GEOMETRIC FLOW, MULTIPLIER IDEAL SHEAVES AND OPTIMAL DESTABILIZER FOR A FANO MANIFOLD

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Abstract. In [D05], it was asked whether the lower bound of the Calabi functional is achieved by a sequence the normalized Donaldson-Futaki invariants. We answer to the question for the Ricci curvature formalism, in place of the scalar curvature. The principle is that the stability indicator is optimized by the multiplier ideal sheaves of certain weak geodesic ray asymptotic to the geometric flow. We actually prove it in the two cases: the inverse Monge-Ampère flow and the Kähler-Ricci flow.

Contents

1. Introduction

Let X be a Fano manifold. We are motivated to study how X is far from Kähler-Einstein. To examine the curvature of each Kähler metric ω in the first Chern class $c_1(X)$ we may make use of the normalized Ricci potential function ρ which is characterized by

$$\mathrm{Ric}\,\omega - \omega = dd^c\rho,\ \int_X (e^\rho - 1)\omega^n = 0. \tag{1.1}$$

The volume $V = \int_X \omega^n$ is independent of ω. The metric is Kähler-Einstein iff $\rho = 0$ and it is equivalent to say that the scalar curvature is constant. For a general polarized manifold (X, L), the famous Calabi functional measures how ω is far from constant scalar curvature and [D05] gives the lower bound in terms of the Donaldson-Futaki invariant. For a Fano polarization $(X, -K_X)$ Ricci potential may work in place of the scalar curvature. In fact in analogy with Donaldson's lower bound, we have the inequality

$$\inf_\omega \left[\frac{1}{V}\int_X (e^\rho - 1)^2\omega^n\right]^{\frac{1}{2}} \geqslant \sup_{(\mathcal{X},\mathcal{L})} \frac{-D^{\mathrm{NA}}(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|_2}. \tag{1.2}$$

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Here $(\mathcal{X}, \mathcal{L})$ runs through arbitrary test configurations of $(X, -K_X)$, $\|(\mathcal{X}, \mathcal{L})\|_2$ is the L^2-norm, and $D^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ is the non-Archimedean D-energy introduced in [B16], [BHJ15]. We review the terminologies and a proof of (1.2), in the next section. Such an inequality already appears in geometric invariant theory (GIT for short), where it confronts square of the moment map with Hilbert-Mumford weights. For this reason we may call (1.2) *moment-weight inequality*. The precise moment map picture was explained in [D15]. The Kemp-Ness functional in GIT is translated into the (Archimedean) D-energy (2.2).

In the scalar curvature setting Donaldson asked whether the equality holds in the above. In our setting, [Y17] recently proved that the equality in (1.2) actually holds for toric Fano manifolds. If there exists a test configuration accomplishes the identity, it should be the optimal destabilizer which has an analogy with the Harder-Narasimhan and the Jordan-Hölder filtration for the vector bundles. The pioneering work [N90] of Nadel already predicted that the certain multiplier ideal sheaf should serve as the destabilizing subsheaf of the vector bundle. See also [PSS06], [R09].

In this paper we show that the equality holds in (1.2) for general Fano manifolds, in virtue of adopting the Ricci potential formulation. Our new ingredient is the gradient flow of the D-energy. Using the dd^c-lemma we fix the reference metric ω_0 and represent any other metric by a function φ so that $\omega = \omega_0 + dd^c\varphi$ holds. The function φ is determined up to a constant and we consider $\rho = \rho_\varphi$ or other quantities as functions in φ. In terms of φ we introduce the *inverse Monge-Ampère flow*

$$\frac{\partial}{\partial t}\varphi = 1 - e^{\rho}, \tag{1.3}$$

which imitates the Calabi flow in the scalar curvature setting. Although the long-time existence of the Calabi flow is still open question, we have the solution for (1.3). This is one of the main results in our previous work [CHT17]. Building on the Mabuchi geometry of space of Kähler metrics, especially on the technique exploited by [DH17], one can construct a weak geodesic ray Φ asymptotic to the flow. Blowing up the multiplier ideal sheaves $\mathcal{J}(m\Phi)$, we obtain a sequence of test configurations, which canonically approximates the geodesic ray. The technology here was paved by [BBJ18] where they gave a variational approach to the celebrated result [CDS15]. We will show that the equality of (1.2) is then naturally achieved by the flow and these test configurations.

Theorem A (moment-weight equality)**.** Greatest lower bound of the Ricci-Calabi functional is given by a sequence of L^2-normalized non-Archimedean Ding energies, that is,

$$\inf_{\omega} \left[\frac{1}{V}\int_X (e^{\rho} - 1)^2 \omega^n\right]^{\frac{1}{2}} = \sup_{(\mathcal{X},\mathcal{L})} \frac{-D^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})}{\|(\mathcal{X}, \mathcal{L})\|_2}.$$

In fact the infimum is achieved by the inverse Monge-Ampère flow (1.3). The supremum is achieved by the test configurations which are defined as the blow-up of the associated multiplier ideal sheaves $\mathcal{J}(m\Phi)$.

Conjecturally the right-hand side would be the *maximum* attained by the optimal $(\mathcal{X}, \mathcal{L})$, provided we slightly stretches the meaning of test configurations. Actually for toric Fano manifolds [Y17] constructed the optimal destabilizer as a possibly irrational but piecewise-linear convex function on the moment polytope. It implies that a single ideal sheaf can not generally optimize the stability indicator. Our proof shows that the weak geodesic ray attains the maximum in a suitable sense (see Remark 4.8). It might be challenging to clarify whether the ray, constructed transcendentally in the above, interpreted into certain algebraic singularities.

Replacing the Ricci-Calabi functional with the H-functional $H(\omega)$, [DS17] established the parallel equality and the corresponding optimal test configuration. In this formalism non-Archimedean D-energy is replaced with H-invariant of the test configurations. The idea of the present paper as well applied to this setting. In the final section we serve another simple proof of [DS17], Theorem 1.2, without using the deep result of [CW14], [CSW15].

Theorem B. Greatest lower bound of H-functional is given by a sequence of H-invariants:

$$\inf_{\omega} H(\omega) = \sup_{(\mathcal{X},\mathcal{L})} H(\mathcal{X},\mathcal{L}).$$

The infimum is achieved by the Kähler-Ricci flow. The supremum is achieved by the test configurations which is defined as the normalized blow-up of the associated multiplier ideal sheaves.

In the H-functional setting the maximum is attained by the test configuration constructed by [CSW15]. Strictly speaking it is not a genuine test configuration but endowed with an irrational $\mathbb{C}^*$-action. In the terminology of [DS17] it is called $\mathbb{R}$-*degeneration*. Our argument does not construct the $\mathbb{R}$-degeneration in the limit, while it shows that the maximum is effectively approximated by the associated multiplier ideal sheaves.

It is known that $H(\mathcal{X},\mathcal{L}) > 0$ for all $(\mathcal{X},\mathcal{L})$ iff X is D-semistable so that the H-invariant is weaker than the non-Archimedean D-energy. For example in the toric case the optimal destabilizer for the entropy gives a product family while the optimal destabilizer for the Ricci-Calabi functional has jut two components in the central fiber. It indicates that H-optimizer corresponds to the Harder-Narasimhan filtration for vector bundles and D-optimizer even takes on a role of the Jordan-Hölder filtration for semistable ones. See [CHT17] for the detail. The construction and comparison of these two destabilizers could be interesting from the viewpoint of birational geometry and should be investigated in the future work.

Just when the author was going to post the preprint, he was informed the appearing work [X19] of M. Xia. It solves the *metrized* version of Theorem A, and of Donaldson's original conjecture for arbitrary compact Kähler manifolds, admitting finite energy geodesic rays in the supremum (so that the non-Archimedean D-energy is replaced with the *radial D-energy* of the geodesic ray). Note that in the scalar curvature setting the infimum requires singular φ because we do not have a smooth solution of the Calabi flow yet. Not a few ideas are in common and we even need [X19], Lemma 5.1 critically in proving Theorem A. We focus on the Fano case but instead answer to the original version of the question and moreover clarify the relation with multiplier ideals.

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2. Preliminary

2.1. **Ricci curvature formulation.** We first give the variational setting for the Kähler-Einstein problem. Throughout the paper X is an n-dimensional Fano manifold and ω denotes a Kähler metric whose cohomology class is the first Chern class $c_1(X)$. As in the introduction, we fix a reference metric to represent the metric by a function φ. In words of the anti-canonical line bundle $-K_X$, one has a fiber metric h_0 with the Chern curvature ω_0. Then $h_0 e^{-\varphi}$ defines another smooth fiber metric so that $\omega_\varphi = \omega_0 + dd^c\varphi$ gives the curvature. We freely chose appropriate description of the metric going back and forth between ω, φ and the fiber metric $h_0 e^{-\varphi}$. Let us denote by $\mathcal{H} = \mathcal{H}(X, \omega_0)$, the collection of all smooth φ for which $\omega = \omega_\varphi$ is strictly positive. We first introduce the Ricci-Calabi functional in $\varphi \in \mathcal{H}$, which is the curvature integration

$$R(\varphi) := \frac{1}{V}\int_X (e^\rho - 1)^2 \omega^n. \tag{2.1}$$

This gives the analogue of the classical functional

$$C(\varphi) := \frac{1}{V}\int_X (S_\omega - \hat{S})^2 \omega^n$$

introduced by E. Calabi. In the above $\hat{S}$ denotes the mean value of the scalar curvature S_ω. Compared to the Calabi functional, it is relatively recent result [D15] where the infinite-dimensional moment map picture for the Ricci-Calabi functional was given. The picture regards this functional as the square of the moment map and provides a natural prospect for the variational approach to the Kähler-Einstein problem. Role of the Kemp-Ness functional in finite-dimensional GIT is then played by the D-energy:

$$D(\varphi) = L(\varphi) - E(\varphi) := -\log \frac{1}{V}\int_X e^{-\varphi + \rho_0}\omega_0^n - E(\varphi). \tag{2.2}$$

The functional first appeared in [BM85] and was written down to the above form by [D88]. Here the second term

$$E(\varphi) := \frac{1}{(n+1)V}\sum_{i=0}^{n}\int_X \varphi\omega^i \wedge \omega_0^{n-i} \tag{2.3}$$

is called Aubin-Mabuchi energy, or the Monge-Ampère energy, because the differential is designed to be the Monge-Ampère measure: $(dE)_\varphi = V^{-1}\omega^n = V^{-1}\omega_\varphi^n$. As a consequence, Kähler-Einstein metric is characterized as the critical point of the D-energy. As we review in the next subsection, the D-energy is convex with respect to the natural metric structure. Asking when the energy functional is proper we are naturally lead to the definition of D-stability.

More recently in [CHT17], we studied the gradient flow of the D-energy

$$\frac{\partial}{\partial t}\varphi = 1 - e^\rho$$

and particularly proved that the long-time solution exists.

Theorem 2.1 ([CHT17], Theorem)**.** *Given an initial data, the inverse Monge-Ampère flow (1.3) has the unique solution $\varphi = \varphi_t$ for all $t \in [0, \infty)$. Moreover, $E(\varphi_t)$ is constant, $D(\varphi_t)$ and $R(t) = \frac{d}{dt}D(\varphi_t)$ are non-increasing.*

This is our key tool for speculating in which direction the D-energy *worstly* decays. In our notation the normalized Kähler-Ricci flow is written as

$$\frac{\partial}{\partial t}\varphi = -\rho.$$

The both flows converge to the Kähler-Einstein metric if it exists. On the other hand, the two flows show different behaviors when ρ tends to be big. It is precisely the situation we are interested in.

2.2. **Geodesic of finite energy metrics.** One remarkable property is that the Monge-Ampère energy is affine and the D-energy is convex along any geodesic for Mabuchi's L^2 structure. It strongly motivate us to exploit the general framework of convex optimization. In fact we may consider general L^p structure for the space of Kähler metrics and especially need to consider L^1 geometry. First from dd^c-lemma any smooth function u can be seen as a tangent vector at φ. The L^p-norm

$$\|u\|_p := \left[\frac{1}{V}\int_X |u|^p\,\omega^n\right]^{\frac{1}{p}} \tag{2.4}$$

hence defines the distance d_p on $\mathcal{H}(X, \omega_0)$. The metric space is not complete so that even if the energy is proper existence of a minimizer is not guaranteed. Therefore the completion $\mathcal{E}^p = \mathcal{E}^p(X, \omega_0)$ comes to the forefront in the variational approach to the Kähler-Einstein problem. This is the main reason that we need to handle with a singular fiber metric $h_0 e^{-\varphi}$ for which φ is only assumed to be locally integrable. Such an L^1-function is called ω_0-plurisubharmonic function (psh for short) if the curvature current $\omega = \omega_0 + dd^c\varphi$ is semipositive. One can see [BBGZ13], [BBEGZ11], [BBJ18], [D15], [D17a], [D17b], and the textbook [GZ17] for the developments in this area.

Let us especially present the construction of $\mathcal{E}^1$ which is indeed closely related with the Monge-Ampère energy. It is well-known that we have the satisfactory definition of the product current ω_φ^n and hence $E(\varphi)$ for any bounded ω_0-psh function φ, by the celebrated work of Bedford-Taylor. To go further, for any ω_0-psh φ we define the Monge-Ampère energy as

$$E(\varphi) := \inf\left\{E(\psi) : \psi \in L^\infty \cap \mathrm{PSH}(X, \omega_0), \psi \geqslant \varphi\right\} \in \mathbb{R} \cup \{-\infty\}. \tag{2.5}$$

The function is called *finite energy* if $E(\varphi) > -\infty$. We define the distance $d_1(\varphi, \psi)$ of finite energy metrics approximating by decreasing sequences of smooth ω_0-psh functions.

Theorem 2.2 (Special case of [D15], Theorem 2)**.** *The space $(\mathcal{E}^1(X, \omega_0), d_1)$ of all finite energy psh functions gives the completion of $(\mathcal{H}(X, \omega_0), d_1)$. Moreover d_1 gives the coarsest refinement of the L^1-topology for psh functions so that the Monge-Ampère energy is continuous. It follows that the D-energy is also continuous in this strong topology.*

Note that [D15] gave a similar construction for general $(\mathcal{E}^p(X,\omega_0), d_p)$.

We next review a certain construction of geodesics. Henceforth we distinguish the geodesic φ^t from the inverse Monge-Ampère flow φ_t, by using the superscript. The singularity of the metric again inevitably appears if one considers a geodesic. Indeed the L^2-geodesic segment φ^t $(t \in [0,1])$ in $\mathcal{E}^2(X,\omega_0)$ has at best $C^{1,1}$-regularity even if the endpoints are assumed to be smooth. For L^1-geodesic it is not even unique, as it was observed in [D17a]. Given smooth endpoints there however exists a path φ^t which is geodesic for all d_p. We follow [B11] for the construction. Let $\varphi, \psi \in \mathcal{H}$ and $a, b \in \mathbb{R}$. Take the complex variable τ of the annulus $A := \{\tau \in \mathbb{C} : e^{-b} < |\tau| < e^{-a}\}$, as the translation of the time parameter $t = -\log|\tau|$. Let us consider a function $\Psi \in \mathrm{PSH}(X \times A, p_1^*\omega_0)$ with the boundary condition $\Psi(x, e^{-a}) \leqslant \varphi(x), \Psi(x, e^{-b}) \leqslant \psi(x)$ and define the Peron-Bremermann type upper-semicontinuous envelope as

$$\Phi(x,\tau) := \sup{}^*\Psi(x,\tau). \tag{2.6}$$

The construction is also equivalent to the terminology *psh geodesic* in [BBJ18]. As a standard fact, we have $\Phi(x, e^{-a}) = \varphi(x), \Phi(x, e^{-b}) = \psi(x)$. Since we assume φ, ψ bounded Φ is also bounded. A standard argument of the pluripotential theory deduces that the $(n+1)$-variable Monge-Ampère measure $(p_1^*\omega_0 + dd^c_{x,\tau}\Phi)^{n+1}$ vanishes over $X \times A$. By the computation of [S92] this is equivalent to say that $E(\varphi^t)$ is affine. It follows that φ^t is weak geodesic for the L^2-structure. Moreover, by [D15], Theorem 4.17, φ^t defines a geodesic in the L^p-Finsler metric space $(\mathcal{E}^p, d_p)$ for an arbitrary $p \geqslant 1$.

It is rather recently proved by [CTW17] that Φ has optimal $C^{1,1}$-regularity for the smooth boundary data. From [D15], Remark 2.5, this geodesic of envelope form has a constant speed in d_p. It means that

$$d_p(\varphi^t, \varphi^s) = d_p(\varphi, \psi)\left|\frac{t-s}{b-a}\right| \tag{2.7}$$

for all t, s. Not all geodesics in $(\mathcal{E}^1, d_1)$ satisfies the property. See also the discussion in [D17a], [BBJ18].

2.3. **Non-Archimedean energies and norms of the test configuration.** The famous Hilbert-Mumford criterion in GIT tells that properness of the Kemp-Ness functional is examined in each direction for a one-parameter subgroup. Given a polarized manifold (X, L) each one-parameter subgroup of the projective transformation induces a degeneration $(\mathcal{X}, \mathcal{L})$ called *test configuration*. It is then natural to ask the asymptotic behavior of D-energy along the degeneration. In the scalar curvature setting for a general polarized manifold [D02] first gave the intrinsic definition of a test configuration and introduced the Donaldson-Futaki invariant in relation to asymptotic behavior of the K-energy functional.

In this paper we first assure that any test configuration $(\mathcal{X}, \mathcal{L})$ is a $\mathbb{G}_m$-equivariant family of $\mathbb{Q}$-polarized schemes, which is defined over the affine line $\mathbb{A}^1$. More generally we take account of the case when $\mathcal{L}$ is relatively semiample. As an assumption the family is trivial outside of the origin and the generic fiber is isomorphic to the anti-canonical polarization $(X, -K_X)$. In terms of the equivariant isomorphism $\mathcal{X}|_{\mathbb{A}^1\setminus\{0\}} \simeq \mathcal{X} \times (\mathbb{A}^1 \setminus \{0\})$, it is convenient to represent a point of $\mathcal{X}|_{\mathbb{A}^1\setminus\{0\}}$ as (x, τ), where $x \in X$

and τ is the affine coordinate centered on $0 \in \mathbb{A}^1$. Moreover, $\mathcal{X}$ may be assumed to be a normal variety. See e.g. [BHJ15] for the detail discussion for the singularities.

Gluing $(\mathcal{X}, \mathcal{L})$ with the trivial family we have the unique $\mathbb{G}_m$-equivariant family $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ defined over $\mathbb{P}^1$, so that the action is as well trivial in neighborhood of $\infty \in \mathbb{P}^1$. As it was compactified one can take the self-intersection number $\bar{\mathcal{L}}^{n+1}$ which in fact gives the non-Archimedean counterpart of the Monge-Ampère energy:

$$E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)V}. \tag{2.8}$$

For the substantial non-Archimedean treatment, we refer [BHJ15], [BHJ16], [BFJ16], [BJ18], and the survey article [B18]. For our purpose it is sufficient to recall that it gives the slope of the Monge-Ampère energy. Namely,

$$E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{t\to\infty} \frac{E(\varphi^t)}{t} \tag{2.9}$$

holds for any ray $\varphi^t \in \mathcal{H}$ compatible with $(\mathcal{X}, \mathcal{L})$. The compatibility requires that for the function

$$\Phi(x, \tau) := \varphi^{-\log|\tau|}(x),$$

the fiber metric $h_0(x)e^{-\Phi(x,\tau)}$ of $\mathcal{L}$ has the semipositive curvature and can be smoothly extended over the unit disk $\mathbb{B} = \{|\tau| < 1\}$. The isomorphism $\mathcal{X}|_{\mathbb{B}\setminus\{0\}} \simeq X \times (\mathbb{B} \setminus \{0\})$ translates the curvature form into $p_1^*\omega_0 + dd^c_{x,\tau}\Phi$. Any two compatible rays φ^t and ψ^t share the same slope because of the bound $|\Phi - \Psi| \leqslant C$ uniform in t. For the same reason, one may even have the same slope for a non-smooth but bounded Φ for which $\omega^n = \omega^n_{\varphi^t}$ and $E(\varphi^t)$ is properly defined as we already mentioned. In particular the above slope formula is still valid for the *weak geodesic ray* φ^t associated with $(\mathcal{X}, \mathcal{L})$.

In [B16], inspired by [B11], the associated weak geodesic ray Φ was in fact constructed as the Peron-Bremermann envelope with the prescribed boundary value. Let us take a function Ψ on $X \times (\mathbb{B}\setminus\{0\})$ for which $h_0(x)e^{-\Psi(x,\tau)}$ is extended to a singular fiber metric of $\mathcal{L}$, so that the curvature is semipositive in the sense of current. The associated weak geodesic ray is defined as the upper-semicontinuous envelope of Ψ with the boundary condition $\Psi(x, 1) \leqslant \varphi_0(x)$, which we denote

$$\Phi(x, \tau) := \sup{}^*\Psi(x, \tau). \tag{2.10}$$

Compare with the construction of the weak geodesic segment (2.6). One can see that it is equivalent to the rays previously constructed in [PS07], [CT08], and [RWN11].

Non-Archimedean D-energy is described as the log-canonical threshold

$$\begin{aligned} D^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= L^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) - E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \\ &:= \mathrm{lct}_{(\bar{\mathcal{X}}, \mathcal{B})}(\mathcal{X}_0) - 1 - \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)V}. \end{aligned}$$

Here the boundary divisor $\mathcal{B}$ is uniquely determined by the property $\mathcal{B} \sim_{\mathbb{Q}} -K_{\bar{\mathcal{X}}/\mathbb{P}^1} - \bar{\mathcal{L}}$ and $\operatorname{supp}\mathcal{B} \subset \mathcal{X}_0$. As a consequence of [B16], Theorem 3.11, we have the slope formula

$$D^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{t\to\infty} \frac{D(\varphi^t)}{t}. \tag{2.11}$$

See also the milestone works [DT92a], [T97]. It shows that the non-Archimedean D-energy for the Ricci curvature formulation just plays the role of the Donaldson-Futaki invariant defined by [D02] (equivalently, non-Archimedean K-energy defined by [BHJ15]) for the scalar curvature formulation. In terms of the positivity of $D^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$, one may define *D-stability* of $(X,-K_X)$ and prove that X admits a Kähler-Einstein metric iff it is D-polystable. We refer [BBJ18] for the variational approach to this problem. At any rate, we do not need the concrete description of D^{NA} or E^{NA} in the proof of Theorem A.

In terms of the $\mathbb{G}_m$-action, $E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ can be described as follows. Let us fix $k\in\mathbb{N}$ and write the weights $\lambda_1,\dots\lambda_{N_k}$ for the induced action on $H^0(\mathcal{X}_0,k\mathcal{L}_0)$. It then holds

$$E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})=\lim_{k\to\infty}\frac{\sum_{j=0}^{N_k}\lambda_j}{kN_k}. \tag{2.12}$$

In particular we observe that replacing $\mathcal{L}$ with line bundle $\mathcal{L}+c\mathcal{X}_0$ one has $E^{\mathrm{NA}}(\mathcal{X},\mathcal{L}+c\mathcal{X}_0)=E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})+c$. Letting $\hat{\lambda}:={N_k}^{-1}\sum_{j=0}^{N_k}\lambda_j$ we may further define the L^p-norm

$$\|(\mathcal{X},\mathcal{L})\|_p=\lim_{k\to\infty}\left[\frac{\sum_{j=0}^{N_k}\left|\lambda_j-\hat{\lambda}\right|^p}{k^pN_k}\right]^{\frac{1}{p}},$$

which is preserved by the above rescaling $\mathcal{L}\mapsto\mathcal{L}+c\mathcal{X}_0$. The main result of [H16] shows that these norms are equivalent to the L^p-norm of the associated weak geodesic ray:

$$\|(\mathcal{X},\mathcal{L})\|_p=\left[\frac{1}{V}\int_X\left|\dot{\varphi}^t-E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})\right|^p\omega_{\varphi^t}^n\right]^{\frac{1}{p}}.$$

Notice that for the ray associated to the test configuration the best possible $C^{1,1}$-regularity was established by [CTW18]. Thus we may have the time-derivative $\dot{\varphi}^t$ well-defined in the integrand.

Once the above results are accepted, the proof of the inequality (1.2) is immediate. Indeed by constant rescaling we may assume $E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})=0$ and the geodesic convexity implies

$$\begin{aligned}-D^{\mathrm{NA}}(\mathcal{X},\mathcal{L})&=\lim_{t\to\infty}\frac{-D(\varphi^t)}{t}\\&\leqslant-\frac{d}{dt}\bigg|_{t=0}D(\varphi^t)=-\frac{1}{V}\int_X\dot{\varphi}^0(e^\rho-1)\omega_{\varphi^0}^n.\end{aligned}$$

For the singular metrics the convexity result was established by [B11]. Applying the Cauchy-Schwartz inequality we obtain (1.2).

3. Construction of a weak geodesic ray and test configurations

3.1. **Estimates for the inverse Monge-Ampère flow.** In the sequel we assume that X admits no Kähler-Einstein metric, otherwise the identity of Theorem A is trivial. We denote the solution of the inverse Monge-Ampère flow (1.3) by φ_t ($t\in[0,\infty)$) and fix any sequence $t_j\to\infty$. Although the solution of the flow is smooth, to consider geodesics we need the space $\mathcal{E}^1$ of finite energy metrics.

We will take a geodesic segment $\varphi_j^t \in \mathcal{E}^1$ $(0 \leqslant t \leqslant t_j)$, which joins φ_0 to φ_{t_j}. Our normalization of the Ricci potential yields that $E(\varphi_t)$ is constant in t. For the supremum we have

Lemma 3.1 (Lemma 4.1 of [CHT17]). *The flow is linearly bounded from above:*

$$\varphi_t \leqslant t + A.$$

It follows that for any fixed T Aubin's J-functional

$$J(\varphi_t) := \frac{1}{V}\int_X \varphi_t \omega_0^n - E(\varphi_t)$$

is bounded in $t \in [0, T]$. Notice that $J(\varphi_t)$ is not bounded in $t \in [0, \infty)$, otherwise the flow converges to a weak minimizer of D-energy in $\mathcal{E}^1$, namely the Kähler-Einstein metric. It achieves the equality in (1.2). In other words, we have $\sup_X \varphi_{t_j} \to +\infty$. In fact by [D17b], Corollary 4.14, d_1 is explicitly described in terms of E as

$$d_1(\varphi, \psi) = E(\varphi) + E(\psi) - 2E(P(\varphi, \psi)),$$

where $P(\varphi, \psi) \in \mathcal{E}^1$ is the upper envelope of ω_0-functions u such that $u \leqslant \varphi, \psi$. The formula implies that

$$\sup_X \varphi_{t_j} - C \leqslant d_1(\varphi_0, \varphi_{t_j}) \leqslant \sup_X \varphi_{t_j} + C.$$

This is comparable with [DH17], Theorem 1 for the Kähler-Ricci trajectory.

3.2. **Construction of a weak geodesic ray asymptotic to the flow.** Following the argument of [DH17] we will show that, taking a subsequence if necessary, a particular choice (2.6) of geodesics φ_j^t converges to a ray φ^t in $(\mathcal{E}^1, d_1)$.

The convergence argument is based on the observation of [B13].

Proposition 3.2. *The relative entropy for the probabilistic measures ν and μ is the Legendre dual of the log-part of D-energy:*

$$\begin{aligned} H(\nu|\mu) &:= \int_X \log(\frac{\nu}{\mu})\nu \\ &= \sup_{f \in C^0(X;\mathbb{R})} \Big[\int_X f\nu - \log \int_X e^f \mu\Big]. \end{aligned}$$

The one-side inequality is obvious from Jensen's inequality and it is actually true for arbitrary lower-semicontinuous function f. For $\nu = V^{-1}\omega_\varphi^n$ the supremum is achieved by $f = -\varphi$. It follows that for any $\varphi \in \mathcal{E}^1$

$$H(V^{-1}\omega_\varphi^n | V^{-1}\omega_0^n) = D(\varphi) + E(\varphi) - \frac{1}{V}\int_X \varphi\omega_\varphi^n.$$

Note that for general non-positive ω_0-psh φ we have

$$(n+1)E(\varphi) \leqslant \frac{1}{V}\int_X \varphi\omega_\varphi^n \leqslant E(\varphi).$$

Since the Monge-Ampère flow non-increases $D(\varphi_t)$ and conserves $E(\varphi_t)$, we deduce from the fact that for any fixed T the entropy $H(V^{-1}\omega^n_{\varphi_t}|V^{-1}\omega^n_0)$ is bounded in $t \in [0,T]$. Now we recall the d_1-compactness of the set

$$\left\{\varphi \in \mathcal{E}^1 : H(V^{-1}\omega^n_{\varphi_t}|V^{-1}\omega^n_0) \leqslant C,\ \sup_X \varphi = 0\right\},$$

which was established in [BBEGZ11]. From the compactness and (2.7) Ascoli's theorem implies that (after passing to a subsequence) φ^t_j converges to a ray φ^t in $(\mathcal{E}^1, d_1)$. Moreover, for any fixed T, the convergent is uniform in $t \in [0,T]$. Since $t_j \to \infty$, φ^t is defined for $t \in [0,\infty)$. In fact by [BBJ18], Theorem 1.7, φ^t restricted to any interval $[a,b]$ is of envelope form (2.6). Consequently the limit ray inherits the constant speed property:

$$d_1(\varphi^t, \varphi^s) = d_1(\varphi^0, \varphi^1)\,|t-s|\,. \tag{3.1}$$

From the normalization we obtain

$$\sup_X \varphi^t \leqslant t + A \tag{3.2}$$

for any $t \in [0,\infty)$. On the other hand

$$\lim_{t\to\infty} \frac{E(\varphi^t)}{t} = 0. \tag{3.3}$$

For $p > 1$, it is not clear from the construction whether φ^t is asymptotic to the flow, in the sense of [DH17]. In general a ray φ^t is asymptotic to the curve φ_t, if there exists $t_j \to \infty$ and constant speed geodesic segments φ^t_j $(t \in [0,t_j])$ connecting φ_0 and φ_{t_j} such that for all t

$$\lim_{j\to\infty} d_p(\varphi^t_j, \varphi^t) = 0.$$

For the Kähler-Ricci flow [DH17] derive the property from the Harnack estimate which is not estabilished for the inverse Monge-Ampère flow. At present setting we will settle for the restrictive estimate:

Proposition 3.3 ([X19], Lemma 5.1)**.** *For each t we have $\varphi^t \in \mathcal{E}^2$ and*

$$d_2(\varphi^0, \varphi^t) \leqslant \liminf_{j\to\infty} d_2(\varphi^0, \varphi^t_j).$$

Proof. We sketch the proof. It fully exploits the CAT(0)-property of $\mathcal{E}^2$, which cannot be expected for other complete length space $\mathcal{E}^p$. In particular, we may generalize the notion of weak convergence in a Hilbert space to any complete CAT(0)-space. See [Bac14] for the general exposition.

Let us fix any t. A standard argument of pluripotential theory shows that the non-increasing sequence

$$\psi^t_j := \sup_{k\geqslant j}{}^* \varphi^t_k$$

converges almost everywhere to φ^t. Since φ^t_j $(j = 1, 2, \dots)$ are bounded in $\mathcal{E}^2$, one can prove that so does ψ^t_j. The boundedness with monotonicity implies $\varphi^t \in \mathcal{E}^2$, by [D15], Lemma 4.16. Now [BDL15], Theorem 5.3 asserts that φ_j^t weakly converges to φ^t. The point here is that the d_1-ball

$$B_\varepsilon(\varphi) := \left\{\psi \in \mathcal{E}^2 : d_1(\varphi, \psi) < \varepsilon\right\}$$

is d_2-closed and d_2-convex. It follows that for any weakly convergent subsequence $\varphi_{j_k}^t \to u^t$ $(k = 1, 2, \dots)$ we have $u^t = \varphi^t$. (From the $\mathcal{E}^2$-boundedness we have at least one weakly convergent subsequence, by [Bac14], Proposition 3.1.2.) Indeed $\varphi_{j_k}^t \in B_\varepsilon(\varphi^t)$ for any sufficiently large k. Since $B_\varepsilon(\varphi^t)$ is d_2-closed and d_2-convex, we conclude $u^t \in B_\varepsilon(\varphi^t)$ by [Bac14], Lemma 3.2.1. The desired inequality follows from the fact that the distance function is lower-semicontinuous with respect to the weak convergence (e.g. [Bac14], Corollary 3.2.4). □

In case $\varphi^t \in \mathcal{E}^p$, by [DL18], Theorem 1.2, we have φ^t as a geodesic ray for any $(\mathcal{E}^p, d_p)$. Moreover, each segment defines a unique geodesic ray when $p > 1$. In particular it then has the constant speed for d_p. Such φ^t is distinguished as *finite energy geodesic* in [DL18] and studied in view of geodesic stability.

Summarizing up we obtain:

Theorem 3.4. *Let φ_t be the inverse Monge Ampère flow and φ_j^t $(t \in [0, t_j])$ be the weak geodesic ray of the envelope form (2.6) so as to connect φ_0 to φ_{t_j}. Then there exists a ray φ^t of envelope form such that $\lim_{j\to\infty} d_1(\varphi_j^t, \varphi^t) = 0$ for each t. As a result φ^t is a geodesic for all $(\mathcal{E}^p(X, \omega_0), d_p)$ and satisfies (3.1), (3.2) and (3.3).*

Remark 3.5. *Provided the Harnack-type estimate for the inverse Monge-Ampère flow was established we may apply* [DH17], *Theorem* 3.2 *and obtain the ray directly. Such an estimate is highly non-trivial, as it implies the linear lower bound of φ_t, or equivalently, the upper bound of the Ricci potential ρ.*

3.3. **Approximative test configurations.** Next we follow [BBJ18] to construct a canonical sequence of test configurations which approximates φ^t. It can be seen as the non-Archimedean analogue of Demailly's approximation [D92] for a psh function.

Changing variables as

$$\Phi(x, e^{-t}) := \varphi^t(x),$$

we obtain the $\mathbb{S}^1$-invariant function Φ on $X \times (\mathbb{B} \setminus \{0\})$, which is actually $p_2^*\omega_0$-psh. From (3.2) $\hat{\Phi} := \Phi + \log|\tau|$ is uniquely extended to a $p_1^*\omega_0$-psh function on $X \times \mathbb{B}$. Since $\varphi^t \in \mathcal{E}^1$, the Lelong number is concentrated in $X \times \{0\}$. Moreover (3.2) implies that even the generic Lelong number along $X \times \{0\}$ is zero. Therefore, support of the $\mathbb{S}^1$-invariant multiplier ideal sheaf $\mathcal{J}(m\hat{\Phi})$ is properly contained in $X \times \{0\}$, so that we have the normalized blow-up $\rho_m \colon \mathcal{X}_m \to X \times \mathbb{C}$. It would be remarkable that the argument really requires the definition of multiplier ideal sheaves for general plurisubharmonic functions, since Φ has non-algebraic singularities. Let E_m be the exceptional divisor. We fix some $m_0 \in \mathbb{N}$ and set the line bundle as

$$\mathcal{L}_m := \rho_m^* p_1^*(-K_X) - \frac{1}{m + m_0} E_m + \frac{m}{m + m_0} \rho_m^* \mathcal{X}_{m,0}. \tag{3.4}$$

The number m_0 is chosen so that $\mathcal{O}(-(m+m_0)p_1^*K_X)\otimes\mathcal{J}(m\hat{\Phi})$ is globally generated for all $m\geqslant 1$. See [BBJ18], Lemma 5.6. The term involving the central fiber $\mathcal{X}_{m,0}$ preserves the linearly equivalence of $\mathcal{L}_m$ and only adjusts the $\mathbb{G}_m$-action. The constructed semiample test configuration $(\mathcal{X}_m,\mathcal{L}_m)$ satisfies the following continuity property, which is crucial for their variational approach to the Kähler-Einstein problem.

Theorem 3.6. *(*[BBJ18]*, Theorem 5.4, Lemma 5.7 and 5.8) For the above constructed weak geodesic ray and test configurations the upper-semicontinuity*

$$\limsup_{m\to\infty} D^{\mathrm{NA}}(\mathcal{X}_m,\mathcal{L}_m)\leqslant\lim_{t\to\infty}\frac{D(\varphi^t)}{t}$$

holds. Moreover, if φ^t is maximal in the sense of [BBJ18]*, Definition 6.5, we have the continuity*

$$\lim_{m\to\infty} D^{\mathrm{NA}}(\mathcal{X}_m,\mathcal{L}_m)=\lim_{t\to\infty}\frac{D(\varphi^t)}{t}.$$

Even the lower-semicontinuity of E^{NA} is special for our choice of test configurations. Since our setting looks slight different from [BBJ18], let us repeat this part of the proof. A similar idea will appear when we compare the L^2-norms in the last part of the proof of Theorem A. We take an $\mathbb{S}^1$-invariant, non-negatively curved smooth (or more generally bounded) fiber metric of the $\mathbb{Q}$-line bundle $\mathcal{L}_m$ on $X\times\mathbb{B}$. It defines a $p_2^*\omega_0$-psh function Φ_m endowed with the analytic singularity of $\mathcal{J}(m\Phi)^{\frac{1}{m+m_0}}$. This is the reason why we adjusted the line bundle by $\mathcal{X}_0$, in (5.6). Using Demailly's approximation theorem locally, we have the estimate

$$\Phi_m\geqslant\Phi-C_{m,r}$$

on the shrunken area $\mathbb{B}(0,r)\times X$. The positive constants C and r are independent of m. Since the Monge-Ampère energy is non-decreasing, it follows

$$\begin{aligned}E^{\mathrm{NA}}(\mathcal{X}_m,\mathcal{L}_m)&=\lim_{t\to\infty}\frac{E(\varphi_m^t)}{t}\\&\geqslant\lim_{t\to\infty}\frac{E(\varphi^t-C_{m,r})}{t}=\lim_{t\to\infty}\frac{E(\varphi^t)}{t}=0.\end{aligned}$$

The key point in the above is the Ohsawa-Takegoshi L^2-extension theorem [OT87] used in Demailly's approximation. Such a uniform lower bound estimate of the Bergman kernel already forms a basis of the celebrated work [CDS15] (see also [Tia15]).

Remark 3.7. *We may ask whether the constructed weak geodesic ray asymptotic to the inverse Monge-Ampère flow is maximal in the sense of* [BBJ18]. *For the proof of Theorem A, however, we do not require the maximality.*

4. Proof of the moment-weight equality

4.1. **Test configurations almost destabilize X.** The inverse Monge-Ampère flow satisfies

$$\frac{d}{dt}D(\varphi_t)=-\frac{1}{V}\int_X(e^{\rho}-1)^2\omega_\varphi^n=R(\varphi_t)$$

and $R(\varphi_t)$ is non-increasing, as a property of the gradient flow. In particular $\frac{d}{dt}D(\varphi_t) \leqslant 0$ and the convexity assures $\lim_{t\to\infty}\frac{D(\varphi_t)}{t} \in [-\infty, 0]$ exists. It then follows

$$\lim_{t\to\infty}\frac{D(\varphi_t)}{t} = \lim_{j\to\infty}\frac{D(\varphi_{t_j})}{t_j} = \lim_{j\to\infty}\frac{D(\varphi_j^{t_j})}{t_j}.$$

Since D-energy is convex along any geodesic, for any fixed T we have

$$\lim_{j\to\infty}\frac{D(\varphi_j^{t_j})}{t_j} \geqslant \frac{D(\varphi_j^T)}{T}.$$

The convergence of φ_j^t to φ^t in $(\mathcal{E}^1, d_1)$ then yields

$$\lim_{t\to\infty}\frac{D(\varphi_t)}{t} \geqslant \frac{D(\varphi^T)}{T}.$$

Letting $T \to \infty$, Theorem 3.6 now implies

Proposition 4.1. *Let φ_t be the inverse Monge-Ampère flow and φ^t be a weak geodesic ray asymptotic to the flow. For the test configurations which canonically approximates φ^t we have*

$$0 \geqslant \lim_{t\to\infty}\frac{D(\varphi_t)}{t} \geqslant \limsup_{m\to\infty} D^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m).$$

The proposition already shows that $(\mathcal{X}_m, \mathcal{L}_m)$ almost destabilize X. To get more precise upper bound of $D^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m)$, we prepare computing the differential of the energy along the flow.

Lemma 4.2. *Along the inverse Monge-Ampère flow we have*

$$\begin{aligned} -\frac{d}{dt}D(\varphi_t) &= -\frac{1}{V}\int_X \dot{\varphi}_t(e^{\rho_t}-1)\omega_{\varphi_t}^n \\ &= \left[\frac{1}{V}\int_X (\dot{\varphi}_t)^2\omega_{\varphi_t}^n\right]^{\frac{1}{2}}\left[\frac{1}{V}\int_X (e^{\rho_t}-1)^2\omega_{\varphi_t}^n\right]^{\frac{1}{2}}. \end{aligned}$$

Proof is immediate. Indeed, from the very definition of the inverse Monge-Ampère flow, the equality holds in the Cauchy-Schwartz inequality.

Remark 4.3. *It is natural to expect the optimal destabilizer for general L^p-norm. See also* [DL18]*, Theorem 1.6. In our argument, however, Lemma 4.2 apparently requires L^2-norm. In addition, the proof of Proposition 3.3 relies on the* $\mathrm{CAT}(0)$*-property of d_2. For a Fano manifold with no zero holomorphic vector fields, existence of the Kähler-Einstein metric is equivalent to the uniform stability with respect to the L^1-norm, as a result of* [BBJ18]*. Note that existence of L^p-destabilizer does not contradicts to the fact.*

4.2. **Comparison of the norms.** In regard with Lemma 4.2 we thus finally should study the L^2-norm

$$\|\dot{\varphi}_t\|_2 := \left[\frac{1}{V}\int_X (\dot{\varphi}_t)^2\omega_{\varphi_t}^n\right]^{\frac{1}{2}}.$$

For the inverse Monge-Ampère flow we have $\dot{\varphi}_t = 1 - e^\rho$ so that $\|\dot{\varphi}_t\|_2 = R(\varphi_t)^{\frac{1}{2}}$ is non-increasing. Note that the weak geodesic ray $\varphi^t \in \mathcal{E}^2$ is possibly apart from any test configurations and it might be not even C^1. For this reason we make use of the choice of φ^t and regard the norm $\|\dot{\varphi}^t\|_2$ as follows.

Definition 4.4. *For a weak geodesic ray $\varphi^t \in \mathcal{E}^2(X, \omega_0)$ with constant speed, we define the L^2-norm as*

$$\left\|\dot{\varphi}^t\right\|_2 := \lim_{t\to\infty} \frac{d_2(\varphi^0, \varphi^t)}{t} = \frac{d_2(\varphi^0, \varphi^t)}{t}.$$

It is of course consistent with the definition for a differentiable φ^t. Observe that $\|\dot{\varphi}^t\|_2$ is constant in t and moreover it is independent of the initial metric φ^0. Let us now take φ^t as in section 3.2.

Lemma 4.5. *For the above norms we have*

$$\|\dot{\varphi}_t\|_2 \geqslant \left\|\dot{\varphi}^t\right\|_2.$$

Proof. If $\|\dot{\varphi}_t\|_2 < \|\dot{\varphi}^t\|_2$ for some t, the above monotonicity implies that $\|\dot{\varphi}_t\|_2 < \|\dot{\varphi}^t\|_2$ holds for any sufficiently large $t \geqslant T$. Proposition 3.3 implies that the right hand side is bounded from above as

$$\left\|\dot{\varphi}^t\right\|_2 = \frac{d_2(\varphi^0, \varphi^t)}{t} \leqslant \liminf_{j\to\infty} \frac{d_2(\varphi^0, \varphi^t_j)}{t} = \liminf_{j\to\infty} \left\|\dot{\varphi}^t_j\right\|_2.$$

They are all independent of t. Therefore we may take $\varepsilon > 0$ such that $\|\dot{\varphi}_t\|_2 + \varepsilon < \left\|\dot{\varphi}^t_j\right\|_2$ for all j and $t \geqslant T$. It implies $d_2(\varphi_0, \varphi_{t_j}) < d_2(\varphi_0, \varphi^{t_j}_j)$ for a sufficiently large j. On the other hand, the L^2-geodesic connecting two metrics is unique by [D17b], Lemma 6.12, so that it has minimal length in all paths. It contradicts to our choice of φ^t_j which is L^p-geodesic for any $p \geqslant 1$. We conclude $\|\dot{\varphi}_t\|_2 \geqslant \|\dot{\varphi}^t\|_2$. □

Now we take a $p_1^*\omega_0$-psh function Φ_m as the weak geodesic ray associated to $(\mathcal{X}_m, \mathcal{L}_m)$, and compare $\|\dot{\varphi}^t\|_2$ with $\|\dot{\varphi}^t_m\|_2$. Recall that the weak geodesic ray associated to the test configuration has $C^{1,1}$-regularity by [PS10], [CTW18]. It implies that the norm $\|\dot{\varphi}^t_m\|_2$ is well-defined. Let us invoke the following Lidskii type inequality.

Theorem 4.6 ([DLR18], Theorem 5.1)**.** *For any $u, v, w \in \mathcal{E}^p(X, \omega_0)$ with $u \geqslant v \geqslant w$ we have*

$$d_p(v, w) \leqslant d_p(u, w) - d_p(u, v).$$

Lemma 4.7. *For the associated weak geodesic rays φ^t_m we have*

$$\left\|\dot{\varphi}^t\right\|_2 \geqslant \left\|\dot{\varphi}^t_m\right\|_2.$$

Proof. Since Φ_m comes from a bounded fiber metric of $\mathcal{L}_m$, it encodes the analytic singularity $\mathcal{J}(m\Phi)^{\frac{1}{m+m_0}}$. Again by using Demailly's approximation theorem locally, we have $\Phi_m \geqslant \Phi - C_{m,r}$. Since φ^t_m is bounded from above and φ^0 is smooth there exists a constant B_m such that $\varphi^0 + B_m \geqslant \varphi^t_m$. We are ready to apply Lidskii type inequality: Theorem 4.6 to these functions and get

$$d_2(\varphi^0 + B_m + C_{m,r}, \varphi^t_m + C_{m,r}) \leqslant d_2(\varphi^0 + B_m + C_{m,r}, \varphi^t). \tag{4.1}$$

It follows from the triangle inequality that

$$\left\|\dot{\varphi}^t\right\|_2 = \lim_{t\to\infty} \frac{d_2(\varphi^0, \varphi^t)}{t} \geqslant \lim_{t\to\infty} \frac{d_2(\varphi^0, \varphi^t_m)}{t} = \left\|\dot{\varphi}^t_m\right\|. \tag{4.2}$$

$\square$

Combining the results all together, we obtain

$$\begin{aligned}\liminf_{m\to\infty} -D^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) &\geqslant \lim_{t\to\infty} \frac{-D(\varphi_t)}{t} \\ &= \lim_{t\to\infty} \|\dot{\varphi}_t\|_2 \left[\frac{1}{V}\int_X (e^{\rho_t}-1)^2 \omega^n_{\varphi_t}\right]^{\frac{1}{2}} \\ &\geqslant \left\|\dot{\varphi}^t_m\right\|_2 \lim_{t\to\infty} \left[\frac{1}{V}\int_X (e^{\rho_t}-1)^2 \omega^n_{\varphi_t}\right]^{\frac{1}{2}}\end{aligned}$$

for all m. For a while we denote $\varepsilon_m := E^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m)$ which is nonnegative, as a consequence of Theorem 3.6. Recall that the norm $\|(\mathcal{X}_m, \mathcal{L}_m)\|_2 = \|\dot{\varphi}^t_m - \varepsilon_m\|_2$ slightly differs from the above $\|\dot{\varphi}^t_m\|_2$, however, we observe

$$\left\|\dot{\varphi}^t_m - \varepsilon_m\right\|_2^2 = \left\|\dot{\varphi}^t_m\right\|_2^2 - \varepsilon_m^2 \leqslant \left\|\dot{\varphi}^t_m\right\|_2^2$$

and hence conclude

$$\liminf_{m\to\infty} \frac{-D^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m)}{\|(\mathcal{X}_m, \mathcal{L}_m)\|_2} \geqslant \lim_{t\to\infty} \left[\frac{1}{V}\int_X (e^{\rho_t}-1)^2 \omega^n_{\varphi_t}\right]^{\frac{1}{2}}.$$

The last inequality completes the proof of Theorem A.

Remark 4.8. *The above proof of Theorem A shows that*

$$\inf_{\omega} \left[\frac{1}{V}\int_X (e^{\rho}-1)^2 \omega^n\right]^{\frac{1}{2}} \leqslant \frac{1}{\|\dot{\varphi}^t\|_2} \lim_{t\to\infty} \frac{-D(\varphi^t)}{t}$$

holds for a weak geodesic ray asymptotic to the inverse Monge-Ampère flow. If the non-Archimedean potential Φ^{NA} induced by φ^t is maximal in the sense of [BBJ18]*, the radial D-energy $\lim_{t\to\infty} t^{-1} D(\varphi^t)$ equals to the non-Archimedean D-energy of Φ^{NA}.*

As a consequence of [L17], the lower bound of the Calabi functional is zero iff X is D-semistable (see also [BBJ18]). We may restate the result in terms of the inverse Monge-Ampère flow.

Corollary 4.9. *Any Fano manifold X admits a Kähler metric with arbitrary small Ricci potential, otherwise a weak geodesic ray φ^t asymptotic to the inverse Monge-Ampère flow has negative slope:*

$$\lim_{t\to\infty} \frac{D(\varphi^t)}{t} < 0.$$

In particular, there exists a test configuration $(\mathcal{X}, \mathcal{L})$ with $D^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) < 0$.

5. The Kähler-Ricci flow case

5.1. **H-functional and H-invariant.** Recall that for $\varphi \in \mathcal{H}(X, \omega_0)$ we define the canonical probability measure

$$\mu_\varphi := e^{-\varphi+\rho_0}\omega_0^n. \tag{5.1}$$

In terms of the canonical measure, H-functional is described as the relative entropy functional:

$$H(\omega_\varphi) := H(\mu_\varphi | V^{-1}\omega_\varphi^n).$$

See Proposition 3.2 for our convention about the relative entropy. The functional first appeared in [DT92b] and has played an important role in the study of Kähler-Ricci flow. As a consequence of Pinsker's inequality it is at least bounded from below by the L^1-version of the Ricci-Calabi functional:

$$\sqrt{2H(\omega)} \geqslant \frac{1}{V}\int_X |e^\rho - 1|\,\omega^n. \tag{5.2}$$

Following [DS17], let us introduce the algebraic H-invariant of a test configuration as

$$\begin{aligned} H(\mathcal{X}, \mathcal{L}) &= -L^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) + F(\mathcal{X}, \mathcal{L}) \\ &:= -L^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) + \lim_{k\to\infty}\left[-\log\frac{1}{N_k}\sum_{j=1}^{N_k} e^{-\frac{\lambda_j}{k}}\right], \end{aligned}$$

where $\lambda_1, \dots, \lambda_{N_k}$ is the weights of the induced $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, k\mathcal{L}_0)$. Comparing with weight description of the non-Archimedean Monge-Ampère energy (2.12), we observe

$$H(\mathcal{X}, \mathcal{L}) \geqslant -D^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}). \tag{5.3}$$

Indeed H-invariant is weaker than the non-Archimedean D-energy. The Fano manifold X satisfies $H(\mathcal{X}, \mathcal{L}) > 0$ for any non-trivial test configurations iff it is D-semistable. Unfortunately, the second term F is in nature more transcendental than E^{NA} and has no numerical description. It even does not correspond to the classical energy of metrics. At least for the associated weak geodesic ray, one may observe that the "virtual slope"

$$F(\dot\varphi^t) := -\log\frac{1}{V}\int_X e^{-\dot\varphi^t}\omega^n$$

gives $F(\mathcal{X}, \mathcal{L})$, due to the following result.

Theorem 5.1 ([H16]). *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration and φ^t the associated $C^{1,1}$-weak geodesic ray. Then the pushed-forward probability measure*

$$\mathrm{DH}(\mathcal{X}, \mathcal{L}) := \dot\varphi^t_*(V^{-1}\omega_{\varphi^t}^n)$$

is independent of the initial data φ^0 and t. Moreover, we have the convergence of the spectral measure:

$$\frac{1}{N_k}\sum_{j=1}^{N_k}\delta_{\frac{\lambda_j}{k}} \to \mathrm{DH}(\mathcal{X}, \mathcal{L}).$$

It then follows from [B16], Theorem 3.11 the slope formula

$$H(\mathcal{X},\mathcal{L}) = \lim_{t\to\infty}\bigg[-\frac{L(\varphi^t)}{t} + F(\dot\varphi^t)\bigg]. \tag{5.4}$$

Lower bound of the H-functional is achieved by the supremum of these (unnormalized) H-invariant.

Theorem 5.2 ([DS17], Theorem 1.2)**.** *For a Fano manifold we have*

$$\inf_\omega H(\omega) = \sup_{(\mathcal{X},\mathcal{L})} H(\mathcal{X},\mathcal{L}).$$

The one-side inequality is easier to see from (5.4). Indeed if we take $f := -\dot\varphi$ in Proposition 3.2 the associated weak geodesic ray satisfies

$$\begin{aligned} H(\omega_{\varphi^0}) &\geqslant -\int_X \dot\varphi^0 \mu_{\varphi^0} - \log\frac{1}{V}\int_X e^{-\dot\varphi^0}\omega^n_{\varphi^0} \\ &\geqslant -\frac{d}{dt}L(\varphi^t) + F(\dot\varphi^t) \end{aligned}$$

for any choice of the initial metric φ^0.

The quantity $\inf_\omega H(\omega)$ is equivalent to the supremum of Perelman's μ-entropy. For a smooth function f satisfying

$$\int_X e^{-f}\omega^n = V,$$

we define the W-functional as

$$W(\omega, f) := \int_X (S_\omega + |\nabla f|^2 + f)e^{-f}\omega^n.$$

Perelman's μ-entropy is then defined to be the infimum:

$$\mu(\omega) := \inf_f W(\omega, f) \leqslant nV.$$

Theorem 5.3 ([DS17], Theorem 4.2)**.**

$$\sup_\omega \mu(\omega) = nV - \inf_\omega H(\omega).$$

We also remark the relation with the greatest lower bound of the Ricci curvature

$$R(X) := \sup\Big\{r \in [0,1] : \mathrm{Ric}\,\omega \geqslant r\omega\Big\}.$$

It was shown in [BBJ18] and [CRZ18] that $R(X) = \min\{\delta_X, 1\}$, where δ_X is the δ-invariant of Berman-Fujita. See [Fuj16] for the original definition and [BJ18] for the non-Archimedean interpretation. In particular, it follows that $R(X) = 1$ iff X is D-semistable.

Proposition 5.4. *If $R(X) > 1/4\pi$, we have $nVR(X) \leqslant \sup_\omega \mu(\omega) \leqslant nV$ and so that*

$$\inf_\omega H(\omega) \leqslant nV(1 - R(X)).$$

Proof. Let us take any $r < R(X)$ close to $R(X)$ and ω such that $\mathrm{Ric}\,\omega \geqslant r\omega$. Changing variables as $u^2 = e^{-f}$ and applying log-Sobolev inequality, we have

$$\begin{aligned} W(\omega, f) &= \int_X (S_\omega u^2 + 4\,|\nabla u|^2 - u^2 \log u^2)\omega^n \\ &\geqslant \int_X S_\omega u^2 \omega^n + (4\pi r - 1)\int_X (u^2 \log u^2)\omega^n. \end{aligned}$$

Since $S_\omega \geq nr$ and $r \geqslant 1/4\pi$ it yields $\mu(\omega) \geqslant nVr$ and hence $\sup_\omega \mu(\omega) \geqslant nVR(X)$. The last claim follows from Theorem 5.3. □

As a consequence of Birkar's uniform bound of the log-canonical thresholds ([B16]), for n-dimensional Fano manifolds $R(X)$ are uniformly bounded from below by a positive constant. The author does not know an example of Fano manifolds with $R(X) \leqslant 1/4\pi$.

5.2. **Weak geodesic ray asymptotic to the flow.** Let us explain how ideas in the previous sections reprove Theorem 5.2. It is natural to take the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega = -\,\mathrm{Ric}\,\omega + \omega$$

in place of the inverse Monge-Ampère flow. We again distinguish the flow φ_t from the geodesic φ^t by using the subscript. In terms of the normalized Ricci potential this can be described as

$$\frac{\partial}{\partial t}\varphi = -\rho. \tag{5.5}$$

In fact the equation (5.5) incorporates the slope into the H-functional in the form

$$H(\omega_{\varphi_t}) = -\frac{d}{dt}L(\varphi_t) + F(\dot{\varphi}_t).$$

As it was shown in [P08], [PSSW09], $H(\omega_{\varphi_t})$ is non-increasing. Notice that in [DH17] another normalization of the Kähler potential

$$r_t := \varphi_t - E(\varphi_t)$$

is adopted. Our choice of φ_t is precisely equal to $\tilde{r}_t$ in their notation. By Perelman's uniform estimate for the Ricci potential we have $\sup_X \varphi_t \leqslant ct + A$. See [P02], [ST08] for the expoundation. For the Monge-Ampère energy, $E(\varphi_t)$ is non-decreasing from Jensen's inequality. In particular, the finite slope $\lim_{t\to\infty} t^{-1}E(\varphi_t)$ exists. For the D-energy we obtain

$$\frac{d}{dt}D(\varphi_t) = \frac{-1}{V}\int_X \rho(e^\rho - 1)\omega_{\varphi_t}^n = -H(\omega_{\varphi_t}) + \frac{1}{V}\int_X \rho\omega_{\varphi_t}^n.$$

Jensen's inequality shows

$$\frac{1}{V}\int_X \rho\omega_{\varphi_t}^n \leqslant \log\frac{1}{V}\int_X e^\rho\omega_{\varphi_t}^n = 0$$

so that $D(\varphi_t)$ is non-increasing. Consequently, we may repeat the argument in subsection 3.3 to deduce the following.

Theorem 5.5 (Renormalization of [DH17], Theorem 2)**.** *Let φ_t be the normalized Kähler-Ricci flow and φ_j^t ($t \in [0, t_j]$) be the weak geodesic ray of the envelope form (2.6) so as to connect φ_0 to φ_{t_j}. Then there exists a ray φ^t such that $\lim_{j\to\infty} d_p(\varphi_j^t, \varphi^t) = 0$ for each t. As a result φ^t is a constant-speed geodesic for all $(\mathcal{E}^p(X, \omega_0), d_p)$. It satisfies $\sup_X \varphi^t \leqslant ct + A$ and $E(\varphi^t)$ constant.*

Proof. The statement was originally proved for $r_t = \varphi_t - E(\varphi_t)$. Let r_j^t, r^t be the corresponding weak geodesic. In the same way as subsection 3.3 we obtain the limit φ^t of φ_j^t. It then easy to check that $\varphi_j^t = r_j^t + \varepsilon_j t$ holds for $\varepsilon_j := t_j^{-1}(E(\varphi_{t_j}) - E(\varphi_0))$ which converges to $\varepsilon := \lim_{t\to\infty} t^{-1} E(\varphi_t)$. We conclude $\varphi^t = r^t + \varepsilon t$. □

We notice that the property $\lim_{j\to\infty} d_p(\varphi_j^t, \varphi^t) = 0$ follows from the Harnack estimate for the Kähler-Ricci flow. Let us extend the definition of $F(\dot{\varphi}^t)$ to the above (possibly not differential) weak geodesic ray. First we recall:

Theorem 5.6 ([D17a], Theorem 1)**.** *For the φ^t constructed from the envelope form (2.6) we have constants m, M such that for any $a, b \in [0, \infty)$*

(1) $\inf_X \frac{\varphi_a - \varphi_b}{a-b} = m$,
(2) $\sup_X \frac{\varphi^a - \varphi^b}{a-b} = M$.

Solution of the Hausdorff moment problem guarantees the following definition.

Definition 5.7. *Let φ^t be the above weak geodesic ray constructed from the envelope form (2.6), which in particular has a constant speed for any d_p. Define the Duistermatt-Heckman measure* $\mathrm{DH}(\varphi^t)$ *as the unique measure supported on $[m, M]$ such that for any $p \geqslant 1$*

$$\left[\int_{\mathbb{R}} |\lambda|^p \,\mathrm{DH}(\varphi^t)\right]^{\frac{1}{p}} = \frac{d_p(\varphi^0, \varphi^t)}{t}$$

holds. By definition $\mathrm{DH}(\varphi^t)$ *does not depend on t. We set*

$$F(\dot{\varphi}^t) := -\log \int_{\mathbb{R}} e^{-\lambda} \mathrm{DH}(\varphi^t).$$

When φ^t is the weak geodesic ray associated to a test configuration, we have

$$\int_{\mathbb{R}} |\lambda|^p \,\mathrm{DH}(\mathcal{X}, \mathcal{L}) = \frac{1}{V}\int_X \left|\dot{\varphi}^t\right|^p \omega_{\varphi^t}^n = \frac{d_p(\varphi^0, \varphi^t)^p}{t^p}.$$

It implies $\mathrm{DH}(\varphi^t) = \mathrm{DH}(\mathcal{X}, \mathcal{L})$. For the flow φ_t we set $\mathrm{DH}(\varphi_t) := (\dot{\varphi}_t)_*(V^{-1}\omega^n)$.

Lemma 5.8. *For the above weak geodesic ray asymptotic to the normalized Kähler Ricci flow we have*

$$F(\dot{\varphi}^t) = F(\dot{\varphi}_t) = 0.$$

Proof. First we observe

$$\int_{\mathbb{R}} e^{-\lambda} \mathrm{DH}(\varphi_t) = \frac{1}{V}\int_X e^{-\dot{\varphi}_t} \omega^n = \frac{1}{V}\int_X e^{\rho} \omega^n = 1.$$

Since $\mathrm{DH}(\varphi_j^t)$ is constant in t, we have

$$\begin{aligned}\int_{\mathbb{R}} |\lambda|^p \,\mathrm{DH}(\varphi_j^t) &= \frac{1}{t_j}\int_0^{t_j}\int_{\mathbb{R}} |\lambda|^p \,\mathrm{DH}(\varphi_j^t)\\ &= \frac{d_p(\varphi_0, \varphi_j^{t_j})^p}{t_j} = \frac{d_p(\varphi_0, \varphi_{t_j})^p}{t_j}\\ &= \frac{1}{t_j}\int_0^{t_j}\int_{\mathbb{R}} |\lambda|^p \,\mathrm{DH}(\varphi_t)\end{aligned}$$

for any $p \geqslant 1$. It means the identity of the probability measures:

$$\mathrm{DH}(\varphi_j^t) = \frac{1}{t_j}\int_0^{t_j} \mathrm{DH}(\varphi_t).$$

In the same way we obtain $\mathrm{DH}(\varphi^t) = \lim_{t\to\infty} \mathrm{DH}(\varphi_j^t)$ from $\lim_{j\to\infty} d_p(\varphi_j^t, \varphi^t) = 0$. In particular

$$\int_{\mathbb{R}} e^{-\lambda}\mathrm{DH}(\varphi^t) = \lim_{j\to\infty}\int_{\mathbb{R}} e^{-\lambda}\left[\frac{1}{t_j}\int_0^{t_j} \mathrm{DH}(\varphi_t)\right] = 1.$$

□

5.3. **Multiplier ideal sheaves for the asymptotic weak geodesic ray.** Totally in parallel with section 3.3 we may further construct approximative test configurations. Set the $\mathbb{S}^1$-invariant $p_2^*\omega_0$-psh function $\Phi(x, e^{-t}) := \varphi^t(x)$. The linear bound $\sup_X \varphi \leqslant ct + A$ implies that $\hat{\Phi} := \Phi + c\log|\tau|$ is uniquely extended to a $p_1^*\omega_0$-psh function on $X \times \mathbb{B}$. We obtain the $\mathbb{S}^1$-invariant multiplier ideal sheaf $\mathcal{J}(m\hat{\Phi})$ and the normalized blow-up $\rho_m \colon \mathcal{X}_m \to X \times \mathbb{A}^1$ with exceptional divisor E_m. The line bundle is given by

$$\mathcal{L}_m := \rho_m^* p_1^*(-K_X) - \frac{1}{m+m_0}E_m + \frac{cm}{m+m_0}\rho_m^*\mathcal{X}_{m,0}. \tag{5.6}$$

Theorem 5.9. *Let φ^t be the above weak geodesic ray for the normalized Kähler-Ricci flow and $(\mathcal{X}_m, \mathcal{L}_m)$ be the canonical sequence of test configurations approximates φ^t. Then we have*

$$\liminf_{m\to\infty} H(\mathcal{X}_m, \mathcal{L}_m) \geqslant \lim_{t\to\infty}\left[\frac{-L(\varphi^t)}{t} + F(\dot{\varphi}^t)\right].$$

Proof. For the part concerned with $L^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m)$ it is due to [BBJ18]. The $F(\mathcal{X}_m, \mathcal{L}_m)$ part follows from essentially the same argument as that for E^{NA}. Indeed, for the weak geodesic ray Φ_m associated with $(\mathcal{X}_m, \mathcal{L}_m)$ we obtain $\Phi_m \geqslant \Phi - C_{m,r}$ by using local Demailly approximation. We again use Theorem 4.6 to compare the p-moments as

$$\begin{aligned}\int_{\mathbb{R}} |\lambda|^p \,\mathrm{DH}(\varphi_m^t) &= \frac{1}{t}\int_0^t\int_{\mathbb{R}} |\lambda|^p \,\mathrm{DH}(\varphi_m^s)\\ &= \frac{d_p(\varphi_0, \varphi_m^t)^p}{t} \geqslant \frac{d_p(\varphi_0, \varphi^t - C_{m,r})^p}{t}.\end{aligned}$$

As $t \to \infty$, just by the definition of $\mathrm{DH}(\varphi^t)$, the last term converges to $\int_{\mathbb{R}} |\lambda|^p \,\mathrm{DH}(\varphi^t)$. It implies $\mathrm{DH}(\varphi_m^t) \geqslant \mathrm{DH}(\varphi^t)$ so that $F(\mathcal{X}_m, \mathcal{L}_m) \geqslant F(\varphi^t)$. □

Combining all together, we obtain

$$\begin{aligned}\liminf_{m\to\infty} H(\mathcal{X}_m,\mathcal{L}_m) &\geqslant \lim_{t\to\infty}\left[\frac{-L(\varphi^t)}{t}+F(\dot{\varphi}^t)\right]\\ &=\lim_{t\to\infty}\left[\frac{-L(\varphi_t)}{t}+F(\dot{\varphi}_t)\right].\end{aligned}$$

Finally the Kähler-Ricci flow equation translates the last term into the limit of $H(\omega_{\varphi_t})$. It completes the proof of Theorem B.

Remark 5.10. *In this setting thanks to Perelman's uniform estimate $\varphi_t \geqslant -Ct$ we may directly consider the multiplier ideal sheaves $\mathcal{J}(mF)$ for the flow $F(x,\tau) := \varphi_{-\log|\tau|}(x)$. We ask whether $\mathcal{J}(mF)$ optimally destabilize X.*

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