 the map $\sigma \mapsto D_\eta(\sigma^* \varphi_j)$ is constant. That is, D_η must be T-invariant. Especially $D_\eta(\sigma_j^* \varphi_j) = D_\eta(\varphi_j)$. The lower-semicontinuity concludes that φ is a minimizer of D_η.

Theorem 2.13 ([LZ17])**.** *A Fano manifold X admits a Mabuchi soliton if and only if $m_X > 0$, $\mathrm{Aut}^0(X, \eta)$ is reductive, and the modified D-energy is coercive.*

We have already explained that the coercivity implies the existence of the metric. The converse direction is based on [DR15]. Since the definition of the coercivity in [LZ17] is slightly different from ours, we briefly sketch the proof.

Proof. Let φ be the Mabuchi soliton. Trivially $K \subset \mathrm{Aut}(M, \varphi)$ so the maximality implies $K = \mathrm{Aut}(M, \varphi)$. Let $G = \mathrm{Aut}^0(X, \eta)$. By Corollary 2.18 $G = K_{\mathbb{C}}$ is reductive. We consider the normalizer and the centralizer

$$N_K(G) := \{g \in G : gkg^{-1} \subset K\},$$
$$C_K(G) := \{g \in G : gkg^{-1} = k \text{ for every } k \in K\}.$$

We first observe $C_K(G) = C(G)$. Indeed any $t \in C_K(G)$ we have the map $\tau \colon G \to G$ defined by $\tau(g) = tgt^{-1}$ and this is identical for $g \in K$. By Corollary 2.18 it implies that τ is identical on G.

From the general theory of Lie groups we know that $N_K(G)/KC_K(G)$ is finite. Let us show $N_K(G) = KC(G)$ in our situation. Since $N_K(G)/KC_K(G)$ is finite, we may write $N_K(G) = K'C_K(G) = K'C(G)$ for some maximal compact subgroup K'. By construction $K \subset K'$ so the maximality of K implies $N_K(G) = KC(G)$.

We now check that $T = C(G)$ acts transitively on the smooth Mabuchi solitons. By Theorem 2.17, for two Mabuchi solitons φ and φ' we have $f \in \mathrm{Aut}^0(X, \eta)$ such that $f^*\varphi = \varphi'$. Since $K = \mathrm{Aut}(M, \varphi) = \mathrm{Aut}(M, \varphi')$ as we have already observed, it follows $f^{-1}Kf \subset K$. Namely, $f \in N_K(G) = KC(G)$.

The above transitivity of T and the regularity of weak minimizers (Theorem 2.14), we may apply [DR15], Theorem 3.4 (with $\mathcal{R} = \mathcal{H}(X, -K_X)^K$, $G = C(\mathrm{Aut}^0(X, \eta))$ there) so that have constants ε, C and

$$D_\eta(\varphi) \geqslant \varepsilon \inf_{\sigma \in T} J(\sigma^*\varphi) - C \tag{2.39}$$

for every $\varphi \in \mathcal{H}(X, -K_X)^K$. □

2.6. **Uniqueness of Mabuchi soliton.** We shall first check the regularity. In [LZ17] the corresponding step is carried out by the continuity method assuming the coercivity. We here introduce a direct argument.

Theorem 2.14. *Assume $m_X = \inf_X (1 + h_\eta)$ is strictly positive. Then the weak Mabuchi soliton of (2.31) is actually smooth.*

Proof. Since φ has finite Monge-Ampère energy it has zero Lelong number (see [GZ17], Exercise 10.7). By the uniform version of Skoda's integrability theorem ([GZ17], Theorem 8.11), μ_φ has L^p-density for any $p > 1$. Noting (2.30) and applying the viscosity theory: [EGZ11], Theorem C to $\mathrm{MA}_g(\varphi) = \mu_\varphi$, we deduce that φ is continuous. We may further show φ is C^∞ essentially using Yau's C^2-estimate. For example one can apply the idea of [ST09] Theorem 1 to the present setting. See [ST19] for the detail exposition. □

For the uniqueness the fact $\varphi \in L^\infty$ is important, because we need the following.

Proposition 2.15 ([B11], Theorem 1.2)**.** *Let φ^t be a weak geodesic which is uniformly bounded in the sense that $|\varphi^t - \varphi_0| \leqslant C$. If the convex function $L(\varphi^t)$ is affine, there exists a $f_t \in \mathrm{Aut}(X, -K_X)$ such that*

$$f_t^* \omega_{\varphi^t} = \omega_{\varphi^0}.$$

Moreover $f_t = \exp(-t \,\mathrm{Re}\, v)$ for some holomorphic vector v lifted to $-K_X$ such that $\mathrm{Im}\, v$ preserves ω_{φ^t}.

Remark 2.16. *By* [B16]*, Proposition 3.3, we may further conclude $f_t^*\varphi^t = \varphi^0$.*

For $\mu \in N_{\mathbb{R}}$ we denote by $\mathrm{Aut}(X, \mu)$ the group of bundle automorphisms preserving μ. Set $\mathrm{Aut}(X, \varphi)$ for a fiber metric φ in a similar manner.

Theorem 2.17. *Let (ω_0, η_0) and (ω_1, η_1) be smooth Mabuchi solitons. Then there exists some $f \in \mathrm{Aut}^0(X, -K_X)$ such that*

$$f^*\omega_1 = \omega_0, \ \ f^*\eta_1 = \eta_0.$$

If $\eta_0 = \eta_1$ we have $f \in \mathrm{Aut}^0(X, \eta_1)$ and one can further take f generated by the imaginary part of $\mathrm{Aut}(X, \varphi_1)_{\mathbb{C}}$.

Proof. First we consider the case $\eta = \eta_0 = \eta_1$ contained in the torus T. Take potentials φ^0, φ^1 of ω_0, ω_1 and bounded geodesic φ^t ($t \in [0,1]$). Since φ^0, φ^1 are minimizers the convex function $D_T(\varphi^t)$ should be affine. In particular $L(\varphi^t)$ is affine. We may apply Theorem 2.15 so that $f_t^* \varphi^t = \varphi^0$. Observe that $\varphi^t = (f_t^{-1})^* \varphi^0$ is a weak Mabuchi soliton, since it is a minimizer of D_T. If we take the extremal vector field v generating η and set $w := (f_t)_* v - v$, it follows $L_w \omega_0 = 0$ and hence $dd^c h_w = 0$. That is, f_t preserves η.

When $\eta_0 \neq \eta_1$ noting that the maximal tori are conjugate to each other we may take some f so that $\eta_1 = f^* \eta_0$, by the uniqueness of the extremal vector field. ☐

The uniqueness argument is closely related to the reductivity result.

Corollary 2.18. *If a Fano manifold X admits a Mabuchi soliton (ω_φ, η) we have*

$$\mathrm{Aut}^0(X, \eta) = \mathrm{Aut}(X, \varphi)_{\mathbb{C}}.$$

That is, $\mathrm{Aut}^0(X, \eta)$ *is a complexification of the compact Lie group* $\mathrm{Aut}(X, \varphi)$.

Proof. From $h_\eta = 1 - e^\rho$ we know $\mathrm{Aut}(X, \varphi) \subset \mathrm{Aut}^0(X, \eta)$. If we take $g \in \mathrm{Aut}^0(X, \eta)$, $g^* \varphi$ is Mabuchi soliton hence some $f \in \mathrm{Aut}(X, \varphi)_{\mathbb{C}}$ satisfies $f^* \varphi = g^* \varphi$. It follows $g = (g \circ f^{-1}) \circ f \in \mathrm{Aut}^0(X, \varphi)_{\mathbb{C}}$. ☐

Mabuchi first showed Theorem 2.17 using the inverse-continuity method of [BM85]. In [M03] Corollary 2.18 is also proved by the twisted Laplacian calculas similarly to [M57]. As it was shown in [N19], one can also derive Corollary 2.18 directly from the second variation of the Ricci-Calabi functional. The present proofs are based on the idea of [B11] for the Kähler-Einstein metric. A virtue of this idea more directly links reductivity to the uniqueness.

2.7. **Thermodynamical formalism and modified K-energy.** Finally, following the thermodynamical formalism of [B13] and its modified version in [BWN14], we introduce the modified K-energy in terms of D-energy.

Recall for two probability measures μ, ν the relative entropy is defined to be

$$H(\mu|\nu) = \int_X \log \left[\frac{d\mu}{d\nu}\right] d\mu. \tag{2.40}$$

Its relation with D-energy is based on the Legendre transformation formula:

$$H(\mu|\mu_0) = \sup_{f \in C^0(X;\mathbb{R})} \left[\int_X f d\mu - \log \int_X e^f d\mu_0\right]. \tag{2.41}$$

Definition 2.19. *Fix a reference $\varphi_0 \in \mathcal{H}(X, -K_X)^K$ and $\mu_0 := \mu_{\varphi_0}$. Let $g \colon P \to \mathbb{R}$ be a positive continuous function on the moment polytope. For μ with finite $E^*(\mu)$ we define the free energy*

$$F_g(\mu) := H(\mu|\mu_0) - E^*_g(\mu). \tag{2.42}$$

For $\varphi \in \mathcal{E}^1(X, -K_X)^K$ we define the modified K-energy

$$M_g(\varphi) := F_g(\mathrm{MA}_g(\varphi)). \tag{2.43}$$

In [BWN14] the Kähler-Ricci soliton case (2.32) was discussed. In this case M_g is equivalent to the energy introduced by [TZ02] for smooth metrics. The treatment is valid for arbitrary g including Mabuchi soliton case.

Remark 2.20. *There are several functionals called modified K-energy in the literatures, which are defined mainly to characterize the extremal Kähler metric. For example the functional $M' := M - E_\eta$ gives one such candidate. An extremal metric might not be a Mabuchi soliton unless it is Kähler-Einstein. Since $M \geqslant D$ clearly implies $M' \geqslant D_\eta$, if D_η is coercive so does M'. It follows that if X admits a Mabuchi soliton it also has an extremal Kähler metric. The converse implication seems to be not known.*

Theorem 2.21 ([BWN14], Proposition 3.2)**.** *We have $M_g \geqslant D_g$ on $\mathcal{E}^1(X, -K_X)^K$ and a metric φ attains the equality iff it is a weak Mabuchi soliton. The modified K-energy M_g is lower bounded iff D_g is. In this case the infimums of the both functionals coincide.*

Proof. For the reader's convenience we give the proof. The one-side inequality (2.41) is a simple consequence of Jensen's inequality and actually holds for lower-semicontinuous function of the form $f = -(\varphi - \varphi_0)$. It immediately shows $M_g \geqslant D_g$, indeed

$$\begin{aligned} F_g(\mu) &= H(\mu|\mu_0) - E_g^*(\mu) \\ &= \sup_f \left[\int_X f d\mu - \log \int_X e^f d\mu_0 \right] - \sup_\varphi \left[E_g(\varphi) - \int_X (\varphi - \varphi_0) d\mu \right] \end{aligned}$$

and the second supremum is attained by the weak solution of $\mathrm{MA}_g(\varphi) = \mu$. At the same time we obtain the formula:

$$M_g(\varphi) = H(\mathrm{MA}_g(\varphi)|\mu_0) - E_g(\varphi) + \int_X (\varphi - \varphi_0)\, \mathrm{MA}_g(\varphi), \tag{2.44}$$

which is analogues to the Chen-Tian formula ([C00]). On the other hand, the first supremum is attained by the solution f of

$$\frac{e^f d\mu_0}{\int_X e^f d\mu_0} = d\mu.$$

Consequently, $M_g(\varphi) = D_g(\varphi)$ iff φ is a weak Mabuchi soliton.

It remains to show $\inf_\varphi M_g = \inf_\varphi D_g \in \mathbb{R} \cup \{-\infty\}$. Set $m := \inf_\varphi M_g$. By the properness result [BBEGZ16], Theorem 2.18, we have

$$H(\mu|\mu_0) \geqslant \alpha E^*(\mu) - C \tag{2.45}$$

for any α smaller than Tian's α-invariant. In particular, $H(\mu|\mu_0) < \infty$ implies that μ has finite energy so that some $\varphi \in \mathcal{E}^1(X, -K_X)^K$ solves $\mathrm{MA}_g(\varphi) = \mu$. Substitution to (2.44) yields

$$H(\mu|\mu_0) \geqslant m + E_g^*(\mu).$$

Since the infimum of the inversion formula

$$L(\varphi) = \inf_\mu \left[H(\mu|\mu_0) + \int_X (\varphi - \varphi_0) d\mu \right]$$

is attained by $\mu = \mu_\varphi$, it follows $D_g \geqslant m$. ∎

Totally in the same manner we observe that the coercivity $M_g(\varphi) \geqslant \varepsilon \inf_{\sigma \in T} J_g(\sigma^*\varphi) - C$ holds on $\mathcal{E}^1(X, -K_X)^K$ iff $D_g \geqslant \varepsilon \inf_{\sigma \in T} J_g(\sigma^*\varphi) - C$ on $\mathcal{E}^1(X, -K_X)^K$.

3. Relative uniform D-stability

Bearing in mind of the last section, we introduce the algebraic (non-Archimedean) counterpart of modified energies and define the right notion of stability which should characterize the existence of Mabuchi soliton. We first recall the notion of D-stability introduced by [B16]. The terminology here is due to [BHJ15], [BHJ16].

3.1. Uniform D-stability.

We first require that a test configuration $\pi\colon (\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ of a polarized manifold (X, L) is a family of polarized schemes defined over the affine line $\mathbb{A}^1$. In fact we may and should allow $\mathcal{L}$ to be only relatively semiample and $\mathbb{Q}$-Cartier divisor. Further, $\mathcal{X}$ is normal variety and endowed with a lifted $\mathbb{G}_m$-action $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(\mathcal{X}, \mathcal{L})$ such that the projection π is equivariant for λ and the standard $\mathbb{G}_m$-action to $\mathbb{A}^1$. Finally, we include into the datum the isomorphism

$$\pi^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq X \times (\mathbb{A}^1 \setminus \{0\})$$

which sends the line bundle $\mathcal{L}$ equivariantly to $L_{\mathbb{A}^1} = p_1^* L$. Although we are concerned with the case $L = -K_X$, $\mathcal{L}$ is still not equivalent to $-K_{\mathcal{X}/\mathbb{P}^1}$. Since we assumed $\mathcal{X}$ is normal, $K_{\mathcal{X}}$ is at least well-defined as a Weil divisor, however, it is even not the line bundle in general. Note that some literatures consider the family over the projective line $\mathbb{P}^1$. This is equivalent to our setting because one can always obtain the unique compactified family $(\bar{\mathcal{X}}, \bar{\mathcal{L}}) \to \mathbb{P}^1$ which is trivial around $\infty \in \mathbb{P}^1$.

Example 3.1. *Every one-parameter subgroup $\mu \in N$ defines a product family $X_{\mathbb{A}^1} = X \times \mathbb{A}^1$ endowed with the non-trivial action: $\lambda(\sigma)(x, \tau) := (\mu(\sigma)x, \sigma\tau)$ for $\sigma \in \mathbb{G}_m$. We call it a product configuration generated by μ. Therefore, test configuration can be seen as a far generalization of one-parameter subgroup. Note that the compactified family is no longer a product space.*

After the compactification $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ we may take the intersection number e.g. $\bar{\mathcal{L}}^{n+1}$. Since we assumed $\mathcal{X}$ is normal, $K_{\mathcal{X}}$ is at least well-defined as a Weil divisor and $K_{\bar{\mathcal{X}}} \bar{\mathcal{L}}^n$ essentially gives the famous Donaldson-Futaki invariant, or equivalently, non-Archimedean K-energy $M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ introduced in [BHJ15].

In terms of the log-canonical threshold we define the non-Archimedean D-energy. Recall that for given divisors $\mathcal{B}, \mathcal{D}$ the log-canonical threshold $\mathrm{lct}_{(\bar{\mathcal{X}}, \mathcal{B})}(\mathcal{D})$ is defined to be the supremum of $c \in \mathbb{R}$ such that the log pair $(\bar{\mathcal{X}}, \mathcal{B} + c\mathcal{D})$ has at worst log canonical singularities. Choosing the boundary divisor $\mathcal{B}$ linearly equivalent to $-K_{\bar{\mathcal{X}}/\mathbb{P}^1} - \bar{\mathcal{L}}$, the quantity reflects the positivity of the canonical divisor. Notice that in this choice the log-canonical divisor $K_{\bar{\mathcal{X}}} + \mathcal{B} \sim_{\mathbb{Q}} -\bar{\mathcal{L}} + \pi^* K_{\mathbb{P}^1}$ is $\mathbb{Q}$-Cartier so that the log discrepancies and log canonical singularities are well defined for any $(\mathcal{X}, \mathcal{L})$.

Definition 3.2. *For a test configuration $\pi\colon (\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ we define*

$$L^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \mathrm{lct}_{(\bar{\mathcal{X}}, \mathcal{B})}(\mathcal{X}_0) - 1, \;\; E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)V},$$

where $\mathcal{X}_0$ is the scheme theoretic central fiber and the boundary divisor is chosen $\mathcal{B} \sim_{\mathbb{Q}} -K_{\bar{\mathcal{X}}/\mathbb{P}^1} - \bar{\mathcal{L}}$. We say a Fano manifold X is D-semistable if the non-Archimedean D-energy

$$D^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) := L^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) - E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$$

is semipositive for all test configurations.

If $(\mathcal{X},\mathcal{L})$ is the product test configuration generated by $\mu \in N$, we write $D^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ as $D^{\mathrm{NA}}(\mu)$. It is known to be equivalent to the Futaki character (2.26), i.e.

$$D^{\mathrm{NA}}(\mu) = F(\mu) \tag{3.1}$$

holds for every $\mu \in N$. One can also define D-stability and D-polystability of a Fano manifold. See [B16], [BHJ15], and [F16] for the detail treatment. The uniform version is more important for us. We say that a test configuration $(\mathcal{X}',\mathcal{L}')$ is a pull-back of $(\mathcal{X},\mathcal{L})$ if a birational equivariant morphism $f\colon \mathcal{X}' \to \mathcal{X}$ yields $\mathcal{L}' = f^*\mathcal{L}$. Two test configurations are equivalent if there exists a common pull-back. It is easy to see that the above invariants have the same values for the equivalent test configurations. More substantially, by [BHJ15], any test configurations can be seen as a fiber metric of the Berkovich analytification $(\mathcal{X}^{\mathrm{NA}},\mathcal{L}^{\mathrm{NA}})$ and any two equivalent test configurations define the same non-Archimedean fiber metric, evaluating each valuation on the central fiber. The above L^{NA} and E^{NA} are actually functionals defined on these non-Archimedean fiber metrics. From this reason, taking a pull-back we may assume a domination $\rho\colon (\mathcal{X},\mathcal{L}) \to X_{\mathbb{A}^1}$ to the product family endowed with a possibly non-trivial action. By the projection formula the following definition is actually independent of ρ.

Definition 3.3. *Let*

$$L_0^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) := V^{-1}(\rho^* L_{\mathbb{A}^1})\bar{\mathcal{L}}^n.$$

We define the non-Archimedean counterpart of Aubin's J-functional as $J^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) := L_0^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) - E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$. A Fano manifold is called uniformly D-stable if there exists a constant $\varepsilon > 0$ such that

$$D^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) \geqslant \varepsilon J^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$$

holds for all test configurations.

Let us illustrate the key relation of the functionals E, J, D with their non-Archimedean version. It explains that test configuration gives the algebraic formulation of the geodesic ray on $\mathcal{H}$. In the sequel we denote the fiber of $\tau \in \mathbb{A}^1$ by $\mathcal{X}_\tau$ and the restricted line bundle by $\mathcal{L}_\tau$. As well, for the unit disk Δ we set $\mathcal{X}_\Delta := \pi^{-1}(\Delta)$ and $\mathcal{L}_\Delta := \mathcal{L}|_{\mathcal{X}_\Delta}$. For the punctured disk $\Delta^* = \Delta \setminus \{0\}$ we have the isomorphism $\mathcal{X}_{\Delta^*} \simeq X \times \Delta^*$ so that identify a point of $\mathcal{X}_{\Delta^*}$ with (x,τ). Let Φ be a smooth fiber metric of $\mathcal{L}_\Delta$, having the semipositive curvarture. It defines the ray

$$\varphi^t(x) = \Phi(\lambda(e^{-t})(x,1)) \tag{3.2}$$

so that φ^t for each $t \in [0,\infty)$ defines a fiber metric of L, having the semipositive curvature. This type of ray is said to be compatible with the test configuration. Any two metrics defines the same asymptotic because the difference of the associated rays is bounded uniformly in t. The following type of results is predicted in the origination of K-stability and proved for arbitrary test configurations in [B16], [BHJ16].

Theorem 3.4. *Let $F\colon \mathcal{H} \to \mathbb{R}$ be a functional either E, J, or D. For a test configuration and a ray φ^t compatible with $(\mathcal{X}, \mathcal{L})$ we have*

$$F^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{t\to\infty} \frac{F(\varphi^t)}{t}.$$

The above formula is indeed true for non-smooth but bounded Φ for which the semipositivity $dd^c\Phi \geqslant 0$ holds in the sense of current. In particular the same result holds for the associated *weak geodesic ray* φ^t which is characterized by the degenerate Monge-Ampère equation

$$(dd^c_{\tau,x}\Phi)^{n+1} = 0 \tag{3.3}$$

on $\mathcal{X}_\Delta$. Given a smooth boundary value φ^0, the bounded solution uniquely exists. For example [B16] gives the solution in terms of the Peron-Bremermann type envelope. After [CTW18] it is known to have the best-possible $C^{1,1}$-regularity. We have already emphasized that the consideration of weak geodesic is necessary for the variational approach.

As an immediate consequence the coercivity of the D-energy implies that X is uniformly D-stable. The heart of [BBJ18] is showing the converse direction. It is known that the uniform stability implies that the automorphism group is finite. What we are going to discuss suggests the one of the treatment for general automorphism groups.

3.2. **Associated concave function and Duistermatt-Heckmann measure.** We continuously fix an extremal one-parameter subgroup η and a torus $T \subset \mathrm{Aut}^0(X, \eta)$. As in the previous subsections we denote the lattice of one-parameter subgroups by N and the dual by M. Let $P \subset M_{\mathbb{R}}$ be the moment polytope of the maximal torus and

$$m_\varphi\colon X \to P$$

the moment map. Recall that the Duistermatt-Heckmann measure is the push-forward

$$\mathrm{DH}_T := (m_\varphi)_*(V^{-1}\omega^n) \tag{3.4}$$

which is again independent of the metric. Any $\mu \in N_{\mathbb{R}}$ is identified with the affine function $G_\mu(x) := \langle \mu, x\rangle$ on $M_{\mathbb{R}}$ hence we may integrate by DH_T. Let $k \in \mathbb{N}$ and $\mu_1, \dots, \mu_{N_k}$ be the weight of the $\mathbb{G}_m$-action to $H^0(X, kL)$, induced by μ. For any $p \geqslant 1$, the equivariant Riemann-Roch formula implies

$$\int_P G_\mu^p(x)\mathrm{DH}_T = \lim_{k\to\infty} \frac{1}{k^p N_k} \sum \mu_i^p. \tag{3.5}$$

If set $\mathrm{DH}_\mu := (h_\mu)_*(V^{-1}\omega^n) = (G_\mu)_*\mathrm{DH}_T$, by the Hausdorff moment theorem we obtain the convergence of the measures on $\mathbb{R}$:

$$\mathrm{DH}_\mu = \lim_{k\to\infty} \frac{1}{N_k} \sum \delta_{\frac{\mu_i}{k}}. \tag{3.6}$$

We also obtain that P is the closed convex hull of the set

$$\left\{ \frac{\chi}{k} \in M_{\mathbb{Q}} : \chi \in M, s_\chi \in H^0(X, kL) \text{ with } \sigma \cdot s_\chi = \chi(\sigma) s_\chi \right\} \tag{3.7}$$

and that $\mathrm{DH}_T = \frac{1}{N_k} \sum \delta_{\frac{\chi}{k}}$ where $\chi \in M$ runs for all $s_\chi \in H^0(X, kL)$. As a consequence

Proposition 3.5. *For every $\mu \in N_{\mathbb{Q}}$ we have*

$$m_X := \inf_P G_{1+\mu} = \lim_{k\to\infty} \min \frac{k+\mu_i}{k} = \inf_X (1+h_\mu)$$

and the final representation is independent of the metric.

More generally, a test configuration defines a concave function $G_{(\mathcal{X},\mathcal{L})}$ on P. For the purpose it is convenient to describe the test configuration in terms of the filtration. Given $s \in H^0(X,L)$, we have a rational section $\bar{s}(x,\tau) = \lambda(\tau)\cdot s(\lambda(\tau^{-1})(x,\tau))$ of $\mathcal{L}$. Considering how extent $\bar{s}$ is holomorphic we obtain a filtration of the section ring, which fully recovers the test configuration. For each $\lambda \in \mathbb{R}$ we set

$$F^\lambda H^0(X,kL) := \{s \in H^0(X,L) : \tau^{-\lceil\lambda\rceil}\bar{s} \in H^0(\mathcal{X},\mathcal{L})\}. \tag{3.8}$$

We may easily show that the filtration is monotone, left-continuous, and multiplicative in both k and λ. By [BHJ15] lemma 2.14, λ-weightspace of the induced action to $H^0(\mathcal{X}_0, k\mathcal{L}_0)$ is given by

$$H^0(\mathcal{X}_0, k\mathcal{L}_0)_\lambda \simeq F^\lambda H^0(X,kL)/F^{\lambda+1}H^0(X,kL). \tag{3.9}$$

It will also follow from Proposition 3.12 below. A non-trivial fact proved in [PS07] is the linearly boundedness. Namely there exists a constant $C>0$ such that

$$F^{kt}H^0(X,kL) = \{0\} \ (\textit{resp.}\ H^0(X,kL)) \tag{3.10}$$

for any $t > C$ (*resp.* $t < -C$) and $k \geqslant 1$. It is equivalent to say: $|\lambda| \leqslant Ck$ for the induced $\mathbb{G}_m$-action.

Imitating [WN12], we construct a concave function from the filtration.

Definition 3.6. *For each $t \in \mathbb{R}$ we define P^t as the closed convex hull of the set*

$$\left\{\frac{\chi}{k} \in M_{\mathbb{Q}} : \chi \in M, s_\chi \in F^{kt}H^0(X,kL) \ \textit{with}\ \sigma\cdot s_\chi = \chi(\sigma)s_\chi\right\}.$$

The associated concave function is

$$G_{(\mathcal{X},\mathcal{L})}(x) := \sup\{t \in \mathbb{R} : x \in P^t\}.$$

It is easy to check $G_{(\mathcal{X},\mathcal{L})} = G_\mu$ when $(\mathcal{X},\mathcal{L})$ is the product configuration generated by $\mu \in N$. Indeed from the definitions we compute

$$\begin{aligned}\tau^{-kt}\overline{s_\chi}(x,\tau) &= \tau^{-kt}\mu(\tau)\cdot s_\chi(\mu(\tau^{-1})x)\\ &= \tau^{-kt}(\mu(\tau)\cdot s_\chi)(x)\\ &= \tau^{-kt+\langle\mu,\chi\rangle}s_\chi(x).\end{aligned}$$

In terms of the associated weak geodesic ray (3.3) we may extend (3.6) to T-equivariant test configurations. Notice that the weak geodesic ray φ^t has $C^{1,1}$-regularity and the right-derivative

$$\dot{\varphi}^0 := \inf_{t>0}\frac{\varphi^t-\varphi^0}{t} \tag{3.11}$$

pointwisely defined is in fact a bounded function. It reflects the linearly boundedness (3.10).

Theorem 3.7. *Let $(\mathcal{X},\mathcal{L})$ be a T-equivariant test configuration. For each $k \in \mathbb{N}$ $\lambda_1,\dots,\lambda_{N_k}$ denote the weights of the induced $\mathbb{G}_m$-action to $H^0(\mathcal{X}_0, k\mathcal{L}_0)$. The push-forward*

$$\mathrm{DH}_{(\mathcal{X},\mathcal{L})} := \dot{\varphi}^0_*(V^{-1}\omega^n)$$

defines a probability measure on $\mathbb{R}$, which is independent of the metric. Moreover, it is equal to

$$(G_{(\mathcal{X},\mathcal{L})})_*\mathrm{DH}_T = \lim_{k\to\infty}\frac{1}{N_k}\sum\delta_{\frac{\lambda_i}{k}}.$$

Proof. The identity $\mathrm{DH}_{(\mathcal{X},\mathcal{L})} = \lim_{k\to\infty}\frac{1}{N_k}\sum\delta_{\frac{\lambda_i}{k}}$ is shown in [H16a]. For any $p \geqslant 1$ we have

$$\int_{\mathbb{R}} t^p (G_{(\mathcal{X},\mathcal{L})})_*\sum\delta_{\frac{\chi}{k}} = \int_P G^p_{(\mathcal{X},\mathcal{L})}\sum\delta_{\frac{\chi}{k}} = \sum G^p_{(\mathcal{X},\mathcal{L})}(\frac{\chi}{k}),$$

where the summation is for all $s_\chi \in H^0(X, kL)$. In view of the Hausdorff moment theorem it remains to show

$$\sum G^p_{(\mathcal{X},\mathcal{L})}(\frac{\chi}{k}) = \sum(\frac{\lambda_i}{k})^p.$$

We fix χ and by the linearly boundedness (3.10) take the largest t such that $s_\chi \in F^{\lceil kt\rceil}H^0(X,kL)$ but $s_\chi \notin F^{\lceil kt\rceil+1}H^0(X,kL)$. From (3.9) such kt one-to-one corresponds to λ_i so we complete the proof. ☐

Remark 3.8. *In* [WN12]*, the associated concave function on the* Okounkov body *is in fact defined for any possibly non-equivariant test configuration. If X is toric polarized manifold the Okounkov body and the associated concave function is equivalent to the present construction. In our setting $G_{(\mathcal{X},\mathcal{L})}$ is to be a piecewise-linear function.*

We conclude this subsection describing the invariants $E^{\mathrm{NA}}, J^{\mathrm{NA}}$ in terms of the $\mathbb{G}_m$-action.

Proposition 3.9 ([BHJ15] Proposition 7.8, Theorem 5.16)**.** *For any test configuration, the non-Archimedean Monge-Ampère energy satisfies*

$$E^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \int_{\mathbb{R}} t\mathrm{DH}_{(\mathcal{X},\mathcal{L})} = \lim_{k\to\infty}\frac{1}{N_k}\sum_{i=1}^{N_k}\frac{\lambda_k}{k}.$$

The functional L_0 satisfies

$$L_0^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \sup\operatorname{supp}\mathrm{DH}_{(\mathcal{X},\mathcal{L})} = \lim_{k\to\infty}\max_i\frac{\lambda_i}{k}.$$

Moreover, $\max_i\frac{\lambda_i}{k}$ is stable in k. More precisely, it is enough to take a sufficiently divisible k so that $k\mathcal{L}$ is globally generated. If there exists a domination $\rho\colon \mathcal{X}\to X_{\mathbb{A}^1}$, let E_0 be the strict trandsform of $X\times\{0\}$ and $D := \mathcal{L}-\rho^*L_{\mathbb{A}^1}$ be the unique $\mathbb{Q}$-divisor supported on $\mathcal{X}_0$. We then have

$$L_0^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \operatorname{ord}_{E_0} D.$$

3.3. **Relative setting.** Let us discuss the relative stability. From now on we fix η, $G = \mathrm{Aut}(X, \eta)$, $T = C(G)$, and $K \subset G$ a maximal compact subgroup containing S. The identical one-parameter subgroup to T is denoted by $1 \in N$. Test configurations are assumed to be G-equivariant, as the metrics were K-invariant.

Definition 3.10. *Let G be a reductive algebraic group. A test configuration $(\mathcal{X}, \mathcal{L})$ endowed with $\mathbb{G}_m \times G$-action is G-equivariant if it is compatible with the equipped $\mathbb{G}_m$-action on $(\mathcal{X}, \mathcal{L})$ and the G-action on $(X, L) = (\mathcal{X}_1, \mathcal{L}_1)$.*

Notice that we imposed the commutativity with G, on the $\mathbb{G}_m$-action. In particular G acts on the central fiber $\mathcal{X}_0$. The G-equivariance is not too much restrictive, as one can see [DS16], Theorem 1, in the Kähler-Einstein case. See also Example 3.21 below.

Example 3.11. *Consider a G-invariant ideal $I \subset \mathcal{O}_X$. Let $\rho \colon \mathcal{X} \to X_{\mathbb{A}^1}$ be the normalization of the blow-up along the ideal $\mathcal{J} := I + (\tau)$ and E be the exceptional divisor. We take $\varepsilon > 0$ and set $\mathcal{L} := \rho^* L_{\mathbb{A}^1} - \varepsilon E$. This typical test configuration called* deformation to the normal cone *is intensively studied in* [RT07]*. Blowing-up construction will even play an important role in proving Theorem A. Indeed one can show that $\mathcal{L}$ is ample for every sufficiently small ε. If for example the support V of $\mathcal{J}$ is smooth we may write $E = \mathbb{P}(N_{V/X} \oplus \mathcal{O}_X)$ as the normal cone. The induced $\mathbb{G}_m$-action is trivial on the normal bundle $N_{V/X}$ and the is the simple multiplication on $\mathcal{O}_X$. Since $\mathcal{J}$ is G-invariant, $(\mathcal{X}, \mathcal{L})$ inherits the G-action so that ρ is equivariant. In this construction we observe that the two actions actually commute to each other.*

Let us consider the case $G = \mathrm{Aut}(X, -K_X)$. For example, $\mathbb{P}^2$ does not have any G-invariant ideal. If X is the one point blow-up of $\mathbb{P}^2$ any G-invariant ideal is supported on the exceptional divisor. We may check that the deformation to the normal cone prevent X to be D-semistable.

The starting point here is to take the inner product of such a test configuration with arbitrary one-parameter subgroups, extending the definition of [FM95]. The equipped $\lambda \colon \mathbb{G}_m \to \mathrm{Aut}(\mathcal{X}, \mathcal{L})$ induces the action to $H^0(\mathcal{X}_0, k\mathcal{L}_0)$ for every $k \geqslant 1$. Since $\mathcal{X}$ is normal and is a family over the curve, it is flat. It follows $H^0(\mathcal{X}_0, k\mathcal{L}_0) \simeq H^0(X, kL)$ for any sufficiently large k. In fact we may have a G-equivariant trivialization of the vector bundle $\pi_*(k\mathcal{L})$ over $\mathbb{A}^1$.

Proposition 3.12. *The G-equivariant algebraic vector bundle $E = \pi_*(k\mathcal{L})$ on the affine line $\mathbb{A}^1$ is G-equivariantly isomorphic to $E_0 \times \mathbb{A}^1$.*

Proof. For the case $G = \{\mathrm{id}\}$ we refer [BHJ15] Proposition 1.3. Taking M into the account the same argument works. Indeed from the commutativity of the first component $\mathbb{G}_m$ with the second G, G-action does not effect. Let M_G be the lattice of weights and

$$H^0(\mathbb{A}^1, E) = \bigoplus_{(\lambda,\chi) \in \mathbb{Z} \oplus M_G} H^0(\mathbb{A}^1, E)_{(\lambda,\chi)} \tag{3.12}$$

be the decomposition to the irreducible representations. Set $V := E_1 = H^0(X, kL)$, $V_\chi := H^0(X, kL)_\chi$ and $F^\lambda V_\chi$ as the image of $H^0(\mathbb{A}^1, E)_{(\lambda,\chi)}$ under the restriction map $H^0(\mathbb{A}^1, E) \to E_1$. Definition 3.10 implies $F^\lambda V_\chi \subset V_\chi$. Since τ has weight -1 with respect to the $\mathbb{G}_m$-action on the base $\mathbb{C}$, multiplication by τ induces $F^{\lambda+1} V_\chi \subset F^\lambda V_\chi$.

Since $F^\lambda V_\chi = V_\chi$ for $\lambda \ll 0$ and $V = \oplus_\chi V_\chi$, the above map sending $\sum \tau^{-\lambda} v_\lambda$ to $\sum v_\lambda$ is surjective. On the other hand, if $\sum \tau^{-\lambda} v_\lambda$ lies in the kernel, $w_\lambda := -\sum_{\lambda' \geqslant \lambda} v_{\lambda'}$ in $F^\lambda V := \oplus_\chi F^\lambda V_\chi$ vanishes for $\lambda \ll 0$. Since $v_\lambda = w_{\lambda+1} - w_\lambda$, it means that $\sum \tau^{-\lambda} v_\lambda$ is in $(\tau - 1) H^0(\mathbb{A}^1, E)$. Thus we have $H^0(\mathbb{A}^1, E)/(\tau - 1) H^0(\mathbb{A}^1, E) \simeq V$ and the equivariant isomorphism

$$E|_{\mathbb{A}^1 \setminus \{0\}} \simeq V \times \mathbb{A}^1 \setminus \{0\}.$$

Similarly, by sending $\sum \tau^{-\lambda} v_\lambda$ to v_λ modulo $F^{\lambda+1} V$ we may show

$$E_0 \simeq \bigoplus_{\lambda \in \mathbb{Z}} F^\lambda V / F^{\lambda+1} V.$$

It follows the equivariant isomorphism

$$H^0(X, E) \simeq \bigoplus_{\lambda \in \mathbb{Z}} \tau^{-\lambda} F^\lambda V.$$

By choosing a basis compatible the filtration and $F^\lambda V = \oplus_\chi F^\lambda V_\chi$, we obtain a required equivariant trivialization. □

Now consider $G = \mathrm{Aut}^0(X, \eta)$. Because the test configuration is assumed to be G-equivariant, given $\mu \in N$ we may simultaneously diagonalize the two actions on $H^0(X, kL)$ and $H^0(\mathcal{X}_0, k\mathcal{L}_0)$ so that each weights λ_i and μ_i are assigned for the common vectors under the equivariant trivialization. In the sequel we may take any such λ_i and μ_i.

Definition 3.13 ([H16b]). *Let $(\mathcal{X}, \mathcal{L})$ be a T-equivariant test configuration. For any one-parameter subgroup $\mu \in N$ we have the limit*

$$E^{\mathrm{NA}}_\mu(\mathcal{X}, \mathcal{L}) = \lim_{k \to \infty} \frac{1}{k^2 N_k} \sum_{i=1}^{N_k} \lambda_i \mu_i.$$

For the identical one-parameter subgroup $1 \in N$ we observe $\mu_i = k$ and hence Proposition 3.9 shows

$$E^{\mathrm{NA}}_1(\mathcal{X}, \mathcal{L}) = \lim_{k \to \infty} \frac{1}{k N_k} \sum_{i=1}^{N_k} \lambda_i = E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}).$$

It is easy to check that the homogeneity naturally extends the definition to $\mu \in N_{\mathbb{Q}}$. We may further extend it to $\mu \in N_{\mathbb{Q}}$ by the following description.

Theorem 3.14 ([H16b]). *Let $(\mathcal{X}, \mathcal{L})$ be a T-equivariant test configuration and $\mu \in N_{\mathbb{Q}}$. For the associated weak geodesic ray φ^t and the Hamilton function h_μ we have*

$$E^{\mathrm{NA}}_\mu(\mathcal{X}, \mathcal{L}) = \frac{1}{V} \int_X \dot{\varphi}^0 h_\mu \omega^n.$$

Since E_g is geodesically affine, the right-hand side gives the slope at infinity.

Corollary 3.15. *For any geodesic ray φ^t compatible with $(\mathcal{X}, \mathcal{L})$ we have*

$$E^{\mathrm{NA}}_\mu(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{E_\mu(\varphi^t)}{t}.$$

Now we choose the extremal one-parameter subgroup $\eta \in N_{\mathbb{Q}}$ to define the non-Archimedean counterpart of E_η.

Definition 3.16. *The non-Archimedean counterpart of the modified Monge-Ampère energy E_η is defined to be*

$$E^{\mathrm{NA}}_\eta(\mathcal{X}, \mathcal{L}) := \langle (\mathcal{X}, \mathcal{L}), 1 + \eta \rangle.$$

We introduce the modifed non-Archimedean energies as $D^{\mathrm{NA}}_\eta(\mathcal{X}, \mathcal{L}) := L^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) - E^{\mathrm{NA}}_\eta(\mathcal{X}, \mathcal{L})$ and $J^{\mathrm{NA}}_\eta(\mathcal{X}, \mathcal{L}) := L^{\mathrm{NA}}_0(\mathcal{X}, \mathcal{L}) - E^{\mathrm{NA}}_\eta(\mathcal{X}, \mathcal{L})$.

Note that J^{NA}_η is not necessarily non-negative, just as J_η was. We shall see that $T \subset \mathrm{Aut}(X, \eta)$ is enough to examine the positivity of J^{NA}_η.

Proposition 3.17. *If $m_X > 0$, then $J^{\mathrm{NA}}_\eta(\mathcal{X}, \mathcal{L}) \geqslant 0$ and the equality holds iff the T-equivariant $(\mathcal{X}, \mathcal{L})$ is the trivial test configuration.*

Proof. It is immediate from Theorem 3.14 and 3.9 that

$$J^{\mathrm{NA}}_\eta(\mathcal{X}, \mathcal{L}) = \sup_X \dot{\varphi}^0 - \frac{1}{V} \int_X \dot{\varphi}^0 (1 + h_\eta) \omega^n. \tag{3.13}$$

Since we may rescale the $\mathbb{G}_m$-action to have $\sup_X \dot{\varphi}^0 = 0$, from the formula $m_X = \inf_X (1 + h_\eta) > 0$ implies $J^{\mathrm{NA}}_\eta(\mathcal{X}, \mathcal{L}) > 0$, otherwise $\dot{\varphi}^0$ is identically zero. By [BHJ15] Theorem A, $\dot{\varphi}^0 \equiv 0$ implies that $(\mathcal{X}, \mathcal{L})$ is trivial. That is, the product configuration with the trivial action. □

In terms of the associated concave function, we may write

$$J^{\mathrm{NA}}_\eta(\mathcal{X}, \mathcal{L}) = \max_P G_{(\mathcal{X}, \mathcal{L})} - \frac{1}{V} \int_P G_{(\mathcal{X}, \mathcal{L})} G_{1+\eta} \mathrm{DH}_T. \tag{3.14}$$

In our definition of stability we assume $m_X > 0$. By Proposition 3.5, this additional assumption is very much easier to check than the positivity of D^{NA}_η for all test configurations.

Let us return to a general $\mu \in N$ and take a G-equivariant trivialization so that the weights λ_i and μ_i are assigned for the common vectors. We endow a new $\mathbb{G}_m$-action with the space $(\mathcal{X}, \mathcal{L})$ such that the weights are given by $\lambda_i + \mu_i$. Since $T = C(G)$, it indeed gives a G-equivariant test configuration which we will denote by $(\mathcal{X}_\mu, \mathcal{L}_\mu)$. If Φ is the weak geodesic ray associated with $(\mathcal{X}, \mathcal{L})$, it is easy to see that

$$\varphi^t_\mu(x) := \Phi(\lambda(e^{-t})\mu(e^{-t})(x, 1)) \tag{3.15}$$

gives the geodesic ray associated with $(\mathcal{X}_\mu, \mathcal{L}_\mu)$. The homogeneity naturally extends the definition to arbitrary $\mu \in N_{\mathbb{Q}}$. From Theorem 3.14, we may further observe that $J^{\mathrm{NA}}(\mathcal{X}_\mu, \mathcal{L}_\mu)$ is continuous in $\mu \in N_{\mathbb{Q}}$.

Lemma 3.18. *The functional $J^{\mathrm{NA}}_\eta(\mathcal{X}_\mu, \mathcal{L}_\mu)$ is rationally piecewise-linear convex function in $N_{\mathbb{R}}$. It is moreover strictly convex in $N_{\mathbb{R}}/\mathbb{R}$. Especially the infimum*

$$J^{\mathrm{NA}}_T(\mathcal{X}, \mathcal{L}) := \inf_{\mu \in N_{\mathbb{R}}} J^{\mathrm{NA}}(\mathcal{X}_\mu, \mathcal{L}_\mu) \tag{3.16}$$

is attained by a rational μ.

Proof. The result was observed in [H18]. Indeed by Proposition 3.9 we see that

$$J_\eta^{\mathrm{NA}}(\mathcal{X}_\mu, \mathcal{L}_\mu) = \max_i \frac{\lambda_i + \mu_i}{k} - \frac{1}{N_k}\sum_{i=1}^{N_k}(\lambda_i + \mu_i)(1 + \eta_i).$$

The first term is independent of k, as soon as $k\mathcal{L}$ is globally generated. Note that the condition is independent of μ. The second term is affine in μ. Therefore, as the function in μ, it is the maximum for finite number of affine functions. The function is obviously non-negative and proper in $N_\mathbb{R}/\mathbb{R}$. ☐

The notation J_T and J_T^{NA} are consistent. We indeed have the slope formula.

Theorem 3.19 ([H18], Theorem B)**.** *Let $(\mathcal{X}, \mathcal{L})$ be a T-equivariant test configuration and φ^t be the associated weak geodesic ray. We have*

$$J_T^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{t\to\infty} \frac{J_T(\varphi^t)}{t}.$$

Notice that $g \in T$ attaining the infimum of $J_T(\varphi^t)$ depends on t. It is at least technically crucial to fix one torus in obtaining this sort of slope formulas. See [H18], Remark 1.8. Based on the results, we now arrive at the definition of the desired stability.

Definition 3.20. *A Fano manifold X is uniformly relatively D-stable if $m_X > 0$, $G = \mathrm{Aut}(X, \eta)$ is reductive, and there exists a constant $\varepsilon > 0$ such that*

$$D_\eta^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geqslant \varepsilon J_T^{\mathrm{NA}}(\mathcal{X}_\mu, \mathcal{L}_\mu)$$

holds for any G-equivariant test configuration. We say that X is relatively D-semistable if $D_\eta^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geqslant 0$ for any G-equivariant test configuration.

Example 3.21. *As we have observed, there is no G-invariant ideal when $X = \mathbb{P}^2$. It simply implies that $\mathbb{P}^2$ is uniformly relatively D-stable. When X is the one point blow-up of $\mathbb{P}^2$, we have the deformation to the normal cone $(\mathcal{X}, \mathcal{L})$ for the exceptional divisor. We may check that $(\mathcal{X}, \mathcal{L})$ dominates the product test configuration generated by η. It means that $D_\eta^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = J_T^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = 0$. Indeed X admits a Mabuchi soliton, and hence it is uniformly relatively D-stable, by the following general result. See* [Y17] *for investigation of the general toric Fano manifolds.*

Theorem 3.22. *If a Fano manifold admits a Mabuchi soliton, then it is uniformly relatively D-stable.*

Proof. By Theorem 2.13 we have the coercivity. As a consequence of Theorem 3.4 and Theorem 3.19, the coercivity implies the stability. ☐

4. Variational approach and proof of the main theorem

Standing on the preparation of the last two sections we give a proof of Theorem A. After we organized the formulation, the argument is now a simple extension of the variational approach [BBJ18], to the relative and equivariant setting.

4.1. **Convergence of weak geodesics.** Existence of the metric implies the stability, by Theorem 3.22. Let us assume that a Fano manifold is uniformly relatively D-stable, in the sense of Definition 3.20. Since J and J_η are equivalent, we may use J_η in replace of J. We shall suppose that the modified D-energy is not coercive and lead the contradiction.

The first step is to construct a weak geodesic ray in which direction the modified D-energy is not coercive. If the coercivity of Definition 2.12 fails for a constant $\varepsilon' < \varepsilon$, we have a sequence $\varphi_j \in \mathcal{H}(X, -K_X)^K$ $(j = 1, 2, \dots)$ so that

$$D_\eta(\varphi_j) \leqslant \varepsilon' J_\eta(\sigma^*\varphi_j) - j \tag{4.1}$$

for *any* $\sigma \in T$. Since both sides are preserved by the constant rescaling $\varphi \mapsto \varphi + c$ we may take

$$\sup_X(\varphi_j - \varphi_0) = 0. \tag{4.2}$$

We may moreover assume

$$E_\eta(\varphi_j) \to -\infty, \tag{4.3}$$

otherwise the uniform version of Skoda's integrability and the weak-compactness of the level set $\{\varphi \in \mathcal{E}^1(X, -K_X)^K : E(\varphi) \geqslant -C\}$ imply

$$D_\eta(\varphi_j) \geqslant -\log C - E_\eta(\varphi_j) \geqslant -\log C'.$$

Here we used again the comparison (2.33) of E and E_η. Then (4.1) yields $J_\eta(\varphi_j) \to \infty$, which contradicts to the assumption $E_\eta(\varphi_j) \geqslant -C$ with (4.2).

For the convergence of the weak geodesics connecting the reference φ_0 with φ_j we need the relative entropy and the strong topology.

Theorem 4.1 ([BBEGZ16], Theorem 2.17)**.** *The sublevel set*

$$\left\{\varphi \in \mathcal{E}^1(X, -K_X)^K : H(\mathrm{MA}(\varphi)|\mu_0) \leqslant C,\ \sup_X(\varphi - \varphi_0) = 0\right\}$$

is compact in the d_1-topology.

By the Legendre transform formula (2.41), we have

$$\begin{aligned} D_\eta(\varphi) &= -\log \int_X e^{-\varphi} - E_\eta(\varphi) \\ &= H(\mathrm{MA}(\varphi)|\mu_0) + \frac{1}{V}\int_X (\varphi - \varphi_0)\,\mathrm{MA}(\varphi) - E_\eta(\varphi) \\ &\geqslant H(\mathrm{MA}(\varphi)|\mu_0) + E(\varphi), \end{aligned}$$

which enables us to control the entropy. Let us now take a weak geodesic φ_j^t $(0 \leqslant t \leqslant -E_\eta(\varphi_j))$ which joins φ_0 to φ_j. Since the modified D-energy is geodesically convex, (4.1) yields

$$D_\eta(\varphi_j^t) \leqslant \frac{t}{-E_\eta(\varphi_j)} D_\eta(\varphi_j) \leqslant \frac{t}{-E_\eta(\varphi_j)}(\varepsilon' J_\eta(\sigma^*\varphi_j) - j). \tag{4.4}$$

In particular if $\sigma = \mathrm{id}$ we observe $D_\eta(\varphi_j^t) \leqslant \varepsilon' t$. By the geodecity we have $E_\eta(\varphi_j^t) = -t$. The term $E(\varphi_j^t)$ is then estimated by (2.33). Therefore, for each fixed T, φ_j^t $(0 \leqslant t \leqslant T)$

is contained in a compact subset with respect to the strong topology. The geodecity as well implies

$$d_1(\varphi_j^t, \varphi_j^s) = d_1(\varphi_j^1, \varphi_0)\,|t-s| \leqslant C(J(\varphi_j^1)+1)\,|t-s|$$

for any $t, s \geqslant 0$. By Ascoli's theorem, passing through a subsequence if necessary, we conclude that φ_j^t strongly converges to φ^t. It is immediate from $E_\eta(\varphi_j^t) = -t$ that $E_\eta(\varphi^t) = -t$. Lastly we let $\mu \in N$ and apply (4.4) to $\sigma = \mu(e^{-s_j})$, $s_j \geqslant 0$. The assumption $\varphi_j \leqslant \varphi_0$ implies $\sigma^*\varphi_j \leqslant \sigma^*\varphi_0$. By the slope formula:

$$L_0^{\mathrm{NA}}(\mu) = \lim_{j\to\infty} \frac{L_0(\mu(e^{-s_j})^*\varphi_0)}{s_j},$$

for $L_0^{\mathrm{NA}}(\mu) \leqslant 0$ we have the bound $L_0(\mu(e^{-s_j})^*\varphi_j) \leqslant L_0(\mu(e^{-s_j})^*\varphi_0) \leqslant o(s_j)$. Compare the claim with Lemma 4.3 below. Let us take $s_j := -E_\eta(\varphi_j)$. The lower-semicontinuity and (4.4) implies

$$\frac{D_\eta(\varphi^t)}{t} \leqslant \liminf_{j\to\infty} \frac{D_\eta(\varphi_j^t)}{t} \leqslant \varepsilon' \liminf_{j\to\infty} \frac{E_\eta(\mu(e^{-s_j})^*\varphi_j)}{E_\eta(\varphi_j)}.$$

Noting $E_\eta(\mu(e^{-s_j})^*\varphi_j) = \langle \mu, 1+\eta\rangle s_j + E_\eta(\varphi_j)$ we conclude

$$\frac{D_\eta(\varphi^t)}{t} \leqslant \varepsilon' \lim_{t\to\infty} \frac{-E_\eta(\varphi_\mu^t)}{t}. \tag{4.5}$$

It is obvious that the same inequality is valid for any $\mu \in N_{\mathbb{Q}}$ with $L_0^{\mathrm{NA}}(\mu) \leqslant 0$.

4.2. **Demailly type approximation.** The second step is to approximate φ^t constructed in the above by a sequence of test configurations. It is the non-Archimedean analogue of Demailly's approximation theorem for plurisubharmonic functions. Given φ^t, the relation (3.2) gives the singular K-invariant metric Φ on $L_{\mathbb{A}^1}$, defined over $X_{\mathbb{A}^1\setminus\{0\}} = \mathbb{C}^* \times X$. Since $\sup_X(\varphi^t - \varphi_0) = 0$, the plurisubharmonicity uniquely extends Φ to $\mathbb{A}^1$. Now for a sufficiently large $m \in \mathbb{N}$ we take the multiplier ideal sheaf $\mathcal{J}(m\Phi)$ and the normalized blow-up $\rho_m \colon \mathcal{X}_m \to \mathbb{A}^1$, endowed with the exceptional divisor E_m and the line bundle

$$\mathcal{L}_m := \rho_m^* L_{\mathbb{A}^1} - \frac{1}{m+m_0} E_m. \tag{4.6}$$

We may show that $\mathcal{L}_m$ is relatively semiample line bundle. See [BBJ18], Lemma 5.6 for the proof. We may check that the test configuration $(\mathcal{X}_m, \mathcal{L}_m)$ inherits the equivariant G-action, since $\mathcal{J}(m\Phi)$ is G-invariant. Note that the central fiber $\mathcal{X}_0$ is the union of the strict transform E_0 of $X \times \{0\}$ and the exceptional divisor E_m. The $\mathbb{G}_m$-action of $(\mathcal{X}_m, \mathcal{L}_m)$ is trivial on E_0 so that it commutes with the G-action.

Theorem 4.2 ([BBJ18], Theorem 5.4 and 6.4 for the $T = \{\mathrm{id}\}$ case)**.** *For the above test configurations constructed from φ^t, we have*

$$E_\eta^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) \geqslant \lim_{t\to\infty} \frac{E_\eta(\varphi^t)}{t},$$

$$\lim_{m\to\infty} L^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) = \lim_{t\to\infty} \frac{L(\varphi^t)}{t}.$$

We need E_η in the above, however, the proof is the same as [BBJ18]. Indeed, using Demailly's approximation theorem locally, we have the estimate

$$\Phi_m \geqslant \Phi - C_{m,r} \tag{4.7}$$

on the shrunken area $\mathbb{B}(0,r) \times X$. The constant $C_{m,r}$ is necessarily independent of t. Since the modified Monge-Ampère energy is monotone, we apply Corollary 3.15 to obtain

$$\begin{aligned} E_\eta^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) &= \lim_{t\to\infty} \frac{E_\eta(\varphi_m^t)}{t} \\ &\geqslant \lim_{t\to\infty} \frac{E_\eta(\varphi^t - C_{m,r})}{t} = \lim_{t\to\infty} \frac{E_\eta(\varphi^t)}{t} = -1. \end{aligned}$$

The key point in the above is the Ohsawa-Takegoshi L^2-extension theorem [OT87] used in Demailly's approximation.

Taking $\mu_m \in N_{\mathbb{Q}}$ we twist the action of $(\mathcal{X}_m, \mathcal{L}_m)$ to obtain the test configuration $(\mathcal{X}'_m . \mathcal{L}'_m) := (\mathcal{X}_{m,\mu_m}, \mathcal{L}_{m,\mu_m})$ which attains

$$\inf_{\mu \in N_{\mathbb{R}}} J_\eta^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) = J_\eta^{\mathrm{NA}}(\mathcal{X}'_m, \mathcal{L}'_m).$$

Since the rescaling $\mu \mapsto \mu + c$ preserves J_η^{NA}, we may assume that $L_0^{\mathrm{NA}}(\mu)$: the maximal weights of $\mu_{m,i}$, is just zero. By Proposition 3.9 we know $L_0^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) = \mathrm{ord}_{E_0} D_m$ for $D_m := \mathcal{L}_m - \rho_m^* L_{\mathbb{A}^1}$. It is zero because $\mathcal{L}_m$ does not contain E_0 by the construction.

Lemma 4.3. *If $\mu \in N_{\mathbb{Q}}$ satisfies $L_0^{\mathrm{NA}}(\mu) = 0$ in the above, we have*

$$L_0^{\mathrm{NA}}(\mathcal{X}'_m, \mathcal{L}'_m) = 0.$$

Proof. Note that the canonical morphism $\rho_m \colon \mathcal{X}_m \to X_{\mathbb{A}^1}$ is no longer equivariant for the twisted action. Let us denote the action by $\lambda \colon \mathbb{G}_m \to \mathrm{Aut}(\mathcal{X}_m, \mathcal{L}_m)$. We already observed $\max_i \frac{\lambda_i}{k} = 0$. The assumption $L_0^{\mathrm{NA}}(\mu) = 0$ is equivalent to say $\max_i \frac{\mu_i}{k} = 0$. It implies that there exists $s_\lambda \in H^0(X, kL)$ such that the rational section $(x, \tau) \mapsto s_\lambda(\lambda(\tau)(x,1))$ has zero order along E_0. (See also the proof of Theorem 3.12). For the twisted action, the associated rational section is $(x,\tau) \mapsto s_\lambda((\lambda+\mu)(\tau)(x,1))$. Since the action of μ preserves E_0, we deduce that this rational section as well has zero order along E_0. Therefore $\max_i \frac{\lambda_i + \mu_i}{k} = 0$.

□

For each $m \geqslant 1$, the uniform relative D-stability implies

$$\begin{aligned} L^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) &= D_\eta^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) + E_\eta^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) \\ &\geqslant -\varepsilon E_\eta^{\mathrm{NA}}(\mathcal{X}'_m, \mathcal{L}'_m) + E_\eta^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) \\ &= -\varepsilon \langle \mu_m, 1+\eta \rangle + (1-\varepsilon) E_\eta^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m). \end{aligned}$$

On the other hand, non-coercivity (4.5) yields

$$\begin{aligned}\lim_{m\to\infty} L^{\mathrm{NA}}(\mathcal{X}_m,\mathcal{L}_m) = \lim_{t\to\infty}\frac{L(\varphi^t)}{t} &= \lim_{t\to\infty}\frac{D_\eta(\varphi^t)+E_\eta(\varphi^t)}{t}\\ &\leqslant \lim_{t\to\infty}\frac{-\varepsilon' E_\eta(\varphi^t_{\mu_m})+E_\eta(\varphi^t)}{t}\\ &= -\varepsilon'\langle \mu_m, 1+\eta\rangle + (1-\varepsilon')\lim_{t\to\infty}\frac{E(\varphi^t)}{t}\end{aligned}$$

for each μ_m. Comparison of the two inequalities leads a contradiction and completes the proof of Theorem A.

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