UNIFORM K-STABILITY AND ASYMPTOTICS OF ENERGY FUNCTIONALS IN KÄHLER GEOMETRY

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Abstract. Consider a smooth polarized complex manifold (X, L) and a smooth ray of positive metrics on L defined by a smooth positive metric on a test configuration for (X, L). For most of the common functionals in Kähler geometry, we prove that the slope at infinity along the ray is given by evaluating the non-Archimedean version of the functional (as defined in our earlier paper [BHJ16]) at the non-Archimedean metric on L defined by the test configuration. Using this asymptotic result, we show that coercivity of the Mabuchi functional implies uniform K-stability, as defined in [Der14a, BHJ16]. As a partial converse, we show that uniform K-stability implies coercivity of the K-energy when restricted to Bergman metrics.

Contents

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Introduction

Let (X, L) be a polarized complex manifold. A central problem in Kähler geometry is to give necessary and sufficient conditions for the existence of canonical metrics on L, for example, constant scalar curvature (csc) metrics. To fix ideas, suppose the reduced automorphism group $\mathrm{Aut}(X, L)$ is discrete. In this case, the celebrated Yau-Tian-Donaldson conjecture asserts that L admits a csc metric iff (X, L) is K-stable. That K-stability follows from the existence of a csc metric was proved by Stoppa [Stop09], building upon work by Donaldson [Don05], but the reverse direction is considered wide open in general.

This situation has led people to introduce stronger stability conditions that would hopefully imply the existence of a cscK metric. Building upon ideas of Donaldson [Don05], Székelyhidi [Szé06] proposed to use a version of K-stability in which, for any test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L), the Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ is bounded below by a positive constant times a suitable *norm* of $(\mathcal{X}, \mathcal{L})$. (See also [Szé15] for a related notion.)

Following this lead, we defined in the prequel [BHJ16] to this paper, (X, L) to be *uniformly K-stable* if there exists $\delta > 0$ such that

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

for any normal and ample test configuration $(\mathcal{X}, \mathcal{L})$. Here $J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ is a non-Archimedean analogue of Aubin's J-functional. It is equivalent to the L^1-norm of $(\mathcal{X}, \mathcal{L})$ as well as the minimum norm considered by Dervan [Der14a]. The norm is zero iff the normalization of $(\mathcal{X}, \mathcal{L})$ is the trivial test configuration, so uniform K-stability implies K-stability.

In [BHJ16] we advocated the point of view that a test configuration defines a *non-Archimedean metric* on L, that is, a metric on the Berkovich analytification of (X, L) with respect to the trivial norm on the ground field $\mathbb{C}$. Further, we defined non-Archimedean analogues of many classical functionals in Kähler geometry. One example is the functional J^{NA} above. Another is M^{NA}, a non-Archimedean analogue of the Mabuchi K-energy functional M. It agrees with the Donaldson-Futaki invariant, up to an explicit error term, and uniform K-stability is equivalent to

$$M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

for any ample test configuration $(\mathcal{X}, \mathcal{L})$. In [BHJ16] we proved that canonically polarized manifolds and polarized Calabi-Yau manifolds are always uniformly K-stable.

A first goal of this paper is to exhibit precise relations between the non-Archimedean functionals and their classical counterparts. From now on we do not *a priori* assume that the reduced automorphism group of (X, L) is discrete. We prove

Theorem A. *Let $(\mathcal{X}, \mathcal{L})$ be an ample test configuration for a polarized complex manifold (X, L). Consider any smooth strictly positive S^1-invariant metric Φ on $\mathcal{L}$ defined near the central fiber, and let $(\phi^s)_s$ be the corresponding ray of smooth strictly psh metrics on L. Denoting by M and J the Mabuchi K-energy functional and Aubin J-functional, respectively, we then have*

$$\lim_{s\to+\infty} \frac{M(\phi^s)}{s} = M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \qquad \textit{and} \qquad \lim_{s\to+\infty} \frac{J(\phi^s)}{s} = J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}).$$

The corresponding equalities also hold for several other functionals, see Theorem 3.6.

At least when the total space $\mathcal{X}$ is smooth, the assertion in Theorem A regarding the Mabuchi functional is closely related to several statements appearing in the literature [PRS08, Corollary 2], [PT09, Corollary 1], [Li12, Remark 12, p.38], following the seminal work [Tia97]. However, to the best of our knowledge, neither the precise statement given here nor its proof is available in the literature, and we therefore provide full details. As in [PRS08] we use Deligne pairings, but we also need to estimate integrals of the form $\int_{\mathcal{X}_\tau} e^{2\Psi|_{\mathcal{X}_\tau}}$, where $\mathcal{X} \to \mathbb{C}$ is a semistable test configuration for X and Ψ is a smooth metric on $K_{\mathcal{X}/\mathbb{C}}$ near the central fiber, see Lemma 3.9.

Donaldson [Don99] (see also [Mab87, Sem92]) has advocated the point of view that the space $\mathcal{H}$ of positive metrics on L is an infinite-dimensional symmetric space. One can view the space $\mathcal{H}^{\mathrm{NA}}$ of positive non-Archimedean metrics on L as (a subset of) the associated (conical) Tits building.

We then use the asymptotic formulas in Theorem A to study coercivity properties of the Mabuchi functional. As an immediate consequence of Theorem A, we have

Corollary B. *If the Mabuchi functional is coercive in the sense that*

$$M \geq \delta J - C$$

on $\mathcal{H}$ for some positive constants δ and C, then (X, L) is uniformly K-stable, that is,

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

holds for any normal ample test configuration $(\mathcal{X}, \mathcal{L})$.

Coercivity of the Mabuchi functional is known to hold if X is a Kähler-Einstein manifold without vector fields. This was first established in the Fano case by [PSSW08]; an elegant proof can be found in [DR15]. As a special case of a very recent result of Berman, Darvas and Lu [BDL16], coercivity of the Mabuchi functional also holds for general polarized varieties admitting a metric of constant scalar curvature and having discrete reduced automorphism group. Thus, if (X, L) admits a constant scalar curvature metric and $\mathrm{Aut}(X, L)/\mathbb{C}^*$ is discrete, then (X, L) is uniformly K-stable. The converse statement is not currently known in general, but see below for the Fano case.

Next we study coercivity of the Mabuchi functional when restricted to the space of Bergman metrics. For any $m \geq 1$ such that mL is very ample, let $\mathcal{H}_m$ be the space of Fubini-Study type metrics on L, induced by the embedding of $X \hookrightarrow \mathbb{P}^{N_m}$ via mL.

Theorem C. *Fix m such that (X, mL) is linearly normal, and $\delta > 0$. Then the following conditions are equivalent:*

(i) *there exists $C > 0$ such that $M \geq \delta J - C$ on $\mathcal{H}_m$.*
(ii) $\mathrm{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) \geq \delta J^{\mathrm{NA}}(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ *for all 1-parameter subgroups λ of* $\mathrm{GL}(N_m, \mathbb{C})$*;*
(iii) $M^{\mathrm{NA}}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) \geq \delta J^{\mathrm{NA}}(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ *for all 1-parameter subgroups λ of* $\mathrm{GL}(N_m, \mathbb{C})$.

Here $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ is the test configuration defined by λ.

Here the equivalence of (ii) and (iii) stems from the close relation between the Donaldson-Futaki invariant and the non-Archimedean Mabuchi functional. In view of Theorem A, the equivalence between (i) and (iii) can be viewed as a generalization of the Hilbert-Mumford criterion. The proof uses in a crucial way a deep result of Paul [Pau12], which states that the restrictions to $\mathcal{H}_m$ of the Mabuchi functional and the J-functionals have log norm singularities (see §4).

Since every ample test configuration arises as a 1-parameter subgroup λ of $\mathrm{GL}(N_m, \mathbb{C})$ for some m, Theorem C implies

Corollary D. *A polarized manifold (X, L) is uniformly K-stable iff there exist $\delta > 0$ and a sequence $C_m > 0$ such that $M \geq \delta J - C_m$ on $\mathcal{H}_m$ for all sufficiently divisible m.*

Following Paul and Tian [PT06, PT09], we say that (X, mL) is *CM-stable* when there exist $C, \delta > 0$ such that $M \geq \delta J - C$ on $\mathcal{H}_m$.

Corollary E. *If (X, L) is uniformly K-stable, then (X, mL) is CM-stable for any sufficiently divisible positive integer m. Hence the reduced automorphism group is finite.*

Here the last statement follows from a result by Paul [Pau13, Corollary 1.1].

Let us now comment on the relation of uniform K-stability to the existence of Kähler-Einstein metrics on Fano manifolds. In [CDS15], Chen, Donaldson and Sun proved that a Fano manifold X admits a Kähler-Einstein metric iff it is K-polystable (see also [Tia15]). Since then, several new proofs have a appeared. Datar and Székelyhidi [DSz15] gave a proof of an equivariant version of the conjecture, using Aubin's original continuity method. Chen, Sun and Wang [CSW15] gave a proof using the Kähler-Ricci flow.

In [BBJ15], Berman and the first and last authors of the current paper used a variational method to prove a slightly different statement: in the absence of vector fields, the existence of a Kähler-Einstein metric is equivalent to uniform K-stability. In fact, the direct implication uses Corollary B above.

In §5 we outline a different proof of the fact that a uniform K-stable Fano manifold admits a Kähler-Einstein metric. Our method, which largely follows ideas of Tian, relies on Székelyhidi's partial C^0-estimates [Szé13] along the Aubin continuity path together with Corollary D.

As noted above, uniform K-stability implies that the reduced automorphism group of (X, L) is discrete. In the presence of vector fields, one may hope that there is a notion of uniform K-polystability. We hope to address this in future work.

There have been several important developments since a first draft of the current paper was circulated. First, Z. Sjöström Dyrefeld [SD16] proved a transcendental version of Theorem A. Second, as mentioned above, it was proved in [BBJ15] that in the case of a Fano manifold without holomorphic vector fields, uniform K-stability is equivalent to coercivity of the Mabuchi functional and hence to the existence of a Kähler-Einstein metric. Finally, the results in this paper were used in [BDL16] to prove that an arbitrary polarized pair (X, L) admitting a csc metric must be K-polystable.

The organization of the paper is as follows. In the first section, we review several classical energy functionals in Kähler geometry and their interpretation as metrics on a suitable Deligne pairings. Then, in §2, we recall some non-Archimedean notions from [BHJ16]. Specifically, a non-Archimedean metric is an equivalence class of test configurations, and the non-Archimedean analogues of the energy functionals in §1 are defined using intersection numbers. In §3 we prove Theorem A relating the classical and non-Archimedean functionals via subgeodesic rays. Section 4 is devoted to the relation between uniform K-stability and CM-stability. In particular, we prove Theorem C and Corollaries D and E. Finally, in §5, we show how to use Székelyhidi's partial C^0-estimates along the Aubin continuity path together with CM-stability to prove that a uniformly K-stable Fano manifold admits a Kähler-Einstein metric.

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1. Deligne pairings and energy functionals

In this section we recall the definition and main properties of the Deligne pairing, as well as its relation to classical functionals in Kähler geometry.

1.1. **Metrics on line bundles.** We use additive notation for line bundles and metrics, as follows. If σ is a (holomorphic) section of a line bundle L on a complex analytic space X, then $\log|\sigma|$ stands for the corresponding (possibly singular) metric on L. For any metric ϕ on L, $\log|\sigma| - \phi$ is therefore a function, and

$$|\sigma|_\phi := |\sigma| e^{-\phi} = \exp(\log|\sigma| - \phi)$$

is the length of σ in the metric ϕ.

If, for $i = 1, 2$, ϕ_i is a metric on a line bundle L_i on X and $a_i \in \mathbb{Z}$, then $a_1\phi_1 + a_2\phi_2$ is a metric on $a_1L_1 + a_2L_2$. A metric on the trivial line bundle will be identified with a function on X.

We normalize the operator d^c so that $dd^c = \frac{i}{\pi}\partial\bar\partial$, and set (somewhat abusively)

$$dd^c\phi := -dd^c \log|\sigma|_\phi$$

for any local trivializing section σ of L. The globally defined $(1,1)$-form (or current) $dd^c\phi$ is then the curvature of ϕ, normalized so that it represents the (integral) first Chern class of L.

If X is a complex manifold, there is a natural bijection between smooth metrics on the canonical bundle K_X and (smooth, positive) volume forms on X, which associates to a smooth metric ϕ on K_X the volume form $e^{2\phi}$ locally defined by

$$e^{2\phi} := \frac{i^{n^2}\sigma \wedge \bar\sigma}{|\sigma|^2_\phi}$$

for any local section σ of K_X.

If ω is a positive $(1,1)$-form on X and $n = \dim X$, then ω^n is a volume form, so $\frac{1}{2}\log\omega^n$ is a metric on K_X in our notation. The *Ricci form* of ω is defined as the curvature $\operatorname{Ric}\omega := -dd^c\frac{1}{2}\log\omega^n$ of ω of this metric; it is thus a smooth $(1,1)$-form in the cohomology class $c_1(X)$ of $-K_X$.

If ϕ is a smooth positive metric on a line bundle L on X, we denote by $S_\phi \in C^\infty(X)$ the *scalar curvature* of the Kähler form $dd^c\phi$, which satisfies

$$S_\phi (dd^c\phi)^n = n \operatorname{Ric}(dd^c\phi) \wedge (dd^c\phi)^{n-1}. \tag{1.1}$$

1.2. **Deligne pairings.** While the construction below works in greater generality [Elk89, Zha96, MG00], we will restrict ourselves to the following setting. Let $\pi\colon Y \to T$ be a flat, projective morphism between smooth complex algebraic varieties, and denote by n its relative dimension. Given line bundles $L_0, \dots, L_n$ on Y, we may consider the intersection product

$$L_0 \cdot \ldots \cdot L_n \cdot [Y] \in \mathrm{CH}_{\dim Y - (n+1)}(Y) = \mathrm{CH}_{\dim T - 1}(Y).$$

Its push-forward belongs to $\mathrm{CH}_{\dim T-1}(T) = \operatorname{Pic}(T)$ since T is smooth, and hence defines an *isomorphism class* of line bundle on T. The *Deligne pairing* of $L_0, \dots, L_n$ selects in a canonical way a specific representative of this isomorphism class, denoted by

$$\langle L_0, \dots, L_n \rangle_{Y/T}.$$

The pairing is functorial, multilinear, and commutes with base change. It further satisfies the following key inductive property: if Z_0 is a non-singular divisor in Y, flat over T and defined by a section $\sigma_0 \in H^0(Y, L_0)$, then we have a canonical identification

$$\langle L_0, \dots, L_n \rangle_{Y/T} = \langle L_1|_{Z_0}, \dots, L_n|_{Z_0} \rangle_{Z_0/T} \tag{1.2}$$

For $n = 0$, $\langle L_0 \rangle_{Y/T}$ coincides with the norm of L_0 with respect to the finite flat morphism $Y \to T$. These properties uniquely characterize the Deligne pairing. Indeed, writing each L_i as a difference of very ample line bundles, multilinearity reduces us to the case where L_i is very ample. We may thus find non-singular divisors $Z_i \in |L_i|$ with $\bigcap_{i \in I} Z_i$ non-singular and flat over T for each set I of indices, and we get

$$\langle L_0, \dots, L_n \rangle_{Y/T} = \langle L_n|_{Z_0 \cap \dots \cap Z_{n-1}} \rangle_{Z_0 \cap \dots \cap Z_{n-1}/T}.$$

1.3. **Metrics on Deligne pairings.** We use [Elk90, Zha96, Mor99] as references. Given a smooth metric ϕ_j on each L_j, the Deligne pairing $\langle L_0, \dots, L_n \rangle_{Y/T}$ can be endowed with a continuous metric

$$\langle \phi_0, \dots, \phi_n \rangle_{Y/T},$$

smooth over the smooth locus of π, the construction being functorial, multilinear, and commuting with base change. It is basically constructed by requiring that

$$\langle \phi_0, \dots, \phi_n \rangle_{Y/T} = \langle \phi_1|_{Z_0}, \dots, \phi_n|_{Z_0} \rangle_{Z_0/T} - \int_{Y/T} \log |\sigma_0|_{\phi_0} dd^c \phi_1 \wedge \dots \wedge dd^c \phi_n \tag{1.3}$$

in the notation of (1.2), with $\int_{Y/T}$ denoting fiber integration, i.e. the push-forward by π as a current. By induction, the continuity of the metric $\langle \phi_0, \dots, \phi_n \rangle$ reduces to that of $\int_{Y/T} \log |\sigma_0|_{\phi_0} dd^c \phi_1 \wedge \dots \wedge dd^c \phi_n$, and thus follows from [Stol66, Theorem 4.9].

Remark 1.1. *As explained in* [Elk90, I.1], *arguing by induction, the key point in checking that* (1.3) *is well-defined is the following symmetry property: if* $\sigma_1 \in H^0(Y, L_1)$ *is a section with divisor* Z_1 *such that both* Z_1 *and* $Z_0 \cap Z_1$ *are non-singular and flat over* T*, then*

$$\int_{Y/T} \log |\sigma_0|_{\phi_0} dd^c \phi_1 \wedge \alpha + \int_{Z_0/T} \log |\sigma_1|_{\phi_1} \alpha$$
$$= \int_{Y/T} \log |\sigma_1|_{\phi_1} dd^c \phi_0 \wedge \alpha + \int_{Z_1/T} \log |\sigma_0|_{\phi_0} \alpha$$

with $\alpha = dd^c \phi_2 \wedge \dots \wedge dd^c \phi_n$*. By the Lelong-Poincaré formula, the above equality reduces to*

$$\pi_* \left(\log |\sigma_0|_{\phi_0} dd^c \log |\sigma_1|_{\phi_1} \wedge \alpha \right) = \pi_* \left(\log |\sigma_1|_{\phi_1} dd^c \log |\sigma_0|_{\phi_0} \wedge \alpha \right),$$

which holds by Stokes' formula applied to a monotone regularization of the quasi-psh functions $\log |\sigma_i|_{\phi_i}$*.*

Metrics on Deligne pairings satisfy the following two crucial properties, which are direct consequences of (1.3).

(i) The curvature current of $\langle \phi_0, \dots, \phi_n \rangle_{Y/T}$ satisfies

$$dd^c \langle \phi_0, \dots, \phi_n \rangle_{Y/T} = \int_{Y/T} dd^c \phi_0 \wedge \dots \wedge dd^c \phi_n, \tag{1.4}$$

where again $\int_{Y/T}$ denotes fiber integration.

(ii) Given another smooth metric ϕ_0' on L_0, we have the change of metric formula

$$\langle \phi_0', \phi_1, \dots, \phi_n \rangle_{Y/T} - \langle \phi_0, \phi_1, \dots, \phi_n \rangle_{Y/T} = \int_{Y/T} (\phi_0' - \phi_0) dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n. \tag{1.5}$$

1.4. **Energy functionals.** Let (X, L) be a polarized manifold, i.e. a smooth projective complex variety X with an ample line bundle L, and set $n := \dim X$,

$$V := (L^n) \quad \text{and} \quad \bar{S} := -nV^{-1}(K_X \cdot L^{n-1}).$$

Denote by $\mathcal{H}$ the set of smooth positive metrics ϕ on L. For $\phi \in \mathcal{H}$ set $\mathrm{MA}(\phi) := V^{-1}(dd^c\phi)^n$. Then $\mathrm{MA}(\phi)$ is a probability measure equivalent to Lebesgue measure, and $\int_X S_\phi \, \mathrm{MA}(\phi) = \bar{S}$ by (1.1).

We recall the following functionals in Kähler geometry. Fix a reference metric $\phi_{\mathrm{ref}} \in \mathcal{H}$. Our notation largely follows [BBGZ13, BBEGZ11].

(i) The *Monge-Ampère energy functional* is given by

$$E(\phi) = \frac{1}{n+1} \sum_{j=0}^{n} V^{-1} \int_X (\phi - \phi_{\mathrm{ref}})(dd^c\phi)^j \wedge (dd^c\phi_{\mathrm{ref}})^{n-j}. \tag{1.6}$$

(ii) The *J-functional* is a translation invariant version of E, defined as

$$J(\phi) := \int_X (\phi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_{\mathrm{ref}}) - E(\phi). \tag{1.7}$$

The closely related *I-functional* is defined by

$$I(\phi) := \int_X (\phi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_{\mathrm{ref}}) - \int_X (\phi - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi). \tag{1.8}$$

(iii) For any closed $(1,1)$-form θ, the *θ-twisted Monge-Ampère energy* is given by

$$E_\theta(\phi) = \sum_{j=0}^{n-1} V^{-1} \int_X (\phi - \phi_{\mathrm{ref}})(dd^c\phi)^j \wedge (dd^c\phi_{\mathrm{ref}})^{n-1-j} \wedge \theta. \tag{1.9}$$

Taking $\theta := -\operatorname{Ric}(dd^c\phi_{\mathrm{ref}})$, we obtain the *Ricci energy* $R := E_{-\operatorname{Ric}(dd^c\phi_{\mathrm{ref}})}$.

(iv) The *entropy* of $\phi \in \mathcal{H}$ is defined as

$$H(\phi) := \tfrac{1}{2} \int_X \log \left[\frac{\mathrm{MA}(\phi)}{\mathrm{MA}(\phi_{\mathrm{ref}})} \right] \mathrm{MA}(\phi), \tag{1.10}$$

that is, (half) the relative entropy of the probability measure $\mathrm{MA}(\phi)$ with respect to $\mathrm{MA}(\phi_{\mathrm{ref}})$. We have $H(\phi) \geq 0$, with equality iff $\phi - \phi_{\mathrm{ref}}$ is constant.

(v) The *Mabuchi functional* (or K-energy) can now be defined via the Chen-Tian formula [Che00] (see also [BB14, Proposition 3.1]) as

$$M(\phi) = H(\phi) + R(\phi) + \bar{S}E(\phi). \tag{1.11}$$

These functionals vanish at ϕ_{ref} and satisfy the variational formulas:

$$\begin{aligned}
\delta E(\phi) &= \mathrm{MA}(\phi) = V^{-1}(dd^c\phi)^n\\
\delta E_\theta(\phi) &= nV^{-1}(dd^c\phi)^{n-1}\wedge\theta\\
\delta R(\phi) &= -nV^{-1}(dd^c\phi)^{n-1}\wedge \mathrm{Ric}(dd^c\phi_{\mathrm{ref}})\\
\delta H(\phi) &= nV^{-1}(dd^c\phi)^{n-1}\wedge(\mathrm{Ric}(dd^c\phi_{\mathrm{ref}})-\mathrm{Ric}(dd^c\phi))\\
\delta M(\phi) &= (\bar S - S_\phi)\,\mathrm{MA}(\phi)
\end{aligned}$$

In particular, the critical points of M are csc metrics.

The functionals I, J and $I-J$ are comparable in the sense that

$$\frac{1}{n}J \le I-J \le nJ \tag{1.12}$$

on $\mathcal{H}$. For $\phi\in\mathcal{H}$ we have $J(\phi)\ge 0$, with equality iff $\phi-\phi_{\mathrm{ref}}$ is constant. These properties are thus also shared by I and $I-J$.

The functionals H, I, J, M are translation invariant in the sense that $H(\phi+c)=H(\phi)$ for $c\in\mathbb{R}$. For E and R we instead have $E(\phi+c)=E(\phi)+c$ and $R(\phi+c)=R(\phi)-\bar S c$, respectively.

1.5. **Energy functionals as Deligne pairings.** The functionals above can be expressed using Deligne pairings, an observation going back at least to [PS04]. Any metric $\phi\in\mathcal{H}$ induces a smooth metric $\frac{1}{2}\log\mathrm{MA}(\phi)$ on K_X. The following identities are now easy consequences of the change of metric formula (1.5).

Lemma 1.2. *For any $\phi\in\mathcal{H}$ we have*

$$\begin{aligned}
(n+1)VE(\phi) &= \langle\phi^{n+1}\rangle_X - \langle\phi_{\mathrm{ref}}^{n+1}\rangle_X;\\
VJ(\phi) &= \langle\phi,\phi_{\mathrm{ref}}^n\rangle_X - \langle\phi_{\mathrm{ref}}^{n+1}\rangle_X - \frac{1}{n+1}\left[\langle\phi^{n+1}\rangle_X - \langle\phi_{\mathrm{ref}}^{n+1}\rangle_X\right];\\
VI(\phi) &= \langle\phi-\phi_{\mathrm{ref}},\phi_{\mathrm{ref}}^n\rangle_X - \langle\phi-\phi_{\mathrm{ref}},\phi^n\rangle_X;\\
VR(\phi) &= \langle\tfrac12\log\mathrm{MA}(\phi_{\mathrm{ref}}),\phi^n\rangle_X - \langle\tfrac12\log\mathrm{MA}(\phi_{\mathrm{ref}}),\phi_{\mathrm{ref}}^n\rangle_X;\\
VH(\phi) &= \langle\tfrac12\log\mathrm{MA}(\phi),\phi^n\rangle_X - \langle\tfrac12\log\mathrm{MA}(\phi_{\mathrm{ref}}),\phi^n\rangle_X;\\
VM(\phi) &= \langle\tfrac12\log\mathrm{MA}(\phi),\phi^n\rangle_X - \langle\tfrac12\log\mathrm{MA}(\phi_{\mathrm{ref}}),\phi_{\mathrm{ref}}^n\rangle_X\\
&\quad + \frac{\bar S}{n+1}\left[\langle\phi^{n+1}\rangle_X - \langle\phi_{\mathrm{ref}}^{n+1}\rangle_X\right],
\end{aligned}$$

where $\langle\ \rangle_X$ denotes the Deligne pairing with respect to the constant map $X\to\{\mathrm{pt}\}$.

Remark 1.3. *The formulas above make it evident that instead of fixing a reference metric $\phi_{\mathrm{ref}}\in\mathcal{H}$, we could view E, $H+R$ and M as metrics on suitable multiples of the complex lines $\langle L^{n+1}\rangle_X$, $\langle K_X,L^n\rangle_X$, and $(n+1)\langle K_X,L^n\rangle_X+\bar S\langle L^{n+1}\rangle_X$, respectively.*

1.6. **The Ding functional.** Now suppose X is a Fano manifold, that is, $L:=-K_X$ is ample. Any metric ϕ on L then induces a positive volume form $e^{-2\phi}$ on X. The *Ding functional* [Din88] on $\mathcal{H}$ is defined by

$$D(\phi) = L(\phi) - E(\phi),$$

where

$$L(\phi) = -\tfrac{1}{2} \log \int_X e^{-2\phi}.$$

This functional, first introduced in [Din88], has proven an extremely useful tool for the study of the existence of Kähler-Einstein metrics, which are realized as the critical points of D, see e.g. [Berm15, BBJ15].

2. Test configurations as non-Archimedean metrics

In this section we recall some notions and results from [BHJ16]. Let X be a smooth projective complex variety and L a line bundle on X.

2.1. **Test configurations.**

As in [BHJ16] we adopt the following flexible terminology for test configurations.

Definition 2.1. *A test configuration $\mathcal{X}$ for X consists of the following data:*

(i) *a flat, projective morphism of schemes $\pi\colon \mathcal{X} \to \mathbb{C}$;*
(ii) *a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{C}$;*
(iii) *an isomorphism $\mathcal{X}_1 \simeq X$.*

We denote by τ the coordinate in $\mathbb{C}$, and by $\mathcal{X}_\tau$ the fiber over τ.

Since X is in particular a variety, i.e. reduced and irreducible, so is $\mathcal{X}$ (cf. [BHJ16, Proposition 2.6]). If $\mathcal{X}, \mathcal{X}'$ are test configurations for X, then there is a unique $\mathbb{C}^*$-equivariant birational map $\mathcal{X}' \dashrightarrow \mathcal{X}$ compatible with the isomorphism in (iii). We say that $\mathcal{X}'$ *dominates* $\mathcal{X}$ if this birational map is a morphism; when it is an isomorphism we somewhat abusively identify $\mathcal{X}$ and $\mathcal{X}'$. Any test configuration $\mathcal{X}$ is dominated by its *normalization* $\widetilde{\mathcal{X}}$.

Definition 2.2. *A test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) consists of a test configuration $\mathcal{X}$ for X, together with the following additional data:*

(iv) *a $\mathbb{C}^*$-linearized $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$;*
(v) *an isomorphism $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L)$.*

A *pull-back* of a test configuration $(\mathcal{X}, \mathcal{L})$ is a test configuration $(\mathcal{X}', \mathcal{L}')$ where $\mathcal{X}'$ dominates $\mathcal{X}$ and $\mathcal{L}'$ is the pull-back of $\mathcal{L}$. In particular, the *normalization* $(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}})$ is the pull-back of $(\mathcal{X}, \mathcal{L})$ with $\nu\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ the normalization morphism.

A test configuration $(\mathcal{X}, \mathcal{L})$ is *trivial* if $\mathcal{X} = X \times \mathbb{C}$ with $\mathbb{C}^*$ acting trivially on X, which implies that $(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0) = (X, L) \times \mathbb{C}$ for some constant $c \in \mathbb{Q}$. A test configuration for (X, L) is *almost trivial* if its normalization is trivial.

We say that $(\mathcal{X}, \mathcal{L})$ is ample (resp. semiample, resp. nef) when $\mathcal{L}$ is relatively ample (resp. relatively semiample, resp. nef). The pullback of a semiample (resp. nef) test configuration is semiample (resp. nef).

If L is ample, then for every semiample test configuration $(\mathcal{X}, \mathcal{L})$ there exists a unique ample test configuration $(\mathcal{X}_{\mathrm{amp}}, \mathcal{L}_{\mathrm{amp}})$ that is dominated by $(\mathcal{X}, \mathcal{L})$ and satisfies $\mu_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{\mathrm{amp}}}$, where $\mu\colon \mathcal{X} \to \mathcal{X}_{\mathrm{amp}}$ is the canonical morphism; see [BHJ16, Proposition 2.17].

Note that while $\mathcal{X}$ can often be chosen smooth, $\mathcal{X}_{\mathrm{amp}}$ will not be smooth, in general.

2.2. **One-parameter subgroups.**

Suppose L is ample. Ample test configurations are then essentially equivalent to one-parameter degenerations of X. See [BHJ16, §2.3] for details on what follows. Fix $m \geq 1$ such that mL is very ample, and consider the corresponding closed embedding $X \hookrightarrow \mathbb{P}^{N_m - 1}$ with $N_m := h^0(X, mL)$. Then every 1-parameter subgroup (1-PS for short) $\lambda\colon \mathbb{C}^* \to \mathrm{GL}(N_m, \mathbb{C})$ induces an ample test configuration $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ for (X, L). By definition, $\mathcal{X}_\lambda$ is the Zariski closure in $\mathbb{P}V \times \mathbb{C}$ of the image of the closed embedding $X \times \mathbb{C}^* \hookrightarrow \mathbb{P}V \times \mathbb{C}^*$ mapping (x, τ) to $(\lambda(\tau)x, \tau)$. Note that λ is trivial iff $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ is, while $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ is a product iff λ preserves X. Note that $\mathcal{X}_\lambda$ is not normal in general.

In fact, every ample test configuration may be obtained as above, see e.g. [BHJ16, §2.3]. Using one-parameter subgroups, we can produce test configurations that are almost trivial but not trivial, as observed in [LX14, Remark 5]. See [BHJ16, Proposition 2.12] for an elementary proof of the following result.

Proposition 2.3. *For every m divisible enough, there exists a 1-PS $\lambda\colon \mathbb{C}^* \to \mathrm{GL}(N_m, \mathbb{C})$ such that the test configuration $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ is nontrivial but almost trivial.*

2.3. **Valuations and log discrepancies.** By a *valuation on X* we mean a real-valued valuation v on the function field $\mathbb{C}(X)$ (trivial on the ground field $\mathbb{C}$). The *trivial valuation* v_{triv} is defined by $v_{\mathrm{triv}}(f) = 0$ for $f \in \mathbb{C}(X)^*$. A valuation v is *divisorial* if it is of the form $v = c \operatorname{ord}_F$, where $c \in \mathbb{Q}_{>0}$ and F is a prime divisor on a projective normal variety admitting a birational morphism onto X. We denote by X^{div} the set of valuations on X that are either divisorial or trivial, and equip it with the weakest topology such that $v \mapsto v(f)$ is continuous for every $f \in \mathbb{C}(X)^*$.

The *log discrepancy* $A_X(v)$ of a valuation in X^{div} is defined as follows. First we set $A_X(v_{\mathrm{triv}}) = 0$. For $v = c \operatorname{ord}_F$ a (nontrivial) divisorial valuation as above, we set $A_X = c(1 + \operatorname{ord}_F(K_{Y/X}))$, where $K_{Y/X}$ is the relative canonical (Weil) divisor.

Now consider a normal test configuration $\mathcal{X}$ of X. Since $\mathbb{C}(\mathcal{X}) \simeq \mathbb{C}(X)(\tau)$, any valuation w on $\mathcal{X}$ restricts to a valuation $r(w)$ on X. Let E be an irreducible component of the central fiber $\mathcal{X}_0 = \sum b_E E$. Then ord_E is a $\mathbb{C}^*$-invariant divisorial valuation on $\mathbb{C}(\mathcal{X})$ and satisfies $\operatorname{ord}_E(t) = b_E$. If we set $v_E := r(b_E^{-1} \operatorname{ord}_E)$, then v_E is a valuation in X^{div}. Conversely, every valuation $v \in X^{\mathrm{div}}$ has a unique $\mathbb{C}^*$-invariant preimage w under r normalized by $w(\tau) = 1$, and w is associated to an irreducible component of the central fiber of some test configuration for X, cf. [BHJ16, Theorem 4.6].

Note that ord_E is a divisorial valuation on $X \times \mathbb{C}$. By [BHJ16, Proposition 4.11], the log discrepancies of ord_E and v_E are related as follows: $A_{X\times\mathbb{C}}(\operatorname{ord}_E) = b_E(1 + A_X(v_E))$.

2.4. **Compactifications.** For some purposes it is convenient to compactify test configurations. The following notion provides a canonical way of doing so.

Definition 2.4. *The* compactification $\bar{\mathcal{X}}$ *of a test configuration $\mathcal{X}$ for X is defined by gluing together $\mathcal{X}$ and $X \times (\mathbb{P}^1 \setminus \{0\})$ along their respective open subsets $\mathcal{X} \setminus \mathcal{X}_0$ and $X \times (\mathbb{C} \setminus \{0\})$, which are identified using the canonical $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{C} \setminus \{0\})$.*

The compactification comes with a $\mathbb{C}^*$-equivariant flat morphism $\bar{\mathcal{X}} \to \mathbb{P}^1$, still denoted by π. By construction, $\pi^{-1}(\mathbb{P}^1 \setminus \{0\})$ is $\mathbb{C}^*$-equivariantly isomorphic to $X \times (\mathbb{P}^1 \setminus \{0\})$ over $\mathbb{P}^1 \setminus \{0\}$.

Similarly, a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) admits a compactification $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$, where $\bar{\mathcal{L}}$ is a $\mathbb{C}^*$-linearized $\mathbb{Q}$-line bundle on $\bar{\mathcal{X}}$. Note that $\bar{\mathcal{L}}$ is relatively (semi)ample iff $\mathcal{L}$ is.

2.5. **Non-Archimedean metrics.** Following [BHJ16, §6] (see also [BJ16a]) we introduce:

Definition 2.5. *Two test configurations $(\mathcal{X}_1, \mathcal{L}_1)$, $(\mathcal{X}_2, \mathcal{L}_2)$ for (X, L) are* equivalent *if there exists a test configuration $(\mathcal{X}_3, \mathcal{L}_3)$ that is a pull-back of both $(\mathcal{X}_1, \mathcal{L}_1)$ and $(\mathcal{X}_2, \mathcal{L}_2)$. An equivalence class is called a* non-Archimedean metric *on L, and is denoted by ϕ. We denote by ϕ_{triv} the equivalence class of the trivial test configuration $(X, L) \times \mathbb{C}$.*

A non-Archimedean metric ϕ is called *semipositive* if some (or, equivalently, any) representative $(\mathcal{X}, \mathcal{L})$ of ϕ is nef. Note that this implies that L is nef.

When L is ample, we say that a non-Archimedean metric ϕ on L is *positive* if some (or, equivalently, any) representative $(\mathcal{X}, \mathcal{L})$ of ϕ is semiample. We denote by $\mathcal{H}^{\mathrm{NA}}$ the set of all non-Archimedean positive metrics on L. By [BHJ16, Lemma 6.3], every $\phi \in \mathcal{H}^{\mathrm{NA}}$ is represented by a unique normal, ample test configuration.

The set of non-Archimedean metrics on a line bundle L admits two natural operations:

(i) a *translation action* of $\mathbb{Q}$, denoted by $\phi \mapsto \phi + c$, and induced by $(\mathcal{X}, \mathcal{L}) \mapsto (\mathcal{X}, \mathcal{L} + c\mathcal{X}_0)$;

(ii) a *scaling action* of the semigroup $\mathbb{N}^*$ of positive integers, denoted by $\phi \mapsto \phi_d$ and induced by the base change of $(\mathcal{X}, \mathcal{L})$ by $\tau \mapsto \tau^d$.

When L is ample (resp. nef) these operations preserve the set of positive (resp. semipositive) metrics. The trivial metric ϕ_{triv} is fixed by the scaling action.

As in §1.1 we use additive notation for non-Archimedean metrics. A non-Archimedean metric on $\mathcal{O}_X$ induces a bounded (and continuous) function on X^{div}.

Remark 2.6. *As explained in* [BHJ16, §6.5]*, a non-Archimedean metric ϕ on L as defined above can be viewed as a metric on the Berkovich analytification* [Berk90] *of L with respect to the trivial absolute value on the ground field $\mathbb{C}$. See also* [BJ16a] *for a more systematic analysis, itself building upon* [BFJ16, BFJ15a]*.*

2.6. **Intersection numbers and Monge-Ampère measures.** We can define the intersection number $(\phi_0 \cdot \ldots \cdot \phi_n)$ of non-Archimedean metrics $\phi_0, \ldots, \phi_n$ on line bundles $L_0, \ldots, L_n$ on X as follows. Pick representatives $(\mathcal{X}, \mathcal{L}_i)$ of ϕ_i, $0 \le i \le n$, with the same test configuration $\mathcal{X}$ for X and set

$$(\phi_0 \cdot \ldots \cdot \phi_n) := (\bar{\mathcal{L}}_0 \cdot \ldots \cdot \bar{\mathcal{L}}_n),$$

where $(\bar{\mathcal{X}}, \bar{\mathcal{L}}_i)$ is the compactification of $(\mathcal{X}, \mathcal{L}_i)$. It follows from the projection formula that this is well-defined, independent of the choice of $\mathcal{L}_i$. Note that $(\phi_{\mathrm{triv}}^{n+1}) = 0$. When $L_0 = \mathcal{O}_X$, so that $\mathcal{L}_0 = \mathcal{O}_X(D)$ for a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D = \sum r_E E$ supported on $\mathcal{X}_0$, we can compute the intersection number as $(\phi_0 \cdot \ldots \cdot \phi_n) = \sum_E r_E (\mathcal{L}_1|_E \cdot \ldots \cdot \mathcal{L}_n|_E)$.

An n-tuple $(\phi_1, \ldots, \phi_n)$ of non-Archimedean metrics on line bundles $L_1, \ldots L_n$ induces a signed finite atomic *mixed Monge-Ampère measure* on X^{div} as follows. Pick representatives $(\mathcal{X}, \mathcal{L}_i)$ of ϕ_i, $1 \le i \le n$, with the same test configuration $\mathcal{X}$ for X and set

$$\mathrm{MA}^{\mathrm{NA}}(\phi_1, \ldots, \phi_n) = V^{-1} \sum_E b_E (\mathcal{L}_1|_E \cdot \ldots \cdot \mathcal{L}_n|_E) \delta_{v_E},$$

where E ranges over irreducible components of $\mathcal{X}_0 = \sum_E b_E E$, and $v_E = r(b_E^{-1} \operatorname{ord}_E) \in X^{\mathrm{div}}$. When the ϕ_i are semipositive, the mixed Monge-Ampère measure is a probability measure. As in the complex case, we also write $\mathrm{MA}^{\mathrm{NA}}(\phi)$ for $\mathrm{MA}^{\mathrm{NA}}(\phi, \ldots, \phi)$.

2.7. **Functionals on non-Archimedean metrics.** Following [BHJ16, §7] we define non-Archimedean analogues of the functionals considered in §1.4. Fix a line bundle L.

Definition 2.7. *Let W be a set of non-Archimedean metrics on L that is closed under translation and scaling. A functional $F \colon W \to \mathbb{R}$ is*

(i) homogeneous *if $F(\phi_d) = dF(\phi)$ for $\phi \in W$ and $d \in \mathbb{N}^*$;*

(ii) translation invariant *if $F(\phi + c) = F(\phi)$ for $\phi \in W$ and $c \in \mathbb{Q}$.*

When L is ample, a functional F on $\mathcal{H}^{\mathrm{NA}}$ may be viewed as a function $F(\mathcal{X}, \mathcal{L})$ on the set of all semiample test configurations $(\mathcal{X}, \mathcal{L})$ that is invariant under pull-back, i.e. $F(\mathcal{X}', \mathcal{L}') = F(\mathcal{X}, \mathcal{L})$ whenever $(\mathcal{X}', \mathcal{L}')$ is a pull-back of a $(\mathcal{X}, \mathcal{L})$ (and, in particular, invariant under normalization). Homogeneity amounts to $F(\mathcal{X}_d, \mathcal{L}_d) = d\, F(\mathcal{X}, \mathcal{L})$ for all $d \in \mathbb{N}^*$, and translation invariance to $F(\mathcal{X}, \mathcal{L}) = F(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0)$ for all $c \in \mathbb{Q}$.

For each non-Archimedean metric ϕ on L, choose a normal representative $(\mathcal{X}, \mathcal{L})$ that dominates $X \times \mathbb{C}$ via $\rho\colon \mathcal{X} \to X \times \mathbb{C}$. Then $\mathcal{L} = \rho^*(L \times \mathbb{C}) + D$ for a uniquely determined $\mathbb{Q}$-Cartier divisor D supported on $\mathcal{X}_0$. Write $\mathcal{X}_0 = \sum_E b_E E$ and let $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ be the compactification of $(\mathcal{X}, \mathcal{L})$.

In this notation, we may describe our list of non-Archimedean functionals as follows. Assume L is big and nef. Let ϕ_{triv} and ψ_{triv} be the trivial metrics on L and K_X, respectively.

(i) The *non-Archimedean Monge-Ampère energy* of ϕ is

$$\begin{aligned} E^{\mathrm{NA}}(\phi) :&= \frac{(\phi^{n+1})}{(n+1)V} \\ &= \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)V}. \end{aligned}$$

(ii) The *non-Archimedean I-functional* and *J-functional* are given by

$$\begin{aligned} I^{\mathrm{NA}}(\phi) :&= V^{-1}(\phi \cdot \phi_{\mathrm{triv}}^n) - V^{-1}((\phi - \phi_{\mathrm{triv}}) \cdot \phi^n) \\ &= \frac{1}{V}(\bar{\mathcal{L}} \cdot (\rho^*(L \times \mathbb{P}^1)^n) - \frac{1}{V}(D \cdot \bar{\mathcal{L}}^n). \end{aligned}$$

and

$$\begin{aligned} J^{\mathrm{NA}}(\phi) :&= V^{-1}(\phi \cdot \phi_{\mathrm{triv}}^n) - E^{\mathrm{NA}}(\phi) \\ &= \frac{1}{V}(\bar{\mathcal{L}} \cdot (\rho^*(L \times \mathbb{P}^1)^n) - \frac{1}{(n+1)V}(\bar{\mathcal{L}}^{n+1}). \end{aligned}$$

(iii) The *non-Archimedean Ricci energy* is

$$\begin{aligned} R^{\mathrm{NA}}(\phi) :&= V^{-1}(\psi_{\mathrm{triv}} \cdot \phi^n) \\ &= V^{-1}\left(\rho^* K_{X \times \mathbb{P}^1/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n\right). \end{aligned}$$

(iv) The *non-Archimedean entropy* is

$$\begin{aligned} H^{\mathrm{NA}}(\phi) :&= \int_{X^{\mathrm{div}}} A_X(v)\, \mathrm{MA}^{\mathrm{NA}}(\phi) \\ &= V^{-1}\left(K_{\bar{\mathcal{X}}/X \times \mathbb{P}^1} \cdot \bar{\mathcal{L}}^n\right) - V^{-1}\left((\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}}) \cdot \bar{\mathcal{L}}^n\right). \end{aligned}$$

(v) The *non-Archimedean Mabuchi functional* (or K-energy) is

$$\begin{aligned} M^{\mathrm{NA}}(\phi) :&= H^{\mathrm{NA}}(\phi) + R^{\mathrm{NA}}(\phi) + \bar{S} E^{\mathrm{NA}}(\phi) \\ &= V^{-1}\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n\right) - V^{-1}\left((\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}}) \cdot \bar{\mathcal{L}}^n\right) + \frac{\bar{S}}{(n+1)V}\left(\bar{\mathcal{L}}^{n+1}\right). \end{aligned}$$

Note the resemblance to the formulas in §1.5. All of these functionals are homogeneous. They are also translation invariant, except for E^{NA} and R^{NA} which satisfy

$$E^{\mathrm{NA}}(\phi + c) = E^{\mathrm{NA}}(\phi) + c \quad \text{and} \quad R^{\mathrm{NA}}(\phi + c) = R^{\mathrm{NA}}(\phi) + \bar{S}c \tag{2.1}$$

for all $\phi \in \mathcal{H}^{\mathrm{NA}}$ and $c \in \mathbb{Q}$.

The functionals I^{NA}, J^{NA} and $I^{\mathrm{NA}} - J^{\mathrm{NA}}$ are comparable on semipositive metrics in the same way as (1.12). By [BHJ16, Lemma 7.6], when ϕ is positive, the first term in the definition of J^{NA} satisfies

$$V^{-1}(\phi \cdot \phi_{\mathrm{triv}}^n) = (\phi - \phi_{\mathrm{triv}})(v_{\mathrm{triv}}) = \max_{X^{\mathrm{div}}}(\phi - \phi_{\mathrm{triv}}) = \max_E b_E^{-1} \operatorname{ord}_E(D).$$

Further, $J^{\mathrm{NA}}(\phi) \geq 0$, with equality iff $\phi = \phi_{\mathrm{triv}} + c$ for some $c \in \mathbb{Q}$, and J^{NA} is comparable to both a natural L^1-norm and the minimum norm in the sense of Dervan [Der14a]. For a normal ample test configuration $(\mathcal{X}, \mathcal{L})$ representing $\phi \in \mathcal{H}^{\mathrm{NA}}$ we also denote the J-norm by $J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$.

2.8. **The Donaldson-Futaki invariant.** As explained in [BHJ16], the non-Archimedean Mabuchi functional is closely related to the Donaldson-Futaki invariant. We have

Proposition 2.8. *Assume L is ample. Let $\phi \in \mathcal{H}^{\mathrm{NA}}$ be the class of an ample test configuration $(\mathcal{X}, \mathcal{L})$, and denote by $(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}})$ its normalization, which is thus the unique normal, ample representative of ϕ. Then*

$$M^{\mathrm{NA}}(\phi) = \mathrm{DF}(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}}) - V^{-1}\left((\widetilde{\mathcal{X}}_0 - \widetilde{\mathcal{X}}_{0,\mathrm{red}}) \cdot \tilde{\mathcal{L}}^n\right) \tag{2.2}$$

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \mathrm{DF}(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}}) + 2V^{-1} \sum_E m_E \left(E \cdot \mathcal{L}^n\right), \tag{2.3}$$

where E ranges over the irreducible components of $\mathcal{X}_0$ contained in the singular locus of $\mathcal{X}$ and $m_E \in \mathbb{N}^*$ is the length of $\left(\nu_* \mathcal{O}_{\widetilde{\mathcal{X}}}\right) / \mathcal{O}_{\mathcal{X}}$ at the generic point of E, with $\nu\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ the normalization.

In particular, $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq M^{\mathrm{NA}}(\phi)$, and equality holds iff $\mathcal{X}$ is regular in codimension one and $\mathcal{X}_0$ is generically reduced.

Indeed, (2.2) follows from the discussion in [BHJ16, §7.3].

For a general non-Archimedean metric ϕ on L we can define

$$\begin{aligned}\mathrm{DF}(\phi) &= M(\phi) + V^{-1}\left((\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}}) \cdot \bar{\mathcal{L}}^n\right) \\ &= V^{-1}\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n\right) + \frac{\bar{S}}{(n+1)V}\left(\bar{\mathcal{L}}^{n+1}\right)\end{aligned}$$

for any normal representative $(\mathcal{X}, \mathcal{L})$ of ϕ. Clearly $M^{\mathrm{NA}}(\phi) \leq \mathrm{DF}(\phi)$ when ϕ is semipositive.

2.9. **The non-Archimedean Ding functional.** Suppose X is weakly Fano, that is, $L := -K_X$ is big and nef. In this case, we define the *non-Archimedean Ding functional* on the set of non-Archimedean metrics on L by

$$D^{\mathrm{NA}}(\phi) = L^{\mathrm{NA}}(\phi) - E^{\mathrm{NA}}(\phi),$$

where L^{NA} is defined by

$$L^{\mathrm{NA}}(\phi) = \inf_v \left(A_X(v) + (\phi - \phi_{\mathrm{triv}})(v)\right),$$

the infimum taken over all valuations v on X that are divisorial or trivial. Recall from §2.5 that $\phi - \phi_{\mathrm{triv}}$ is a non-Archimedean metric on $\mathcal{O}_X$ and induces a bounded function on divisorial valuations. Note that $L^{\mathrm{NA}}(\phi + c) = L^{\mathrm{NA}}(\phi) + c$; hence D^{NA} is translation invariant.

We have $D^{\mathrm{NA}} \leq M^{\mathrm{NA}}$ and $D^{\mathrm{NA}} \leq J^{\mathrm{NA}}$, see [BHJ16, Proposition 7.25].

2.10. **Uniform K-stability.** As in [BHJ16] we make the following definition.

Definition 2.9. *A polarized complex manifold (X, L) is* uniformly K-stable *if there exists a constant $\delta > 0$ such that the following equivalent conditions hold.*

(i) $M^{\mathrm{NA}}(\phi) \geq \delta J^{\mathrm{NA}}(\phi)$ *for every* $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$*;*
(ii) $\mathrm{DF}(\phi) \geq \delta J^{\mathrm{NA}}(\phi)$ *for every* $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$*;*
(iii) $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ *for any normal ample test configuration* $(\mathcal{X}, \mathcal{L})$*.*

Here the equivalence between (ii) and (iii) is definitional, and (i) $\Longrightarrow$ (ii) follows immediately from $\mathrm{DF} \leq M^{\mathrm{NA}}$. The implication (ii) $\Longrightarrow$ (i) follows from the homogeneity of M^{NA} together with the fact that $\mathrm{DF}(\phi_d) = M^{\mathrm{NA}}(\phi_d)$ for d sufficiently divisible. See [BHJ16, Proposition 8.2] for details.

The fact that $J^{\mathrm{NA}}(\phi) = 0$ iff $\phi = \phi_{\mathrm{triv}} + c$ implies that uniform K-stability is stronger than K-stability as introduced by [Don02]. Oour notion of uniform K-stability is equivalent to uniform K-stability defined either with respect to the L^1-norm or the minimum norm in the sense of [Der14a], see [BHJ16, ???].

In the Fano case, uniform K-stability is further equivalent to *uniform Ding stability*:

Theorem 2.10. *Assume $L := -K_X$ is ample and fix a number δ with $0 \leq \delta \leq 1$. Then the following conditions are equivalent:*

(i) $M^{\mathrm{NA}} \geq \delta J^{\mathrm{NA}}$ *on* $\mathcal{H}^{\mathrm{NA}}$*;*
(ii) $D^{\mathrm{NA}} \geq \delta J^{\mathrm{NA}}$ *on* $\mathcal{H}^{\mathrm{NA}}$*.*

This is proved in [BHJ16, §8.2] using the Minimal Model Program as in [LX14]. See also [Fuj15, Fuj16].

3. Non-Archimedean limits

In this section we prove Theorem A and Corollary B.

3.1. Rays of metrics and non-Archimedean limits.

For any line bundle L on X, there is a bijection between smooth rays $(\phi^s)_{s>0}$ of metrics on L and S^1-invariant smooth metrics Φ on the pull-back of L to $X \times \Delta^*$, with $\Delta^* = \Delta_1^* \subset \mathbb{C}$ the punctured unit disc. The restriction of Φ to $L|_{X_\tau}$ for $\tau \in \Delta^*$ is given by pullback of $\phi^{\log|\tau|^{-1}}$ under the map $X_\tau \to X$ given by the $\mathbb{C}^*$-action. Similarly, smooth rays $(\phi^s)_{s>s_0}$ correspond to S^1-invariant smooth metrics on the pull-back of L to $X \times \Delta_{r_0}^*$, with $r_0 = e^{-s_0}$.

A *subgeodesic ray* is a ray (ϕ^s) whose corresponding metric Φ is semipositive. Such rays can of course only exist when L is nef.

Definition 3.1. *We say that a smooth ray (ϕ^s) admits a non-Archimedean metric ϕ^{NA} as* non-Archimedean limit *if there exists a test configuration $(\mathcal{X}, \mathcal{L})$ representing ϕ^{NA} such that the metric Φ on $L \times \Delta^*$ extends to a smooth metric on $\mathcal{L}$ over Δ.*

In other words, a non-Archimedean limit exists iff Φ has *analytic singularities* along $X \times \{0\}$, i.e. splits into a smooth part and a divisorial part after pulling-back to a blow-up.

Lemma 3.2. *Given a ray (ϕ^s) in $\mathcal{H}$, the non-Archimedean limit $\phi^{\mathrm{NA}} \in \mathcal{H}^{\mathrm{NA}}$ is unique, if it exists.*

Proof. Let ψ_1 and ψ_2 be non-Archimedean limits of $(\phi^s)_s$ and let Φ be the smooth metric on L_{Δ^*} defined by the ray (ϕ^s). For $i = 1, 2$, pick a representative $(\mathcal{X}_i, \mathcal{L}_i)$ of ψ_i such that Φ extends as a smooth metric on $\mathcal{L}_i$ over Δ. After replacing $(\mathcal{X}_i, \mathcal{L}_i)$ by suitable pullbacks, we may assume $\mathcal{X}_1 = \mathcal{X}_2 =: \mathcal{X}$ and that $\mathcal{X}$ is normal. Then $\mathcal{L}_2 = \mathcal{L}_1 + D$ for a $\mathbb{Q}$-divisor D supported on $\mathcal{X}_0$. Now a smooth metric on $\mathcal{L}_1$ induces a singular metric on $\mathcal{L}_1 + D$ that is smooth iff $D = 0$. Hence $\mathcal{L}_1 = \mathcal{L}_2$, so that $\psi_1 = \psi_2$. □

Remark 3.3. *Following* [Berk09, §2] *(see also* [Jon16, BJ16b]*) one can construct a compact Hausdorff space X^{An} fibering over the interval $[0,1]$ such that the fiber X_ρ^{An} over any point $\rho \in (0,1]$ is homeomorphic to the complex manifold X, and the fiber X_0^{An} over 0 is homeomorphic to the Berkovich analytification of X with respect to the trivial norm on $\mathbb{C}$. Similarly, the line bundle L induces a line bundle L^{An} over X^{An}. If a ray $(\phi^s)_{s>0}$ admits a non-Archimedean limit ϕ^{NA}, then it induces a continuous metric on L^{An} whose restriction to L_ρ^{An} is given by $\phi^{\log\rho^{-1}}$ and whose restriction to X_0^{an} is given by ϕ^{NA}. In this way, ϕ^{NA} is indeed the limit of ϕ^s as $s \to \infty$.*

3.2. Non-Archimedean limits of functionals.

Now assume that L is ample.

Definition 3.4. *A functional $F \colon \mathcal{H} \to \mathbb{R}$ admits a functional $F^{\mathrm{NA}} \colon \mathcal{H}^{\mathrm{NA}} \to \mathbb{R}$ as a* non-Archimedean limit *if, for every smooth subgeodesic ray (ϕ^s) in $\mathcal{H}$ admitting a non-Archimedean limit $\phi^{\mathrm{NA}} \in \mathcal{H}^{\mathrm{NA}}$, we have*

$$\lim_{s \to +\infty} \frac{F(\phi^s)}{s} = F^{\mathrm{NA}}(\phi^{\mathrm{NA}}). \tag{3.1}$$

Proposition 3.5. *If $F \colon \mathcal{H} \to \mathbb{R}$ admits a non-Archimedean limit $F^{\mathrm{NA}} \colon \mathcal{H}^{\mathrm{NA}} \to \mathbb{R}$, then F^{NA} is homogeneous.*

Proof. Consider a semiample test configuration $(\mathcal{X},\mathcal{L})$ representing a non-Archimedean metric $\phi^{\mathrm{NA}}\in\mathcal{H}^{\mathrm{NA}}$, and let $(\phi^s)_s$ be a smooth subgeodesic ray compatible with $(\mathcal{X},\mathcal{L})$. For $d\ge 1$, let $(\mathcal{X}_d,\mathcal{L}_d)$ be the normalized base change induced by $\tau\to\tau^d$. The associated non-Archimedean metric ϕ_d^{NA} is then the non-Archimedean limit of the subgeodesic ray (ϕ^{ds}), so $\lim_{s\to\infty}s^{-1}F(\phi_{ds})=F^{\mathrm{NA}}(\phi_d^{\mathrm{NA}})$. On the other hand, we clearly have $\lim_{s\to\infty}(ds)^{-1}F(\phi^{ds})=\lim_{s\to\infty}s^{-1}F(\phi^s)=F^{\mathrm{NA}}(\phi^{\mathrm{NA}})$. The result follows. ☐

3.3. **Asymptotics of the functionals.** The following result immediately implies Theorem A and Corollary B.

Theorem 3.6. *The functionals E, H, I, J, M and R on $\mathcal{H}$ admit non-Archimedean limits on $\mathcal{H}^{\mathrm{NA}}$ given, respectively, by E^{NA}, H^{NA}, I^{NA}, J^{NA}, M^{NA} and R^{NA}.*

In addition, we have the following result due to Berman. For the proof, see [Berm15, Proposition 3.8] and also [BBJ15, Theorem 3.1] for a more general result.

Theorem 3.7. *If $L:=-K_X$ is ample, then the Ding functional D on $\mathcal{H}$ admits D^{NA} on $\mathcal{H}^{\mathrm{NA}}$ as non-Archimedean limit.*

The main tool in the proof of Theorem 3.6 is the following result.

Lemma 3.8. *For $i=0,\dots,n$, let L_i be a line bundle on X with a smooth reference metric $\phi_{i,\mathrm{ref}}$. Let also $(\mathcal{X},\mathcal{L}_i)$ be a smooth test configuration for (X,L_i), Φ_i an S^1-invariant smooth metric on $\mathcal{L}_i$ near $\mathcal{X}_0$, and denote by (ϕ_i^s) the corresponding ray of smooth metrics on L_i. Then*

$$\langle\phi_0^s,\dots,\phi_n^s\rangle_X-\langle\phi_{0,\mathrm{ref}},\dots,\phi_{n,\mathrm{ref}}\rangle_X=s\left(\bar{\mathcal{L}}_0\cdot\ldots\cdot\bar{\mathcal{L}}_n\right)+O(1)$$

as $s\to\infty$. Here $(\bar{\mathcal{X}},\bar{\mathcal{L}}_i)$ is the compactification of $(\mathcal{X},\mathcal{L}_i)$ for $0\le i\le n$ and $\langle\cdot,\dots,\cdot\rangle_X$ denotes the Deligne pairing with respect to the constant morphism $X\to\{\mathrm{pt}\}$.

Proof. The Deligne pairing $F:=\langle\mathcal{L}_0,\dots,\mathcal{L}_n\rangle_{\mathcal{X}/\mathbb{C}}$ is a line bundle on $\mathbb{C}$, endowed with a $\mathbb{C}^*$-action and a canonical identification of its fiber at $\tau=1$ with the complex line $\langle L_0,\dots,L_n\rangle_X$. It extends to a line bundle $\langle\bar{\mathcal{L}}_0,\dots,\bar{\mathcal{L}}_n\rangle_{\bar{\mathcal{X}}/\mathbb{P}^1}$ on $\mathbb{P}^1$ that is $\mathbb{C}^*$-equivariantly trivial near $\mathbb{P}^1\smallsetminus\{0\}$. Denoting by $w\in\mathbb{Z}$ the weight of the $\mathbb{C}^*$-action on the fiber at 0, we have

$$w=\deg\langle\bar{\mathcal{L}}_0,\dots,\bar{\mathcal{L}}_n\rangle_{\bar{\mathcal{X}}/\mathbb{P}^1}=\left(\bar{\mathcal{L}}_0,\dots,\bar{\mathcal{L}}_n\right).$$

Pick a nonzero vector $v\in F_1=\langle L_0,\dots,L_n\rangle_X$. The $\mathbb{C}^*$-action produces a section $\tau\mapsto\tau\cdot v$ of F on $\mathbb{C}^*$, and $\sigma:=\tau^{-w}(\tau\cdot v)$ is a nowhere vanishing section of F on $\mathbb{C}$, see [BHJ16, Corollary 1.4].

Since the metrics Φ_i are smooth and S^1-invariant, $\Psi:=\langle\Phi_0,\dots,\Phi_n\rangle_{\mathcal{X}/\mathbb{C}}$ is a continuous S^1-invariant metric on F near $0\in\mathbb{C}$. It follows that the function $\log|\sigma|_\Psi$ is bounded near $0\in\mathbb{C}$.

The S^1-invariant metric Ψ defines a ray (ψ^s) of metrics on the line F_1 through $|v|_{\psi^s}=|\tau\cdot v|_{\Psi_\tau}$, for $s=\log|\tau|^{-1}$, where Ψ_τ is the restriction of Ψ to F_τ. Thus

$$\log|v|_{\psi^s}=\log|\tau\cdot v|_{\Psi_\tau}=w\log|\tau|+\log|\sigma|_{\Psi_\tau}=-sw+O(1).$$

By functoriality, the metric ψ^s on F_1 is nothing but the Deligne pairing $\langle\phi_0^s,\dots,\phi_n^s\rangle$. If we set $\psi_{\mathrm{ref}}=\langle\phi_{0,\mathrm{ref}},\dots,\phi_{n,\mathrm{ref}}\rangle_X$, it therefore follows that

$$\langle\phi_0^s,\dots,\phi_n^s\rangle_X-\langle\phi_{0,\mathrm{ref}},\dots,\phi_{n,\mathrm{ref}}\rangle_X=\log|v|_{\psi_{\mathrm{ref}}}-\log|v|_{\psi^s}=sw+O(1),$$

which completes the proof. ☐

Proof of Theorem 3.6. Let $(\phi^s)_s$ be a smooth subgeodesic ray in $\mathcal{H}$ admitting a non-Archimedean limit $\phi^{\mathrm{NA}} \in \mathcal{H}^{\mathrm{NA}}$, and let $(\mathcal{X}, \mathcal{L})$ be a smooth, semiample test configuration representing ϕ^{NA}, such that $(\phi^s)_s$ induces a smooth metric Φ on $\mathcal{L}$ over Δ. By Lemma 1.2, we have

$$(n+1)V\left[E(\phi^s) - E(\phi_{\mathrm{ref}})\right] = \langle \phi^s, \dots, \phi^s \rangle_X - \langle \phi_{\mathrm{ref}}, \dots, \phi_{\mathrm{ref}} \rangle_X.$$

Using Lemma 3.8, it follows that

$$\lim_{s \to +\infty} \frac{E(\phi^s)}{s} = \frac{\left(\bar{\mathcal{L}}^{n+1}\right)}{(n+1)V} = E^{\mathrm{NA}}(\phi^{\mathrm{NA}}),$$

which proves the result for the Monge-Ampère energy E. The case of the functionals I, J and R is similarly a direct consequence of Lemma 1.2 and Lemma 3.8. In view of the Chen-Tian formulas for M and M^{NA}, it remains to treat the case of the entropy functional H. In fact, it turns out to be easier to treat the functional $H + R$.

Using the semistable reduction theorem [KKMS] as in [ADVLN11], we may choose d such that the base change $(\mathcal{X}_d, \mathcal{L}_d)$ admits a semistable pull-back. By homogeneity of $H^{\mathrm{NA}} + R^{\mathrm{NA}}$ with respect to base change, we may thus assume from the outset that $\mathcal{X}$ is semistable. Note, however, that $\mathcal{L}$ is now only relatively semiample and Φ only semipositive.

By Lemma 1.2 we have

$$V(H(\phi^s) + R(\phi^s)) = \langle \tfrac{1}{2}\log \mathrm{MA}(\phi^s), \phi^s, \dots, \phi^s \rangle_X - \langle \tfrac{1}{2}\log \mathrm{MA}(\phi_{\mathrm{ref}}), \phi_{\mathrm{ref}}, \dots, \phi_{\mathrm{ref}} \rangle_X,$$

so since $\mathcal{X}_0$ is reduced, we must show that

$$\langle \tfrac{1}{2}\log \mathrm{MA}(\phi^s), \phi^s, \dots, \phi^s \rangle_X - \langle \tfrac{1}{2}\log \mathrm{MA}(\phi_{\mathrm{ref}}), \phi_{\mathrm{ref}}, \dots, \phi_{\mathrm{ref}} \rangle_X = s\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n\right) + o(s). \quad (3.2)$$

The collection of metrics $\frac{1}{2}\log \mathrm{MA}(\Phi|_{X_\tau})$ with $\tau \neq 0$ defines a smooth metric Ψ on $K_{\mathcal{X}/\mathbb{C}}$ over a punctured neighborhood of $0 \in \mathbb{C}$, but the difficulty here is that Ψ will not a priori extend to a smooth (or even locally bounded) metric on $K_{\mathcal{X}/C}$ because of the singularities of $\mathcal{X}_0$.

Instead, pick a smooth reference metric Ψ_{ref} on $K_{\mathcal{X}/\mathbb{C}}$ near $\mathcal{X}_0$, and denote by $(\psi^s_{\mathrm{ref}})_{s > s_0}$ the corresponding ray of smooth metrics on K_X. By Lemma 3.8 we have

$$\langle \psi^s_{\mathrm{ref}}, \phi^s, \dots, \phi^s \rangle_X - \langle \tfrac{1}{2}\log \mathrm{MA}(\phi_{\mathrm{ref}}), \phi_{\mathrm{ref}}, \dots, \phi_{\mathrm{ref}} \rangle_X = s\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n\right) + O(1).$$

It therefore remains to show that

$$\langle \tfrac{1}{2}\log \mathrm{MA}(\phi^s), \phi^s, \dots, \phi^s \rangle_X - \langle \psi^s_{\mathrm{ref}}, \phi^s, \dots, \phi^s \rangle_X = \tfrac{1}{2}\int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}}\right](dd^c\phi^s)^n$$

is $o(s)$ as $s \to \infty$.

Since $K_{\mathcal{X}/\mathbb{C}} = K_{\mathcal{X}} - \pi^* K_{\mathbb{C}}$, with $\pi\colon \mathcal{X} \to \mathbb{C}$ the projection, $\Psi_{\mathrm{ref}} + \frac{1}{2}\pi^* \log(i d\tau \wedge d\bar{\tau})$ is a smooth metric on $K_{\mathcal{X}}$ near $\mathcal{X}_0$. Let μ be the corresponding volume form on $\mathcal{X}$. For each $\tau \neq 0$ and each (n,n)-form α on $\mathcal{X}$, observe that

$$\alpha|_{X_\tau} = \left.\frac{\alpha \wedge i d\tau \wedge d\bar{\tau}}{\mu}\right|_{X_\tau} e^{2\Psi_{\mathrm{ref}}|_{X_\tau}}$$

In particular, the function $(dd^c\phi^s)^n / e^{2\psi^s_{\mathrm{ref}}}$ on X is the pull-back by the $\mathbb{C}^*$-action of the restriction of the function $g := \frac{(dd^c\Phi)^n \wedge i dt \wedge d\bar{t}}{\mu}$ to X_τ. Note that g is bounded above since μ

is a volume form on $\mathcal{X}$ and $dd^c\Phi$ is smooth and semipositive. Since $\mathrm{MA}(\phi^s)$ is a probability measure,

$$\int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}}\right](dd^c\phi^s)^n = V\int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}}\right]\mathrm{MA}(\phi^s)$$

is uniformly bounded above as a function of s. On the other hand, we have

$$V^{-1}\int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}}\right](dd^c\phi^s)^n$$
$$= \int_X \log\left[\frac{\mathrm{MA}(\phi^s)}{e^{2\psi^s_{\mathrm{ref}}}/\int_X e^{2\psi^s_{\mathrm{ref}}}}\right]\mathrm{MA}(\phi^s) - \log\int_X e^{2\psi^s_{\mathrm{ref}}} \ge -\log\int_X e^{2\psi^s_{\mathrm{ref}}}, \quad (3.3)$$

since the first term is the relative entropy of the probability measure $\mathrm{MA}(\phi^s)$ with respect to the probability measure $e^{2\psi^s_{\mathrm{ref}}}/\int_X e^{2\psi^s_{\mathrm{ref}}}$. Now Lemma 3.9 below yields $\log\int_X e^{2\psi^s_{\mathrm{ref}}} = O(\log s)$, which concludes the proof of (3.2), and hence of Theorem 3.6. ☐

Lemma 3.9. *Let $\mathcal{X}$ be a semistable test configuration for X and Ψ a smooth metric on $K_{\mathcal{X}/\mathbb{C}}$ near $\mathcal{X}_0$. Denote by $e^{2\Psi_\tau}$ the induced volume form on $\mathcal{X}_\tau$ for $\tau \neq 0$. Then*

$$\int_{\mathcal{X}_\tau} e^{2\Psi_\tau} \sim \left(\log|\tau|^{-1}\right)^d \quad \text{as } \tau\to 0, \quad (3.4)$$

with d denoting the dimension of the dual complex of $\mathcal{X}_0$, so that $d+1$ is the largest number of local components of $\mathcal{X}_0$.

Here $A \sim B$ means that A/B is bounded from above and below by positive constants. Much more precise results are proved in [BJ16b].

Proof. Since $\mathcal{X}_0$ is a reduced simple normal crossing divisor, every point of $\mathcal{X}_0$ admits local coordinates $(z_0,\dots,z_n)$ that are defined in a neighborhood of $B := \{|z_i|\le 1\}$ and such that $z_0\dots z_p = \varepsilon t$ with $0\le p\le n$ and $\varepsilon > 0$. The integer d in the statement of the theorem is then the largest such integer p. By compactness of $\mathcal{X}_0$, it will be enough to show that

$$\int_{B\cap\mathcal{X}_\tau} e^{2\Psi_\tau} \sim \left(\log|\tau|^{-1}\right)^p.$$

The holomorphic n-form

$$\eta := \sum_{j=0}^p (-1)^j \frac{dz_0}{z_0}\wedge\dots\wedge\widehat{\frac{dz_j}{z_j}}\wedge\dots\wedge\frac{dz_p}{z_p}\wedge dz_{p+1}\wedge\dots\wedge dz_n$$

satisfies $\eta\wedge dt = c\,dz_0\wedge\dots\wedge dz_n$ for a non-zero constant c. As a result, the holomorphic family $\eta_\tau := \eta|_{\mathcal{X}_\tau}$ of sections of $K_{\mathcal{X}_\tau}$ defines a local frame of $K_{\mathcal{X}/\mathbb{C}}$ on B, and it follows that

$$C^{-1}\eta_\tau\wedge\bar\eta_\tau \le e^{2\Psi_\tau} \le C\eta_\tau\wedge\bar\eta_\tau$$

for a constant $C>0$ independent of τ. Hence it suffices to prove $\int_{B\cap\mathcal{X}_t}\eta_t\wedge\bar\eta_t \sim \left(\log|\tau|^{-1}\right)^p$.

To this end, we parametrize $B\cap\mathcal{X}_t$ in polar coordinates as follows. Consider the p-dimensional simplex

$$\sigma = \{w\in\mathbb{R}^{p+1}_{\ge 0} \mid \sum_{j=0}^p w_j = 1\},$$

the p-dimensional torus

$$T = \{\theta \in (\mathbb{R}/2\pi\mathbb{Z})^{m+1} \mid \sum_j \theta_j = 0\},$$

and the polydisc D^{n-p}. We may cover $\mathbb{C}^*$ by two simply connected open sets, on each of which we fix a branch of the complex logarithm. We then define a diffeomorphism χ_τ from $\sigma \times T \times D^{n-p}$ to $B \cap \mathcal{X}_\tau$ by setting

$$z_j = e^{w_j \log(\varepsilon\tau) + i\theta_j} \quad \text{for } 0 \leq j \leq p.$$

A simple computation shows that

$$\chi_\tau^*(\eta \wedge \bar{\eta}) = \text{const} \left(\log|\varepsilon\tau|^{-1}\right)^p dV,$$

where dV denotes the natural volume form on $\sigma \times T \times D^{n-p}$. It follows that, for $|\tau| \ll 1$,

$$\int_{B \cap \mathcal{X}_\tau} \eta_\tau \wedge \bar{\eta}_\tau \sim \int_{\sigma \times T \times D^{n-p}} \chi_\tau^*(\eta \wedge \bar{\eta}) \sim \left(\log|\tau|^{-1}\right)^p, \tag{3.5}$$

which completes the proof. □

4. Uniform K-stability and CM-stability

In this section we explore the relationship between uniform K-stability and (asymptotic) CM-stability. In particular we prove Theorem C, Corollary D and Corollary E.

4.1. **Functions with log norm singularities.**

In this section, G denotes a reductive complex algebraic group.

Definition 4.1. *We say that a function $f : G \to \mathbb{R}$ has* log norm singularities *if there exist finitely many rational numbers a_i, finite dimensional complex vector spaces V_i endowed with a G-action and non-zero vectors $v_i \in V_i$ such that*

$$f(g) = \sum_i a_i \log \|g \cdot v_i\| + O(1)$$

for some choice of norms on the V_i's.

Remark 4.2. *By the equivalence of norms on a finite dimensional vector space, the description of f is independent of the choice of norms on the V_i. In particular, given a maximal compact subgroup K of G, the norms may be assumed to be K-invariant, so that f descends to a function on the Riemannian symmetric space G/K.*

Remark 4.3. *Taking appropriate tensor products, is is easy to see that every function f on G with log norm singularities may be written as*

$$f(g) = a\left(\log \|g \cdot v\| - \log \|g \cdot w\|\right) + O(1), \tag{4.1}$$

where $a \in \mathbb{Q}_{>0}$ and v, w are vectors in a normed vector space V endowed with a G-action.

The following generalization of the Kempf-Ness/Hilbert-Mumford criterion is closely related to results of [Pau13], which they simplify to some extent. Our elementary argument is inspired by the discussion on pp.241–243 of [Tho06].

Theorem 4.4. *Let f be a function on G with log norm singularities.*

(i) *For each 1-PS $\lambda \colon \mathbb{C}^* \to G$, there exists $f^{\mathrm{NA}}(\lambda) \in \mathbb{Q}$ such that*

$$(f \circ \lambda)(\tau) = f^{\mathrm{NA}}(\lambda) \log |\tau|^{-1} + O(1)$$

for $|\tau| \le 1$.

(ii) *f is bounded below on G iff $f^{\mathrm{NA}}(\lambda) \ge 0$ for all 1-PS λ.*

The chosen notation stems from the fact that f^{NA} induces a function on the (conical) Tits building of G, i.e. the non-Archimedean analogue of G/K (compare [MFF, §2.2]).

Before entering the proof, let us recall some basic facts about representations of algebraic tori. Let $T \simeq (\mathbb{C}^*)^r$ be an algebraic torus, and introduce as usual the dual lattices

$$M := \mathrm{Hom}(T, \mathbb{C}^*) \simeq \mathbb{Z}^r \quad \text{and} \quad N := \mathrm{Hom}(\mathbb{C}^*, T) \simeq \mathbb{Z}^r.$$

Note that N is the group of 1-PS of T. For each finite-dimensional vector space V on which T acts and each $m \in M$, let $V_m \subset V$ be the subspace on which each $t \in T$ acts by multiplication by $m(t)$. The action of T on V being diagonalizable, we have a direct sum decomposition $V = \bigoplus_{m \in M} V_m$, and the set of *weights of* V is defined as the (finite) set $M_V \subset M$ of characters $m \in M$ for which $V_m \neq 0$.

Given a non-zero vector $v \in V$, the set $M_v \subset M_V$ of weights of v is defined as those $m \in M$ for which the projection $v_m \in V_m$ of v is non-zero. The *weight polytope* of v is defined as the convex hull $P_v \subset M_\mathbb{R}$ of M_v in $M_\mathbb{R}$, whose support function $h_v\colon N_\mathbb{R} \to \mathbb{R}$ is the convex, positively homogeneous function defined by

$$h_v(\lambda) = \max_{m \in M_v} \langle m, \lambda \rangle,$$

where the bracket denotes the dual pairing between $M_\mathbb{R}$ and $N_\mathbb{R}$.

Proof of Theorem 4.4. By Remark 4.3 we may assume f is of the form

$$f(g) := \log \|g \cdot v\| - \log \|g \cdot w\|,$$

where v, w are nonzero vectors in a finite dimensional normed vector space V equipped with a G-action.

(i) Let first $\lambda\colon \mathbb{C}^* \to G$ be a 1-parameter subgroup, and denote by $I_v \subset \mathbb{Z}$ the set of weights of v with respect to λ. We then have

$$\lambda(\tau) \cdot v = \sum_{m \in I_v} \tau^m v_m,$$

and hence

$$\log \|\lambda(\tau) \cdot v\| = \max_{m \in I_v} (m \log |\tau| + \log \|v_m\|) + O(1) = -\left(\min_{m \in I_v} m\right) \log |\tau|^{-1} + O(1)$$

for $|\tau| \le 1$, and (i) follows with $f^{\mathrm{NA}}(\lambda) = \min I_w - \min I_v$.

(ii) The direct implication follows immediately from (i). For the reverse implication we use the Cartan (or polar) decomposition $G = KTK$, where $T \subset G$ is any maximal algebraic torus and $K \subset G$ be a maximal compact subgroup. We then get an isomorphism $T/K \cap T \simeq N_\mathbb{R}$, hence a group homomorphism

$$\mathrm{Log}\,|\cdot|\colon T \to N_\mathbb{R},$$

which in compatible bases for $T \simeq \mathbb{C}^{*r}$ and $N_R \simeq \mathbb{R}^r$ is given by

$$(t_1, \dots, t_r) \mapsto (\log |t_1|, \dots, \log |t_r|).$$

Note that

$$\log |m(t)| = \langle m, \mathrm{Log}\,|t| \rangle$$

for all $m \in M$ and $t \in T$, and

$$\mathrm{Log}\,|\lambda(\tau)| = (\log |\tau|)\lambda$$

in $N_\mathbb{R}$ for each 1-PS $\lambda\colon \mathbb{C}^* \to T$ (i.e. $\lambda \in N$).

In this notation, we claim that

$$f(k'tk) = h_{k \cdot v}(\mathrm{Log}\,|t|) - h_{k \cdot w}(\mathrm{Log}\,|t|) + O(1), \tag{4.2}$$

for all $k, k' \in K$ and $t \in T$.

To see that (4.2) holds, we may assume the norm on V is K-invariant. We then have for all $k, k' \in K$ and $t \in T$

$$\log \|(k'tk) \cdot v\| = \log \|t \cdot (k \cdot v)\| = \log \| \sum_{m \in M_{k \cdot v}} m(t)(k \cdot v)_m \|$$
$$= \log \max_{m \in M_{k \cdot v}} \|m(t)(k \cdot v)_m\| + O(1) = \max_{m \in M_{k \cdot v}} (\langle m, \mathrm{Log}\,|t| \rangle + \log \|(k \cdot v)_m)\|) + O(1).$$

By the compactness of K, we further may find $C = C(v) > 0$ such that

$$-C \le \log \|(k \cdot v)_m\| \le C$$

for all $k \in K$ and all $m \in M_{k\cdot v}$. By the definition of the support function $h_{k\cdot v}$, we thus have

$$\max_{m\in M_{k\cdot v}} \left(\langle m, \operatorname{Log}|t|\rangle + \log\|(k\cdot v)_m)\|\right) = h_{k\cdot v}(\operatorname{Log}|t|) + O(1).$$

We have thus proved that

$$\log\|(k'tk)\cdot v\| = h_{k\cdot v}(\operatorname{Log}|t|) + O(1).$$

A similar estimate of course holds with w in place of v, and (4.2) follows.

As a consequence of (4.2), we get

$$f^{\mathrm{NA}}(k^{-1}\lambda k) = h_{k\cdot v}(\lambda) - h_{k\cdot w}(\lambda) \tag{4.3}$$

for all $\lambda \in N$. If we assume that $f^{\mathrm{NA}} \geq 0$ on all 1-PS of G, then $h_{k\cdot v} \geq h_{k\cdot w}$ on N, hence on $N_{\mathbb{Q}}$ by homogeneity, and hence on $N_{\mathbb{R}}$ by density. From (4.2) and the Cartan decomposition $G = KTK$ it follows, as desired, that f is bounded below on G. The proof is now complete. □

4.2. **Proof of Theorem C and Corollaries D and E.** Replacing L with mL, we may assume for notational simplicity that $m = 1$. Set $N := h^0(L)$ and $G := \mathrm{SL}(N,\mathbb{C})$, so that each $\sigma \in G$ defines a Fubini-Study type metric ϕ_σ on L. Note that $M - \delta J$ is bounded below on $\mathcal{H}_1 \simeq \mathrm{GL}(N,\mathbb{C})/\mathrm{U}(N)$ iff $M(\phi_\sigma) - \delta J(\phi_\sigma)$ bounded below for $\sigma \in G$, by translation invariance of M and J.

The key ingredient is the following result of S. Paul [Pau12].

Theorem 4.5. *The functionals E, J and M all have log norm singularities on G.*

Granted this result we can deduce Theorem C. The equivalence of (ii) and (iii) follows from the same argument as Lemma 7.22 in [BHJ16], so it suffices to show that (i) and (iii) are equivalent. By Theorem 4.5, the function $f(\sigma) := M(\phi_\sigma) - \delta J(\phi_\sigma)$ on G has log norm singularities. By Theorem 4.4, it is thus bounded below iff

$$\lim_{s\to+\infty} \frac{(f\circ\lambda)(e^{-s})}{s} \geq 0$$

for each 1-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$. We obtain the desired result since by Theorem B, this limit is equal to $M^{\mathrm{NA}}(\phi_\lambda) - \delta J^{\mathrm{NA}}(\phi_\lambda)$, where $\phi_\lambda \in \mathcal{H}^{\mathrm{NA}}$ is the non-Archimedean metric on L defined by λ.

Corollary D follows since every ample test configuration of (X, L) is induced by a 1-PS, see §2.2. The first assertion of Corollary E follows immediately, and the fact that the reduced automorphism group of (X, L) is finite is a consequence of [Pau13, Corollary 1.1].

Proof of Theorem 4.5. Recall from [Pau12] that to the linearly normal embedding $X \hookrightarrow \mathbb{P}H^0(X,L)^* \simeq \mathbb{P}^{N-1}$ are associated the X-resultant R, i.e. the Chow coordinate of X, and the X-hyperdiscriminant Δ, which cuts out the dual variety of

$$X \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N-1} \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{Nn-1},$$

the second arrow being the Segre embedding.

In our notation, we then have $\deg R = V(n+1)$ and $\deg\Delta = V\left(n(n+1) - \bar{S}\right)$ [Pau12, Proposition 5.7], and [Pau12, Theorem A] becomes

$$M(\phi_\sigma) = V^{-1}\log\|\sigma\cdot\Delta\| - V^{-1}\frac{\deg\Delta}{\deg R}\log\|\sigma\cdot R\| + O(1), \tag{4.4}$$

which proves the assertion for $M(\phi_\sigma)$.

We next consider

$$J(\phi_\sigma) = \int_X (\phi_\sigma - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_{\mathrm{ref}}) - E(\phi_\sigma).$$

On the one hand, by [Pau04, Theorem 1] (or [Zha96, Theorem 1.6, Theorem 3.6]) we have

$$E(\phi_\sigma) = \frac{1}{\deg R} \log \|\sigma \cdot R\| + O(1). \tag{4.5}$$

On the other hand, choosing any norm on the space of complex $N \times N$-matrices (in which G of course embeds), it is observed in the proof of [Tia14, Lemma 3.2] that

$$\int_X (\phi_\sigma - \phi_{\mathrm{ref}}) \, \mathrm{MA}(\phi_{\mathrm{ref}}) = \log \|\sigma\| + O(1).$$

The assertion for $J(\phi_\sigma)$ follows. □

4.3. **Discussion of** [Tia14]**.** The statement of [Tia14, Lemma 3.1] sounds overoptimistic from the GIT point of view, as it would mean that CM-stability can be tested by only considering 1-parameter subgroups of a fixed maximal torus T.

At least, the proof is incorrect, the problem being the estimate (3.1), which claims that $\phi_{\tau k} - \phi_\tau$ is uniformly bounded with respect to $\tau \in T$ and $k \in K$. As the next example shows, this is not even true for a fixed $k \in K$.

Example 4.6. *Assume (s_1, s_2) is a basis of $H^0(X, L)$, let $k \in U(2)$ be the unitary transformation exchanging s_1 and s_2, $\tau = (t, t^{-1})$, and pick a point x with $s_1(x) = 0$. Then*

$$\phi_{\tau k}(x) - \phi_\tau(x) = 4 \log |\tau|$$

is unbounded.

In any case, the methods here do not seem to be able to deduce CM-stability from K-stability, because of the following fact.

Proposition 4.7. *For each polarized manifold (X, L) and each m large and divisible enough, there exists a non-trivial 1-PS λ in $\mathrm{GL}(N_m, \mathbb{C})$ such that J and M remain bounded on the corresponding Fubini-Study ray $\phi^s := \phi_{\lambda(e^{-s})}$.*

Proof. As originally observed in [LX14] (cf. Proposition 2.3), (X, L) admits a non-trivial ample test configuration $(\mathcal{X}, \mathcal{L})$ that is almost trivial, i.e. with trivial normalization. As recalled in §2.2, for each m large and divisible enough, $(\mathcal{X}, \mathcal{L})$ can be realized as the test configuration induced by a 1-PS $\lambda : \mathbb{C}^* \to \mathrm{GL}(N_m, \mathbb{C})$, which is non-trivial since $(\mathcal{X}, \mathcal{L})$ is. Since the normalization of $(\mathcal{X}, \mathcal{L})$ is trivial, the associated non-Archimedean metric is of the form $\phi_{\mathrm{triv}} + c$ for some $c \in \mathbb{Q}$, and hence $M^{\mathrm{NA}}(\phi_\lambda) = J^{\mathrm{NA}}(\phi_\lambda) = 0$. Since M and J have log norm singularities on $\mathrm{GL}(N_m, \mathbb{C})$ by Theorem 4.5, M and J are indeed bounded on ϕ^s by Theorem 4.4. □

5. Remarks on the Yau-Tian-Donaldson conjecture

As explained in the introduction, we will here give a simple argument, following ideas of Tian, for the existence of a Kähler-Einstein metric on a Fano manifold X, assuming $(X, -K_X)$ is uniformly K-stable and the partial C^0-estimates due to Székelyhidi.

5.1. **Partial C^0-estimates and the continuity method.** For the moment, consider an arbitrary polarized manifold (X, L). For each m such that mL is very ample, we have a 'Bergman kernel approximation' map $P_m \colon \mathcal{H} \to \mathcal{H}_m$, defined by setting $P_m(\phi)$ to be the Fubini-Study metric induced by the L^2-scalar product on $H^0(X, mL)$ defined by $m\phi$.

Definition 5.1. *A subset $A \subset \mathcal{H}$ satisfies* partial C^0-estimates at level m *if there exists $C > 0$ such that $|P_m(\phi) - \phi| \le C$ for all $\phi \in A$.*

Now assume X is Fano, and set $L := -K_X$. Given a Kähler form $\alpha \in c_1(X)$, consider Aubin's continuity method

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\alpha. \tag{5.1}$$

It is well-known that there exists a unique maximal solution $(\omega_t)_{t\in[0,T)}$, where $0 < T \le 1$. The following important result, due to Székelyhidi [Szé13], confirms a conjecture of Tian.

Theorem 5.2. *The set $A := \{\omega_t \mid t \in [0, T)\}$ satisfies partial C^0-estimates at level m, for arbitrarily large positive integers m.*

Given this result, we shall prove

Theorem 5.3. *Any uniformly K-stable Fano manifold admits a Kähler-Einstein metric.*

By working (much) harder, Datar and Székelyhidi [DSz15] have in fact been able to deduce from Theorem 5.2 a much better result dealing with K-polystability and allowing a compact group action.

5.2. **CM-stability and partial C^0-estimates.** We first present in some detail well-known ideas due to Tian [Tia12, §4.3]. In this section, (X, L) is an arbitrary polarized manifold.

Proposition 5.4. *Assume that (X, mL) is CM-stable, and that $A \subset \mathcal{H}$ satisfies partial C^0-estimates at level m. Then there exist $\delta, C > 0$ such that $M \ge \delta J - C$ on A.*

The proof is based on two lemmas.

Lemma 5.5. *For any two metrics $\phi, \psi \in \mathcal{H}$, we have*

(i) *$|J(\phi) - J(\psi)| \le 2 \sup(\phi - \psi)$;*
(ii) *$M(\phi) \ge M(\psi) - C \sup |\phi - \psi|$ for some $C > 0$ only depending on a one-sided bound (either upper or lower) for the Ricci curvature of the Kähler metric $dd^c\psi$.*

Proof. Recall that

$$E(\phi) - E(\psi) = \frac{1}{n+1} \sum_{j=0}^{n} V^{-1} \int_X (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j}.$$

As a consequence, $|E(\phi) - E(\psi)| \le \sup |\phi - \psi|$, and (i) follows immediately.

For (ii), we basically argue as in the proof of [Tia14, Lemma 3.1]. By the Chen-Tian formula (1.11), we have

$$M(\phi) - M(\psi) = H_\psi(\phi) + \bar{S}\,(E(\phi) - E(\psi)) + E_{\mathrm{Ric}(dd^c\psi)}(\psi) - E_{\mathrm{Ric}(dd^c\psi)}(\phi).$$

Here the entropy term $H_\psi(\phi)$ is non-negative, and we have

$$E_{\mathrm{Ric}(dd^c\psi)}(\phi) - E_{\mathrm{Ric}(dd^c\psi)}(\psi) = \sum_{j=0}^{n-1} V^{-1} \int_X (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j-1} \wedge \mathrm{Ric}(dd^c\psi).$$

Assume $\mathrm{Ric}(dd^c\psi) \le C dd^c\psi$ for some constant $C > 0$. We may then write

$$\begin{aligned}(dd^c\phi)^j &\wedge (dd^c\psi)^{n-j-1} \wedge \mathrm{Ric}(dd^c\psi) \\ &= C(dd^c\phi)^j \wedge (dd^c\psi)^{n-j} - (dd^c\phi)^j \wedge (dd^c\psi)^{n-j-1} \wedge (C' dd^c\psi - \mathrm{Ric}(dd^c\psi)),\end{aligned}$$

a difference of two positive measures of mass CV and $CV + (L^{n-1} \cdot K_X)$, respectively, and the desired estimate follows.

The case where $\mathrm{Ric}(dd^c\psi) \ge -C' dd^c\psi$ is treated similarly (and will anyway not be used in what follows). □

We next recall a well-known upper bound for the Ricci curvature of restrictions of Fubini-Study metrics.

Lemma 5.6. *We have* $\mathrm{Ric}(dd^c\phi) \le N_m dd^c\phi$ *for all* $\phi \in \mathcal{H}_m$.

Proof. Choose a basis of $H^0(X, mL)$, and let ω be the corresponding Fubini-Study metric on $\mathbb{P} := \mathbb{P}H^0(X, mL)^*$. Its curvature tensor

$$\Theta(T_\mathbb{P}, \omega) \in C^\infty(\mathbb{P}, \Lambda^{1,1} T^*_\mathbb{P} \otimes \mathrm{End}(T_\mathbb{P}))$$

is Griffiths positive and satisfies

$$\mathrm{Tr}_{T_\mathbb{P}} \Theta(T_\mathbb{P}, \omega) = \mathrm{Ric}(\omega) = N_m \omega.$$

For each complex submanifold $Y \subset \mathbb{P}$, the curvature of its tangent bundle T_Y with respect to $\omega|_Y$ satisfies $\Theta(T_Y, \omega|_Y) \le \Theta(T_\mathbb{P}, \omega)|_{T_Y}$ as $(1,1)$-forms on Y with values in the endomorphisms of T_Y, as a consequence of a well-known curvature monotonicity property going back to Griffiths. We thus have

$$\mathrm{Ric}(\omega|_Y) = \mathrm{Tr}_{T_Y} \Theta(T_Y, \omega|_Y) \le \mathrm{Tr}_{T_Y} \Theta(T_\mathbb{P}, \omega)|_{T_Y}.$$

Using now $\Theta(T_\mathbb{P}, \omega) \ge 0$, we have on the other hand

$$\mathrm{Tr}_{T_Y} \Theta(T_\mathbb{P}, \omega)|_{T_Y} \le \mathrm{Tr}_{T_\mathbb{P}} \Theta(T_\mathbb{P}, \omega)|Y = N_m \omega|_Y,$$

and hence

$$\mathrm{Ric}(\omega|_Y) \le N_m \omega|_Y.$$

Applying this to the images of $X \subset \mathbb{P}$ under projective transformations yields the desired result. □

Proof of Proposition 5.4. Since (X, mL) is CM-stable, there exist $\delta, C > 0$ such that

$$M(P_m(\phi)) \ge \delta J(P_m(\phi)) - C \tag{5.2}$$

for all $\phi \in \mathcal{H}$. By assumption on A, we also have $|P_m(\phi) - \phi| \le C$ for all $\phi \in A$, and by Lemma 5.6, the Ricci curvature of $dd^c P_m(\phi)$ is uniformly bounded above. Hence Lemma 5.5 shows, as desired, that there exists $C' > 0$ with $M(\phi) \ge \delta J(\phi) - C'$ for all $\phi \in A$. □

5.3. **Proof of Theorem 5.3.** Assume now that X is a Fano manifold and set $L := -K_X$. Consider the continuity method (5.1). Pick metrics ψ and ϕ_t on $-K_X$ such that $\alpha = dd^c\psi$ and $\omega_t = dd^c\phi_t$, respectively. After adding a constant to ϕ_t, (5.1) may be written

$$(dd^c\phi_t)^n = e^{-2(t\phi_t+(1-t)\psi)}. \tag{5.3}$$

We recall the proof of the following well-known monotonicity property.

Lemma 5.7. *The function $t \to M(\phi_t)$ is non-increasing.*

Proof. We have

$$\begin{aligned}
-\frac{d}{dt}M(\phi_t) &= nV^{-1}\int_X \dot\phi_t\left(\mathrm{Ric}(\omega_t)\wedge\omega_t^{n-1}-\omega_t^n\right)\\
&= nV^{-1}(1-t)\int_X \dot\phi_t dd^c(\psi-\phi_t)\wedge(dd^c\phi_t)^{n-1}\\
&= nV^{-1}(1-t)\int_X (\psi-\phi_t)dd^c\dot\phi_t\wedge(dd^c\phi_t)^{n-1}.
\end{aligned}$$

Since d^c is normalized so that $dd^c = \frac{i}{\pi}\partial\overline{\partial}$, we have

$$n\frac{dd^c\dot\phi_t\wedge\omega_t^{n-1}}{\omega_t^n} = \mathrm{tr}_{\omega_t} dd^c\dot\phi_t = -\tfrac{1}{2\pi}\Delta_t''\dot\phi_t$$

with Δ_t'' denoting the $\bar\partial$-Laplacian with respect to ω_t. On the other hand, differentiating (5.3) yields

$$ndd^c\dot\phi_t\wedge\omega_t^{n-1} = 2(\psi-\phi_t-t\dot\phi_t)\omega_t^n,$$

and hence

$$\psi-\phi_t = \left(t-\tfrac{1}{\pi}\Delta_t''\right)\dot\phi_t.$$

We get

$$\begin{aligned}
-\frac{d}{dt}M(\phi_t) &= \frac{1-t}{2\pi}\int_X \left(\left(\tfrac{1}{\pi}\Delta_t''-t\right)\dot\phi_t\right)\left(\Delta_t''\dot\phi_t\right)\mathrm{MA}(\phi_t)\\
&= \frac{1-t}{2\pi}\int_X \langle\left(\tfrac{1}{\pi}\Delta_t''-t\right)\bar\partial\dot\phi_t,\bar\partial\dot\phi_t\rangle_{\omega_t}\mathrm{MA}(\phi_t).
\end{aligned}$$

Since $\mathrm{Ric}(\omega_t)\ge t\omega_t$, the $\bar\partial$-Laplacian Δ_t'' satisfies $\frac{1}{\pi}\Delta_t''\ge t$ on $(0,1)$-forms, and the last integral is thus nonnegative. This follows indeed from the Bochner-Kodaira-Nakano identity applied to

$$C^\infty(X,\Lambda^{0,1}T_X^*)\simeq C^\infty(X,\Lambda^{n,1}T_X^*\otimes K_X^*)$$

with the fiber metric $\psi_t = -\frac{1}{2}\log\omega_t^n$ on $K_X^* = -K_X$, with curvature $dd^c\psi_t = \mathrm{Ric}(\omega_t)$. □

We may now complete the proof of Theorem 5.3. By Corollary E, $(X, -mK_X)$ is CM-stable for all m divisible enough. Theorem 5.2 and Proposition 5.4 therefore yield $\delta, C>0$ such that $M(\phi_t)\ge\delta J(\phi_t)-C$ along Aubin's continuity method. Since $M(\phi_t)$ is bounded above by Lemma 5.7, it follows that $J(\phi_t)$ remains bounded. By [Tia, Lemma 6.19], the oscillation of ϕ_t is bounded, and well-known arguments allow us to conclude, see [Tia, §6.2].

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