# UNIFORM K-STABILITY, DUISTERMAAT-HECKMAN MEASURES AND SINGULARITIES OF PAIRS

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Abstract. The purpose of the present paper is to set up a formalism inspired from non-Archimedean geometry to study K-stability. We first provide a detailed analysis of Duistermaat-Heckman measures in the context of test configurations, characterizing in particular the trivial case. For any normal polarized variety (or, more generally, polarized pair in the sense of the Minimal Model Program), we introduce and study the non-Archimedean analogues of certain classical functionals in Kähler geometry. These functionals are defined on the space of test configurations, and the Donaldson-Futaki invariant is in particular interpreted as the non-Archimedean version of the Mabuchi functional, up to an explicit error term. Finally, we study in detail the relation between uniform K-stability and singularities of pairs, reproving and strengthening Y. Odaka's results in our formalism. This provides various examples of uniformly K-stable varieties.

## Contents



## Introduction

Let $(X, L)$ be a polarized complex manifold, i.e. a smooth complex projective variety $X$ endowed with an ample line bundle $L$. Assuming for simplicity that $\mathrm{Aut}(X, L)$ is discrete (and hence finite), the Yau-Tian-Donaldson conjecture states that the first Chern class $c_1(L)$ contains a constant scalar curvature Kähler metric (cscK metric for short) iff $(X, L)$ satisfies a certain algebro-geometric condition known as *K-stability*. Building on [Don01, AP06], it was proved in [Sto09] that K-stability indeed follows from the existence of a cscK metric.

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When $c_1(X)$ is a multiple of $c_1(L)$, the converse was recently established ([CDS15a, CDS15b, CDS15c], see also [Tia12]); in this case a cscK metric is the same as a Kähler-Einstein metric.

The notion of K-stability, introduced in [Tia97, Don02], is defined as the positivity of the Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X},\mathcal{L})$ of every non-trivial test configuration $(\mathcal{X},\mathcal{L})$ for $(X,L)$ (where, as pointed out in [LX14], triviality must be carefully defined). However, G. Székelyhidi [Szé06, Szé14] proposed that a *uniform* notion of K-stability should be used to formulate the Yau-Tian-Donaldson conjecture in the general case. In this uniform version, $\mathrm{DF}(\mathcal{X},\mathcal{L})$ is bounded below by a positive quantity measuring how far $(\mathcal{X},\mathcal{L})$ is from being trivial.

The present paper proposes an algebro-geometric study of three topics naturally related to uniform K-stability. First, we study in detail *Duistermaat-Heckman measures* in the context of test configurations. These measures encode in particular the $L^p$-norms used in Székelyhidi's formulation of uniform K-stability. Second, we view test configurations for $(X,L)$ as *non-Archimedean metrics* on $L$, define non-Archimedean analogues of classical functionals from Kähler geometry, and use this formalism to define a version of uniform K-stability which we call *J-uniform K-stability.* Finally, we analyze the interaction between singularities of pairs (in the sense of the Minimal Model Program) and uniform K-stability, revisiting Y. Odaka's work [Oda12, Oda13b, OSa12, OSu11].

**Duistermaat-Heckman measures.** Working, for the moment, over an arbitrary algebraically closed ground field, let $(X,L)$ be a polarized $\mathbb{G}_m$-scheme, i.e. a projective scheme $X$ with a $\mathbb{G}_m$-action, together with a $\mathbb{G}_m$-linearized ample line bundle $L$. The Duistermaat-Heckman measure $\mathrm{DH}_{(X,L)}$ is the probability measure on $\mathbb{R}$ describing the asymptotic distribution as $m\to\infty$ of the (scaled) weights of the $\mathbb{G}_m$-action on $H^0(X,mL)$, counted with multiplicity, i.e.

$$\mathrm{DH}_{(X,L)}=\lim_{m\to\infty}\sum_{\lambda\in\mathbb{Z}}\frac{\dim H^0(X,mL)_\lambda}{\dim H^0(X,mL)}\delta_{\lambda/m},$$

with $H^0(X,mL)=\bigoplus_{\lambda\in\mathbb{Z}}H^0(X,mL)_\lambda$ the weight space decomposition.

When $X$ is a variety (i.e. reduced and irreducible), Okounkov proved the existence of $\mathrm{DH}_{(X,L)}$, expressing it as a linear projection of the restriction of the Lebesgue measure of a convex body of dimension $n=\dim X$ [Ok96]. As a result, $\mathrm{DH}_{(X,L)}$ is then absolutely continuous, its support is a line segment $[\lambda_{\min},\lambda_{\max}]$, and the Brunn-Minkowski inequality yields the following log concavity property: $\lambda\mapsto\mathrm{DH}_{(X,L)}(x>\lambda)^{1/n}$ is concave on $(-\infty,\lambda_{\max})$.

Using the equivariant Riemann-Roch theorem for schemes due to Edidin and Graham, we establish in §6 below the existence of the Duistermaat-Heckman measure $\mathrm{DH}_{(X,L)}$ of an arbitrary polarized $T$-scheme $(X,L)$, and show that it can be expressed as a convex combination of the Duistermaat-Heckman measures of the top-dimensional irreducible components of $X$ (with their reduced structure).

Duistermaat-Heckman measures can be more generally defined for a polarized scheme $(X,L)$ without a $\mathbb{G}_m$-action, by introducing certain $\mathbb{G}_m$-equivariant degenerations. Recall that a *test configuration* $(\mathcal{X},\mathcal{L})$ for $(X,L)$ is a $\mathbb{G}_m$-equivariant partial compactification of $(X,L)\times\mathbb{G}_m$. It comes with a proper, flat, $\mathbb{G}_m$-equivariant morphism $\pi:\mathcal{X}\to\mathbb{A}^1$, together with a $\mathbb{Q}$-line bundle $\mathcal{L}$ extending $p_1^*L$ on $X\times\mathbb{G}_m$. When $(\mathcal{X},\mathcal{L})$ is ample, i.e. $\mathcal{L}$ is $\pi$-ample, the central fiber $(\mathcal{X}_0,\mathcal{L}_0)$ is a polarized $\mathbb{G}_m$-scheme, and we define the Duistermaat-Heckman measure $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ to be that of $(\mathcal{X}_0,\mathcal{L}_0)$. In case $(X,L)$ itself is given a $\mathbb{G}_m$-action, its Duistermaat-Heckman measure coincides with that of the corresponding product test configuration $(\mathcal{X},\mathcal{L})$.

As observed in [WN12], every ample test configuration $(\mathcal{X},\mathcal{L})$ for $(X,L)$ defines a *filtration* of the graded ring $R(X,L)=\bigoplus_{m\in\mathbb{N}}H^0(X,mL)$. In fact, this amounts to a classical construction by Rees, and yields a one-to-one correspondence between ample test configurations and finitely generated filtrations of $R(X,L)$.

When $X$ is a variety, the results of [BC11] may be applied to the filtration. Building on this, our first main result may be summarized as follows.

**Theorem A.** *Let $(X,L)$ be a polarized variety defined over an arbitrary algebraically closed field, and let $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ be the Duistermaat-Heckman measure of an ample test configuration $(\mathcal{X},\mathcal{L})$ for $(X,L)$. Then:*

- (i) *The support $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ is a line segment $[\lambda_{\min},\lambda_{\max}]$, and the tail distribution $\lambda\mapsto\mathrm{DH}_{(\mathcal{X},\mathcal{L})}(x>\lambda)^{1/n}$ is log concave on $(-\infty,\lambda_{\max})$. In particular, $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ is the sum of an absolutely continuous measure and a point mass at $\lambda_{\max}$.*
- (ii) *The density of the absolutely continuous part of $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ is piecewise polynomial.*
- (iii) *If $X$ is normal, then the normalization $(\widetilde{\mathcal{X}},\tilde{\mathcal{L}})$ of $(\mathcal{X},\mathcal{L})$ satisfies $\mathrm{DH}_{(\widetilde{\mathcal{X}},\tilde{\mathcal{L}})}=\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$, and $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ is a Dirac mass iff $(\widetilde{\mathcal{X}},\tilde{\mathcal{L}})$ is trivial.*

As we shall see, (i) is in fact a direct consequence of the results of [BC11]. Assertion (ii) generalizes a well-known property of Duistermaat-Heckman measures for smooth complex polarized $\mathbb{C}^*$-varieties $(X,L)$ [DH82], and relies on a result of [ELMNP06].

For each $p\in[1,\infty]$, the $L^p$-norm $\|(\mathcal{X},\mathcal{L})\|_p$ of an ample test configuration $(\mathcal{X},\mathcal{L})$ is defined as the $L^p$ norm of $\lambda-\bar\lambda$ with respect to $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$, with $\bar\lambda$ the barycenter of this measure. Then (iii) asserts in particular that $\|(\mathcal{X},\mathcal{L})\|_p=0$ iff $(\widetilde{\mathcal{X}},\tilde{\mathcal{L}})$ is trivial.

Recall that $(X,L)$ is K-stable iff $\mathrm{DF}(\mathcal{X},\mathcal{L})\ge 0$ for all normal, ample test configurations, with equality iff $(\mathcal{X},\mathcal{L})$ is trivial. Following Székelyhidi, we say that $(X,L)$ is *$L^p$-uniformly K-stable* if there exists $\delta>0$ such that $\mathrm{DF}(\mathcal{X},\mathcal{L})\ge\delta\|(\mathcal{X},\mathcal{L})\|_p$ for all normal, ample test configurations. The above result therefore shows that uniform K-stability indeed implies K-stability (!). We show that $L^p$-uniform K-stability can only hold for $p\le\frac{n}{n-1}$ (cf. Proposition 7.25). One of the points of the present paper is to introduce a different notion of uniform K-stability, based on an analogy between functionals in Kähler geometry and their non-Archimedean counterparts.

**Non-Archimedean functionals and J-uniform K-stability.** Assume that $(X,L)$ is a normal polarized variety. A test configuration for $(X,L)$ then induces a *non-Archimedean metric* on $L$, when the ground field is equipped with the *trivial* norm, see §5. In this language, ample test configurations become (semi)positive metrics.

Several classical functionals on the space of Archimedean (Kähler) metrics now have natural counterparts in the non-Archimedean setting. For example, the *non-Archimedean Monge-Ampère energy* is

$$E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})=\frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)V}=\int_{\mathbb{R}}\lambda\,\mathrm{DH}_{(\mathcal{X},\mathcal{L})}(d\lambda),$$

where $V = (L^n)$, $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ is the natural compactification of $(\mathcal{X}, \mathcal{L})$ over $\mathbb{P}^1$ and $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ is the Duistermaat-Heckman measure of $(\mathcal{X}, \mathcal{L})$. The *non-Archimedean J-energy* is

$$J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lambda_{\max} - E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lambda_{\max} - \int_{\mathbb{R}} \lambda \, \mathrm{DH}_{(\mathcal{X},\mathcal{L})}(d\lambda),$$

with $\lambda_{\max}$ the upper bound of the support of $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$.

Suppose we are also given a *boundary* $B$ in the sense of the Minimal Model Program, i.e. a $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. We then say that $((X, B), L)$ is a polarized pair, and define the *non-Archimedean Ricci energy* $R_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ in terms of intersection numbers on a test configuration dominating $(\mathcal{X}, \mathcal{L})$. The *non-Archimedean entropy* $H_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ is defined in terms of the log discrepancies with respect to $(X, B)$ of certain divisorial valuations, and will be described in more detail below.

The *non-Archimedean Mabuchi functional* is now defined so as to satisfy the analogue of the *Chen-Tian* formula (see [Che00] and also [BB14, Proposition 3.1])

$$M_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = H_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) + \bar{S}_B E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) + R_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

with

$$\bar{S}_B := -nV^{-1}\left((K_X + B) \cdot L^{n-1}\right).$$

The whole point of these constructions is that $M_B^{\mathrm{NA}}$ is essentially the same as the Donaldson-Futaki invariant.[1] We show more precisely that every normal, ample test configuration $(\mathcal{X}, \mathcal{L})$ satisfies

$$M_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \mathrm{DF}_B(\mathcal{X}, \mathcal{L}) + V^{-1}\left((\mathcal{X}_{0,\mathrm{red}} - \mathcal{X}_0) \cdot \mathcal{L}^n\right), \tag{0.1}$$

with $\mathrm{DF}_B$ denoting the log Donaldson-Futaki invariant. Further, $M_B^{\mathrm{NA}}$ is homogeneous with respect to $\mathbb{G}_m$-equivariant base change, a property which is particularly useful in relation with semistable reduction, and fails for the Donaldson-Futaki invariant when the central fiber is non-reduced.

We say that a polarized pair $((X, B), L)$ is *J-uniformly K-stable* if there exists $\delta > 0$ such that $\mathrm{DF}_B(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ for all normal, semiample test configurations. This condition implies $L^1$-uniform stability, because $J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \frac{1}{2}\|(\mathcal{X}, \mathcal{L})\|_1$. Using the homogeneity of $M_B^{\mathrm{NA}}$ and a weak form of semistable reduction, we show that J-uniform K-stability is equivalent to the apparently stronger condition $M_B^{\mathrm{NA}} \geq \delta J^{\mathrm{NA}}$, which we interpret as a counterpart to the *coercivity* of the K-energy in the Archimedean case.

The relation between the non-Archimedean functionals above and their classical counterparts will be systematically studied in [BHJ15]. Let us indicate the main idea. Assume $(X, L)$ is a smooth polarized complex variety, and $B = 0$. Denote by $\mathcal{H}$ the space of Kähler metrics on $L$ and by $\mathcal{H}^{\mathrm{NA}}$ the space of non-Archimedean metrics. The general idea is that $\mathcal{H}^{\mathrm{NA}}$ plays the role of the 'Tits boundary' of the (infinite dimensional) symmetric space $\mathcal{H}$. Given an ample test configuration $(\mathcal{X}, \mathcal{L})$ (viewed as an element of $\mathcal{H}^{\mathrm{NA}}$) and a smooth ray $(\phi_s)_{s\in(0,+\infty)}$ corresponding to a smooth $S^1$-invariant metric on $\mathcal{L}$, we shall prove in [BHJ15] that

$$\lim_{s\to+\infty} \frac{F(\phi_s)}{s} = F^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}), \tag{0.2}$$

[1] The interpretation of the Donaldson-Futaki invariant as a non-Archimedean Mabuchi functional has been known to Shou-Wu Zhang for quite some time, cf. [PRS08, Remark 6].

where $F$ denotes the Monge-Ampère energy, $J$-energy, entropy, or Mabuchi energy functional and $F^{\mathrm{NA}}$ is the corresponding non-Archimedean functional defined above. In the case of the Mabuchi energy, this result is closely related to [PT06, PT09, PRS08], but to the best of our knowledge, the precise expression for the difference between the slope at infinity of $M$ and the Donaldson-Futaki invariant has not appeared so far in the literature.

**Singularities of pairs and uniform K-stability.** A key point in our approach to K-stability is to relate the birational geometry of $X$ and that of its test configurations using the language of *valuations*.

More specifically, let $(X, L)$ be a normal polarized variety, and $(\mathcal{X}, \mathcal{L})$ a normal test configuration. Every irreducible component $E$ of $\mathcal{X}_0$ defines a divisorial valuation $\mathrm{ord}_E$ on the function field of $\mathcal{X}$. Since the latter is canonically isomorphic to $k(X \times \mathbb{A}^1) \simeq k(X)(t)$, we may consider the restriction $r(\mathrm{ord}_E)$ of $\mathrm{ord}_E$ to $k(X)$, which is proved to be a divisorial valuation as well when $E$ is non-trivial, i.e. not the strict transform of the central fiber of the trivial test configuration.

This correspondence between irreducible components of $\mathcal{X}_0$ and divisorial valuations on $X$ is analyzed in detail in §4. In particular, we prove that the Rees valuations of a closed subscheme $Z \subset X$, i.e. the divisorial valuations associated to the normalized blow-up of $X$ along $Z$, coincide with the valuations induced on $X$ by the normalization of the deformation to the normal cone of $Z$.

Given a boundary $B$ on $X$, we define the *non-Archimedean entropy* of a normal test configuration $(\mathcal{X}, \mathcal{L})$ as

$$H_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = V^{-1} \sum_E A_{(X,B)}(r(\mathrm{ord}_E))(E \cdot \mathcal{L}^n),$$

the sum running over the non-trivial components of $\mathcal{X}_0$ and $A_{(X,B)}(v)$ denoting the log discrepancy of a divisorial valuation $v$ with respect the pair $(X, B)$. Recall that the pair $(X, B)$ is log canonical (lc for short) if $A_{(X,B)}(v) \geq 0$ for all divisorial valuations on $X$, and Kawamata log terminal (klt for short) if the inequality is everywhere strict. Our main result here is a characterization of these singularity classes in terms of the non-Archimedean entropy functional.

**Theorem B.** *Let $(X, L)$ be a normal polarized variety, and $B$ an effective boundary on $X$. Then $(X, B)$ is lc (resp. klt) iff $H_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq 0$ (resp. $> 0$) for every non-trivial normal, ample test configuration $(\mathcal{X}, \mathcal{L})$. In the klt case, there automatically exists $\delta > 0$ such that $H_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ for all $(\mathcal{X}, \mathcal{L})$.*

The strategy to prove the first two points is closely related to that of [Oda13b]. In fact, we also provide a complete proof of the following mild generalization (in the normal case) of the main result of *loc.cit*:

$$((X, B), L) \text{ K-semistable} \implies (X, B) \text{ lc.}$$

If $(X, B)$ is not lc (resp. not klt), then known results from the Minimal Model Program allow us to construct a closed subscheme $Z$ whose Rees valuations have negative (resp. non-positive) discrepancies; the normalization of the deformation to the normal cone of $Z$ then provides a test configuration $(\mathcal{X}, \mathcal{L})$ with $H_B^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) < 0$ (resp. $\leq 0$). To prove uniformity in the klt case, we exploit the the strict positivity of the *global log canonical threshold* $\mathrm{lct}((X, B), L)$ of $((X, B), L)$.

As a consequence, we are able to analyze J-uniform K-stability in the 'log Kähler-Einstein case', i.e. when $K_X + B$ is numerically proportional to $L$.

**Corollary C.** *Let $(X, L)$ be a normal polarized variety, $B$ an effective boundary, and assume that $K_X + B \equiv \lambda L$ with $\lambda \in \mathbb{Q}$.*

(i) *If $\lambda > 0$, then $((X, B), L)$ is J-uniformly K-stable iff $(X, B)$ is lc;*
(ii) *If $\lambda = 0$, then $((X, B), L)$ is J-uniformly K-stable iff $(X, B)$ is klt;*
(iii) *If $\lambda < 0$ and $\mathrm{lct}((X, B), L) > \frac{n}{n+1}|\lambda|$, then $((X, B), L)$ is J-uniformly K-stable.*

**Relation to other works.** Since we aim to give a systematic introduction to uniform $K$-stability, and to set up some non-Archimedean terminology, we have tried to make the exposition as self-contained as possible. This means that we reprove or mildly generalize some already known results [Oda12, Oda13b, OSa12, Sun13, OSu11].

During the preparation of the present work, we were informed of R. Dervan's independent work [Der14a](see also [Der14b]), which has a substantial overlap with the present paper. First, test configurations with trivial $L^2$-norm were also characterized in [Der14a, Theorem 1.2]. Next, the *minimum norm* introduced in *loc.cit* turns out to coincide (up to a unimportant normalizing constant) with our non-Archimedean J-functional. As a result, *uniform K-stability with respect to the minimum norm* as in [Der14a] is the same as our concept of J-uniform K-stability. Finally, Corollary C above is to a large extent contained in [Der14a, §3].

**Structure of the paper.** Section 1 gathers a number of preliminary facts on filtrations and valuations, with a special emphasis on the Rees construction and the relation between Rees valuations and integral closure.

Sections 2 and 3 give a fairly self-contained treatment of test configurations and Donaldson-Futaki invariants. We discuss in particular some scheme theoretic aspects, and Proposition 3.7 provides an explicit expression for the difference between the Donaldson-Futaki invariant of a test configuration and of its normalization.

The correspondence between irreducible components of the central fiber of a normal test configurations and divisorial valuations on $X$ is considered in Section 4. In particular, Theorem 4.8 relates Rees valuations and the deformation to the normal cone.

In Section 5 we define a notion of non-Archimedean metric on $L$ as an equivalence class of test configurations. This is inspired by [BFJ12, BFJ15a].

Section 6 is dedicated to Duistermaat-Heckman measures. Existence is established in Theorem 6.1, and Theorem A is a consequence of Proposition 6.15, Theorem 6.16 and Theorem 6.19.

Section 7 introduces the non-Archimedean analogues of the usual energy functionals, viewed as functionals on the space of non-Archimedean metrics. We define J-uniform K-stability, and compare it to $L^p$-uniform K-stability in the sense of Székelyhidi.

Section 8 is concerned with the interaction between uniform K-stability and singularities. Theorem 8.1 and Theorem 8.2 establish Theorem B as well as the generalization of [Oda13b] mentioned above. Corollary C is a combination of Corollary 8.3, Corollary 8.4 and Proposition 8.16.

Finally, Appendix A provides a proof of the two-term Riemann-Roch theorem on a normal variety, whose complete proof we could not locate in the literature, and Appendix B summarizes Edidin and Graham's equivariant version of the Riemann-Roch theorem for schemes.

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## 1. Preliminary facts on filtrations and valuations

We work over an algebraically closed field $k$, whose characteristic is arbitrary unless otherwise specified, and write $\mathbb{G}_m$ for the multiplicative group over $k$ and $\mathbb{A}^1 = \operatorname{Spec} k[t]$ for the affine line.

All schemes are assumed to be separated and of finite type over $k$, and a *variety* is an integral (i.e. reduced and irreducible) scheme. By an *ideal* on a scheme $X$ we mean a coherent ideal sheaf, whereas a *fractional ideal* is a coherent $\mathcal{O}_X$-submodule of the sheaf of rational functions.

1.1. **Norms and filtrations.** Let $V$ be a finite dimensional $k$-vector space. In this paper, a *filtration* of $V$ will mean a decreasing, left-continuous, separating and exhaustive $\mathbb{R}$-indexed filtration $F^\bullet V$. In other words, it is a family of subspaces $(F^\lambda V)_{\lambda\in\mathbb{R}}$ of $V$ such that

(i) $F^\lambda V \subset F^{\lambda'} V$ when $\lambda \ge \lambda'$;
(ii) $F^\lambda V = \bigcap_{\lambda'<\lambda} F^{\lambda'} V$;
(iii) $F^\lambda V = 0$ for $\lambda \gg 0$;
(iv) $F^\lambda V = V$ for $\lambda \ll 0$.

A $\mathbb{Z}$-*filtration* is a filtration $F^\bullet V$ such that $F^\lambda V = F^{\lceil\lambda\rceil} V$ for $\lambda \in \mathbb{R}$. Equivalently, it is a family of subspaces $(F^\lambda V)_{\lambda\in\mathbb{Z}}$ satisfying (i), (iii) and (iv) above.

With these conventions, filtrations are in one-to-one correspondence with non-Archimedean norms on $V$ compatible with the trivial absolute value on $k$, i.e. functions $\|\cdot\|\colon V \to \mathbb{R}_+$ such that

(i) $\|s+s'\| \le \max\{\|s\|, \|s'\|\}$ for all $s, s' \in V$;
(ii) $\|cs\| = \|s\|$ for all $s \in V$ and $c \in k^*$;
(iii) $\|s\| = 0 \iff s = 0$.

The correspondence is given by

$$-\log\|s\| = \sup\left\{\lambda\in\mathbb{R} \mid s \in F^\lambda V\right\} \quad \text{and} \quad F^\lambda V = \left\{s \in V \mid \|s\| \le e^{-\lambda}\right\}.$$

The *successive minima* of the filtration $F^\bullet V$ is the decreasing sequence

$$\lambda_{\max} = \lambda_1 \ge \cdots \ge \lambda_N = \lambda_{\min}$$

where $N = \dim V$, defined by

$$\lambda_j = \max\left\{\lambda \in \mathbb{R} \mid \dim F^\lambda V \ge j\right\}.$$

From the point of view of the norm, they are indeed the analogues of the (logarithmic) successive minima in Minkowski's geometry of numbers.

The graded pieces of a filtration $F^\bullet V$ are defined as $\mathrm{Gr}^F_\lambda V = F^\lambda V / F^{>\lambda} V$ with $F^{>\lambda} V = \bigcup_{\lambda' > \lambda} F^{\lambda'} V$. We then have

$$-\frac{d}{d\lambda} \dim F^\lambda V = \sum_{\lambda \in \mathbb{R}} \left(\dim \mathrm{Gr}^F_\lambda V\right) \delta_\lambda = \sum_{j=1}^{N} \delta_{\lambda_j}. \tag{1.1}$$

in the sense of distributions.

Next let $R := \bigoplus_{m \in \mathbb{N}} R_m$ be a graded $k$-algebra with finite dimensional graded pieces $R_m$. A filtration $F^\bullet R$ of $R$ is defined as the data of a filtration $F^\bullet R_m$ for each $m$, satisfying

$$F^\lambda R_m \cdot F^{\lambda'} R_{m'} \subset F^{\lambda + \lambda'} R_{m+m'}$$

for all $\lambda, \lambda' \in \mathbb{R}$ and $m, m' \in \mathbb{N}$. The data of $F^\bullet R$ is equivalent to the data of a non-Archimedean submultiplicative norm $\|\cdot\|$ on $R$, i.e. a non-Archimedean norm $\|\cdot\|_m$ as above on each $R_m$, satisfying

$$\|s \cdot s'\|_{m+m'} \le \|s\|_m \|s'\|_{m'}$$

for all $s \in R_m$, $s' \in R_{m'}$. We will use the following terminology.

**Definition 1.1.** *We say that a $\mathbb{Z}$-filtration $F^\bullet R$ of a graded algebra $R$ is* finitely generated *if the bigraded algebra*

$$\bigoplus_{(\lambda, m) \in \mathbb{Z} \times \mathbb{N}} F^\lambda R_m$$

*is finitely generated over $k$.*

The condition equivalently means that the graded $k[t]$-algebra

$$\bigoplus_{m \in \mathbb{N}} \left( \bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} F^\lambda R_m \right)$$

is finitely generated.

1.2. **The Rees construction.** We review here a classical construction due to Rees, which yields a geometric interpretation of $\mathbb{Z}$-filtrations.

Consider first a $\mathbb{Z}$-filtration $F^\bullet V$ of a $k$-vector space $V$. Then

$$\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} F^\lambda V$$

is a torsion free, finitely generated $k[t]$-module. It can thus be written as the space of global sections of a unique vector bundle $\mathcal{V}$ on $\mathbb{A}^1 = \operatorname{Spec} k[t]$. The grading provides a $\mathbb{G}_m$-linearization of $\mathcal{V}$. We claim that we have $\mathbb{G}_m$-equivariant isomorphisms

$$\mathcal{V}|_{\mathbb{A}^1 \setminus \{0\}} \simeq V \times \left(\mathbb{A}^1 \setminus \{0\}\right) \tag{1.2}$$

with $t^{-\lambda} F^\lambda V$ corresponding to the weight-$\lambda$ part of $\mathcal{V}$, and

$$\mathcal{V}_0 \simeq \mathrm{Gr}^F_\bullet V = \bigoplus_{\lambda \in \mathbb{Z}} F^\lambda V / F^{\lambda+1} V. \tag{1.3}$$

Intuitively, $\mathcal{V}$ may be thought of as a way to degenerate the filtration to its graded object.

To see that (1.2) holds, consider the $k$-linear map $\pi \colon H^0(\mathbb{A}^1, \mathcal{V}) \to V$ sending $\sum_\lambda t^{-\lambda} v_\lambda$ to $\sum_\lambda v_\lambda$. This map is surjective since $F^\lambda V = V$ for $\lambda \ll 0$. If $\sum_\lambda t^{-\lambda} v_\lambda$ lies in the kernel, then $v_\lambda = w_{\lambda+1} - w_\lambda$ for all $\lambda$, where $w_\lambda = -\sum_{\mu\geq\lambda} v_\mu \in F^\lambda V$. Conversely, any element of the form $\sum_\lambda t^{-\lambda}(w_{\lambda+1} - w_\lambda)$, where $w_\lambda \in F^\lambda V$, is in the kernel of $\pi$, and the set of such elements is equal to $(t-1)H^0(\mathbb{A}^1, \mathcal{V})$. Thus $\pi$ induces an isomorphism between $\mathcal{V}_1 = H^0(\mathbb{A}^1, \mathcal{V})/(t-1)H^0(\mathbb{A}^1, \mathcal{V})$ and $V$, which induces (1.2) using the $\mathbb{G}_m$-action. The proof of (1.3) is similar.

If we start conversely with a $\mathbb{G}_m$-linearized vector bundle $\mathcal{V}$ on $\mathbb{A}^1$, we get a $\mathbb{Z}$-filtration on $V := \mathcal{V}_1$, by letting $F^\lambda V$ be the image of the weight-$\lambda$ part of $H^0(\mathbb{A}^1, \mathcal{V})$ under the restriction map $H^0(\mathbb{A}^1, \mathcal{V}) \to V$. Since $t$ has weight $-1$ with respect to the $\mathbb{G}_m$-action on $\mathbb{A}^1$, multiplication by $t$ induces an injection $F^{\lambda+1}V \subset F^\lambda V$, so that this is indeed a decreasing filtration. Further, $t^{-\lambda}F^\lambda V$ can be identified with the weight-$\lambda$ part of $H^0(\mathbb{A}^1, \mathcal{V})$. Using the above facts, it is straightforward to deduce:

**Proposition 1.2.** *The above constructions define an equivalence of categories between $\mathbb{Z}$-filtered, finite dimensional vector spaces and $\mathbb{G}_m$-linearized vector bundles on $\mathbb{A}^1$.*

Every filtered vector space admits a basis compatible with the filtration, and is thus (non-canonically) isomorphic to its graded object. On the geometric side, this yields (compare [Don05, Lemma 2]):

**Corollary 1.3.** *Every $\mathbb{G}_m$-linearized vector bundle $\mathcal{V}$ on $\mathbb{A}^1$ is $\mathbb{G}_m$-equivariantly trivial,* i.e. *$\mathbb{G}_m$-isomorphic to $\mathcal{V}_0 \times \mathbb{A}^1$ with $\mathcal{V}_0$ the fiber at $0$.*

For line bundles, the trivialization admits the following particularly simple description.

**Corollary 1.4.** *Let $\mathcal{L}$ be a $\mathbb{G}_m$-linearized line bundle on $\mathbb{A}^1$, and let $\lambda \in \mathbb{Z}$ be the weight of the $\mathbb{G}_m$-action on $\mathcal{L}_0$. For each non-zero $v \in \mathcal{L}_1$, setting $s(t) := t^{-\lambda}(t \cdot v)$ defines a weight-$\lambda$ trivialization of $\mathcal{L}$.*

*Proof.* While this is a special case of the above construction, it can be directly checked as follows. The section $s' \in H^0(\mathbb{A}^1 \setminus \{0\}, \mathcal{L})$ defined by $s'(t) := t \cdot v$ defines a rational section of $\mathcal{L}$. If we set $\mu := \mathrm{ord}_0(s')$, then $v_0 := \lim_{z\to 0} z^{-\mu} s'(z)$ is a non-zero element of $\mathcal{L}_0$, which satisfies

$$t \cdot v_0 = \lim_{z\to 0} z^{-\mu}\left((tz)\cdot v\right) = t^\mu \lim_{z\to 0}(tz)^{-\mu}\left((tz)\cdot v\right) = t^\mu v_0.$$

It follows that $\mu$ coincides with the weight $\lambda$ of the $\mathbb{G}_m$-action on $\mathcal{L}_0$. $\square$

**Remark 1.5.** *Much more generally, given a reductive algebraic group $G$, the equivariant Serre problem asks whether every $G$-linearized vector bundle on an affine space $\mathbb{A}^n$ with a linear action of $G$ is necessarily $G$-equivariantly trivial. The famous Quillen-Suslin theorem gives a positive answer when $G$ is trivial, and this is more generally true whenever $G$ is commutative,* i.e. *the product of an algebraic torus with a finite abelian group, cf.* [MMP96]. *On the other hand, the answer is negative for all non-commutative connected reductive groups, and also for some (non-commutative) finite groups.*

1.3. **Valuations.** Let $K$ be a finitely generated field extension of $k$, with $n := \mathrm{tr.deg}\, K/k$, so that $K$ may be realized as the function field of a (normal, projective) $n$-dimensional variety.

Since we only consider real-valued valuations, we simply call *valuation* $v$ on $K$ a group homomorphism $v : K^* \to (\mathbb{R}, +)$ such that $v(f+g) \geq \min\{v(f), v(g)\}$ and $v|_{k^*} \equiv 0$ [ZS]. It is convenient to set $v(0) = +\infty$. The *trivial valuation* $v_{\mathrm{triv}}$ is defined by $v_{\mathrm{triv}}(f) = 0$ for all $f \in K^*$. To each valuation $v$ is attached the following list of invariants. The *valuation ring* of $v$ is $\mathcal{O}_v := \{f \in K \mid v(f) \geq 0\}$. This is a local ring with maximal ideal $\mathfrak{m}_v := \{f \in K \mid v(f) > 0\}$, and the *residue field* of $v$ is $k(v) := \mathcal{O}_v/\mathfrak{m}_v$. The *transcendence degree* of $v$ (over $k$) is $\operatorname{tr.deg}(v) := \operatorname{tr.deg} k(v)/k$. Finally, the *value group* of $v$ is $\Gamma_v := v(K^*) \subset \mathbb{R}$, and the *rational rank* of $v$ is $\operatorname{rat.rk}(v) := \dim_{\mathbb{Q}} (\Gamma_v \otimes \mathbb{Q})$.

If $k \subset K' \subset K$ is an intermediate field extension, $v$ is a valuation on $K$ and $v'$ is its restriction to $K'$, the Abhyankar-Zariski inequality states that

$$\operatorname{tr.deg}(v) + \operatorname{rat.rk}(v) \leq \operatorname{tr.deg}(v') + \operatorname{rat.rk}(v') + \operatorname{tr.deg} K/K'. \tag{1.4}$$

Taking $K' = k$, we get $\operatorname{tr.deg}(v) + \operatorname{rat.rk}(v) \leq n$, and we say that $v$ is an *Abhyankar valuation* if equality holds; such valuations can be geometrically characterized, see [KK05, ELS03, JM12]. In particular, the trivial valuation is Abhyankar; it is the unique valuation with transcendence degree $n$. We say that $v$ is *divisorial* if $\operatorname{rat.rk}(v) = 1$ and $\operatorname{tr.deg}(v) = n - 1$. By a theorem of Zariski, this is the case iff there exists a normal projective variety $Y$ with $k(Y) = K$ and a prime divisor $F$ of $Y$ such that $v = c \operatorname{ord}_F$ for some $c > 0$. We then have $k(v) = k(F)$ and $\Gamma_v = c\mathbb{Z}$.

If $X$ is a variety with $k(X) = K$, a valuation $v$ is *centered on* $X$ if there exists a scheme point $\xi \in X$ such that $v \geq 0$ on the local ring $\mathcal{O}_{X,\xi}$ and $v > 0$ on its maximal ideal. We also say $v$ is a valuation on $X$ in this case. By the valuative criterion of separatedness, the point $\xi$ is unique, and is called the *center* of $v$ on $X$. If $X$ is proper, the valuative criterion of properness guarantees that any $v$ is centered on $X$. If a divisorial valuation $v$ is centered on $X$, then $v = c \operatorname{ord}_F$ where $F$ is a prime divisor on a normal variety $Y$ with a proper birational morphism $\mu : Y \to X$; the center of $v$ on $X$ is then the generic point of $\mu(F)$.

For any valuation $v$ centered on $X$, we can make sense of $v(s) \in \mathbb{R}_+$ for a (non-zero) section $s \in H^0(X, L)$ of a line bundle $L$ on $X$ by trivializing $L$ at the center $\xi$ of $v$ on $X$ and evaluating $v$ on the local function corresponding to $s$ in this trivialization. Since any two such trivializations differ by a unit at $\xi$, $v(s)$ is well-defined, and $v(s) > 0$ iff $s(\xi) = 0$.

Similarly, given an ideal $\mathfrak{a} \subset \mathcal{O}_X$ we set

$$v(\mathfrak{a}) = \inf\{v(f) \mid f \in \mathfrak{a}_\xi\}.$$

It is in fact enough to take the min over any finite set of generators of $\mathfrak{a}_\xi$. We also set $v(Z) := v(\mathfrak{a})$, where $Z$ is the closed subscheme defined by $\mathfrak{a}$.

Finally, for later use we record the following simple variant of [HS, Theorem 10.1.6].

**Lemma 1.6.** *Assume that $X = \operatorname{Spec} A$ is affine. Let $S$ be a finite set of valuations on $X$, which is irredundant in the sense that for each $v \in S$ there exists $f \in A$ with $v(f) < v'(f)$ for all $v' \in S \setminus \{v\}$. Then $S$ is uniquely determined by the function $h_S(f) := \min_{v \in S} v(f)$.*

*Proof.* Let $S$ and $T$ be two irredundant finite sets of valuations with $h_S = h_T =: h$. For each $v \in S$, $w \in T$ set $C_v := \{f \in A \mid h(f) = v(f)\}$ and $D_w := \{f \in A \mid h(f) = w(f)\}$, and observe that these sets are stable under finite products. For each $v \in S$, we claim that there exists $w \in T$ with $C_v \subset D_w$. Otherwise, for each $w$ there exists $f_w \in C_v \setminus D_w$, i.e. $v(f_w) = h(f_w) < w(f_w)$. Setting $f = \prod_w f_w$, we get for each $w' \in T$

$$w'(f) = \sum_{w \in T} w'(f_w) > \sum_{w \in S} h(f_w) = \sum_{w \in S} v(f_w) = v(f) \geq h(f),$$

and taking the min over $w' \in T$ yields a contradiction.

We next claim that $C_v \subset D_w$ implies that $v = w$. This will prove that $S \subset T$, and hence $S = T$ by symmetry. Note first that $v(f) = h(f) = w(f)$ for each $f \in C_v$. Now choose $g_v \in A$ with $v(g_v) < v'(g_v)$ for all $v' \neq v$ in $S$, so that $g_v \in C_v \subset D_w$. For each $f \in A$, we then have $v(g_v^m f) < v'(g_v^m f)$ for $m \gg 1$, and hence $g_v^m f \in C_v \subset D_w$. It follows that

$$mv(g_v) + v(f) = v(g_v^m f) = w(g_v^m f) = mw(g_v) + w(f) = mv(g_v) + w(f),$$

and hence $v(f) = w(f)$. □

1.4. **Integral closure and Rees valuations.** Let $X$ be a scheme and $Z \subset X$ a closed subscheme with ideal $\mathfrak{a} \subset \mathcal{O}_X$. On the one hand, the *normalized blow-up* $\pi : \widetilde{X} \to X$ along $Z$ is the composition of the blow-up of $Z$ in $X$ with the normalization morphism. On the other hand, the *integral closure* $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is the set of elements $f \in \mathcal{O}_X$ satisfying a monic equation $f^d + a_1 f^{d-1} + \cdots + a_d = 0$ with $a_j \in \mathfrak{a}^j$.

The following well-known connection between normalized blow-ups and integral closures shows in particular that $\overline{\mathfrak{a}}$ is a coherent ideal sheaf.

**Lemma 1.7.** *Let $Z \subset X$ be a closed subscheme, with ideal $\mathfrak{a} \subset \mathcal{O}_X$, and let $\pi : \widetilde{X} \to X$ be the normalized blow-up along $Z$. Then $D := \pi^{-1}(Z)$ is an effective Cartier divisor with $-D$ $\pi$-ample, and we have for each $m \in \mathbb{N}$:*

(i) *$\mathcal{O}_{\widetilde{X}}(-mD)$ is $\pi$-globally generated;*
(ii) *$\pi_* \mathcal{O}_{\widetilde{X}}(-mD) = \overline{\mathfrak{a}^m}$;*
(iii) *$\mathcal{O}_{\widetilde{X}}(-mD) = \mathcal{O}_{\widetilde{X}} \cdot \overline{\mathfrak{a}^m} = \mathcal{O}_{\widetilde{X}} \cdot \mathfrak{a}^m$;*

*In particular, $\pi$ coincides with the normalized blow-up of $\overline{\mathfrak{a}}$, and also with the (usual) blow-up of $\overline{\mathfrak{a}^m}$ for any $m \gg 1$.*

We recall the brief argument for the convenience of the reader.

*Proof.* Let $\mu : X' \to X$ be the blow-up along $Z$, so that $\mu^{-1}(Z) = D'$ is a Cartier divisor on $X'$ with $-D'$ $\mu$-very ample, and hence $\mathcal{O}_{X'}(-mD')$ $\mu$-globally generated for all $m \in \mathbb{N}$. Denoting by $\nu : \widetilde{X} \to X'$ the normalization morphism, we have $\nu^* D' = D$. Since $\nu$ is finite, it follows that $-D$ is $\pi$-ample and satisfies (i), which reads $\mathcal{O}_{\widetilde{X}}(-mD) = \mathcal{O}_{\widetilde{X}} \cdot \mathfrak{a}_m$ with

$$\mathfrak{a}_m := \pi_* \mathcal{O}_{\widetilde{X}}(-mD).$$

It therefore remains to establish (ii). By normality of $\widetilde{X}$, $\mathcal{O}_{\widetilde{X}}(-mD)$ is integrally closed, hence so is $\mathfrak{a}_m = \pi_* \mathcal{O}_{\widetilde{X}}(-mD)$. As $\mathfrak{a} \subset \mathfrak{a}_1$, we have $\mathfrak{a}^m \subset \mathfrak{a}_1^m \subset \mathfrak{a}_m$, and hence $\overline{\mathfrak{a}^m} \subset \mathfrak{a}_m$.

The reverse inclusion requires more work; we reproduce the elegant geometric argument of [Laz, II.11.1.7]. Fix $m \geq 1$. As the statement is local over $X$, we may choose a system of generators $(f_1, \dots, f_p)$ for $\mathfrak{a}^m$. This defines a surjection $\mathcal{O}_X^{\oplus p} \to \mathfrak{a}^m$, which induces, after pull-back and twisting by $-lD$, a surjection

$$\mathcal{O}_{\widetilde{X}}(-lD)^{\oplus p} \to \mathcal{O}_{\widetilde{X}}\left(-(m+l)D\right) = \mathfrak{a}^m \cdot \mathcal{O}_{\widetilde{X}}(-lD)$$

for any $l \geq 1$. Since $-D$ is $\pi$-ample, Serre vanishing implies that the induced map

$$\mathfrak{a}_l^{\oplus p} = \pi_* \mathcal{O}_{\widetilde{X}}\left(-lD\right)^{\oplus p} \to \mathfrak{a}_{(m+l)} = \pi_* \mathcal{O}_{\widetilde{X}}\left(-(m+l)mD\right)$$

is also surjective for $l \gg 1$, i.e. $\mathfrak{a}^m \cdot \mathfrak{a}_l = \mathfrak{a}_{m+l}$. But since $\mathfrak{a}_{m+l} \supset \mathfrak{a}_m \cdot \mathfrak{a}_l \supset \mathfrak{a}^m \cdot \mathfrak{a}_l$, $\mathfrak{a}_m$ acts on the finitely generated $\mathcal{O}_X$-module $\mathfrak{a}_l$ by multiplication by $\mathfrak{a}^m$, and the usual determinant trick therefore yields $\mathfrak{a}_m \subset \overline{\mathfrak{a}^m}$. □

Assume from now on that $X$ is integral, i.e. a variety.

**Definition 1.8.** *Let $Z \subset X$ be a closed subscheme with ideal $\mathfrak{a}$, and let $\pi : \widetilde{X} \to X$ be the normalized blow-up of $Z$, with $D := \pi^{-1}(Z)$. The* Rees valuations *of $Z$ (or $\mathfrak{a}$) are the divisorial valuations $v_E = \frac{\mathrm{ord}_E}{\mathrm{ord}_E(D)}$, where $E$ runs over the irreducible components of $D$.*

Note that $v_E(Z) = v_E(\mathfrak{a}) = v_E(D) = 1$ for all $E$. We now show that the present definition of Rees valuations coincides with the standard one in valuation theory (see for instance [HS, Chapter 5]). The next result is a slightly more precise version of [HS, Theorem 2.2.2, (3)].

**Theorem 1.9.** *The set of Rees valuations of $\mathfrak{a}$ is the unique finite set $S$ of valuations such that:*

(i) $\overline{\mathfrak{a}^m} = \bigcap_{v \in S} \{f \in \mathcal{O}_X \mid v(f) \geq m\}$ *for all $m \in \mathbb{N}$;*

(ii) *$S$ is minimal with respect to (i).*

*Proof of Theorem 1.9.* For each finite set of valuations $S$, set $h_S(f) := \min_{v \in S} v(f)$. Using that $h_S(f^m) = m h_S(f)$, it is straightforward to check that any two sets $S$, $S'$ satisfying (i) have $h_S = h_{S'}$. If $S$ and $S'$ further satisfy (ii), then they are irredundant in the sense of Lemma 1.6, which therefore proves that $S = S'$.

It remains to check that the set $S$ of Rees valuations of $Z$ satisfies (i) and (ii). The first property is merely a reformulation of Lemma 1.7. Now pick an irreducible component $E$ of $D$. It defines a fractional ideal $\mathcal{O}_{\widetilde{X}}(E)$. Since $-D$ is $\pi$-ample, $\mathcal{O}_{\widetilde{X}}(-mD)$ and $\mathcal{O}_{\widetilde{X}}(-mD) \cdot \mathcal{O}_{\widetilde{X}}(E)$ both become $\pi$-globally generated for $m \gg 1$. Since $\mathcal{O}_{\widetilde{X}}(-mD)$ is strictly contained in $\mathcal{O}_{\widetilde{X}}(-mD) \cdot \mathcal{O}_{\widetilde{X}}(E)$, it follows that $\overline{\mathfrak{a}^m} = \pi_* \mathcal{O}_{\widetilde{X}}(-mD)$ is strictly contained in

$$\pi_* \left( \mathcal{O}_{\widetilde{X}}(-mD) \cdot \mathcal{O}_{\widetilde{X}}(E) \right) \subset \bigcap_{E' \neq E} \{f \in \mathcal{O}_X \mid v_{E'}(f) \geq m\},$$

which proves (ii). □

**Example 1.10.** *The Rees valuations of an effective Weil divisor $D = \sum_{i=1}^m a_i D_i$ on a normal variety $X$ are given by $v_i := \frac{1}{a_i} \mathrm{ord}_{D_i}$, $1 \leq i \leq m$.*

We end this section on Rees valuations with the following result.

**Proposition 1.11.** *Let $\pi : Y \to X$ be a projective birational morphism between normal varieties, and assume that $Y$ admits a Cartier divisor that is both $\pi$-exceptional and $\pi$-ample. Then $\pi$ is isomorphic to the blow-up of $X$ along a closed subscheme $Z$ of codimension at least 2, and the divisorial valuations $\mathrm{ord}_F$ defined by the $\pi$-exceptional prime divisors $F$ on $Y$ coincide, up to scaling, with the Rees valuations of $Z$.*

This is indeed a direct consequence of the following well-known facts.

**Lemma 1.12.** *Let $\pi : Y \to X$ be a projective birational morphism between varieties with $X$ normal. If $G$ is a $\pi$-exceptional, $\pi$-ample Cartier divisor, then:*

(i) *$-G$ is effective;*

(ii) $\mathrm{supp}\, G$ *coincides with the exceptional locus of $\pi$;*

(iii) *for $m$ divisible enough, $\pi$ is isomorphic to the blow-up of the ideal $\mathfrak{a}_m := \pi_* \mathcal{O}_Y(mG)$, whose zero locus has codimension at least 2.*

*Conversely, the blow-up of $X$ along a closed subscheme of codimension at least 2 admits a $\pi$-exceptional, $\pi$-ample Cartier divisor.*

*Proof.* Set $\mathfrak{a}_m := \pi_* \mathcal{O}_Y(mG)$, viewed as a fractional ideal on $X$. Since $G$ is $\pi$-exceptional, every rational function in $\pi_*\mathcal{O}_Y(mG)$ is regular in codimension 1, and $\mathfrak{a}_m$ is thus an ideal whose zero locus has codimension at least 2, by the normality of $X$.

If we choose $m \gg 1$ such that $\mathcal{O}_Y(mG)$ is $\pi$-globally generated, then we have $\mathcal{O}_Y(mG) = \mathcal{O}_Y \cdot \mathfrak{a}_m \subset \mathcal{O}_Y$, which proves (i).

By assumption, $\operatorname{supp} G$ is contained in the exceptional locus $E$ of $\pi$. Since $X$ is normal, $\pi$ has connected fibers by Zariski's main theorem, so $E$ is the union of all projective curves $C \subset Y$ that are mapped to a point of $X$. Any such curve satisfies $G \cdot C > 0$ by the relative ampleness of $G$, and hence $C \subset \operatorname{supp} G$ since $-G$ is effective. Thus $\operatorname{supp} G = E$, proving (ii).

Finally, the relative ampleness of $G$, implies that the $\mathcal{O}_X$-algebra $\bigoplus_{m\in\mathbb{N}} \mathfrak{a}_m$ is finitely generated, and its relative Proj over $X$ is isomorphic to $Y$. The finite generation implies $\bigoplus_{l\in\mathbb{N}} \mathfrak{a}_{ml} = \bigoplus_{l\in\mathbb{N}} \mathfrak{a}_m^l$ for all $m$ divisible enough, and applying $\operatorname{Proj}_X$ shows that $X$ is isomorphic to the blow-up of $X$ along $\mathfrak{a}_m$. □

1.5. **Boundaries and log discrepancies.** Let $X$ be a normal variety. In the Minimal Model Program (MMP) terminology, a *boundary* $B$ on $X$ is a $\mathbb{Q}$-Weil divisor (i.e. a codimension one cycle with rational coefficients) such that $K_X + B$ is $\mathbb{Q}$-Cartier. Alternatively, one says that $(X, B)$ is a *pair* to describe this condition, and $K_X + B$ is called the *log canonical divisor* of this pair. In particular, 0 is a boundary iff $X$ is $\mathbb{Q}$-Gorenstein.

To any divisorial valuation $v$ on $X$ is associated its *log discrepancy* with respect to the pair $(X, B)$, denoted by $A_{(X,B)}(v)$ and defined as follows. For any normal birational model $\mu : Y \to X$ and prime divisor $F$ of $Y$ such that $v = c \operatorname{ord}_F$, we set

$$A_{(X,B)}(v) := c\left(1 + \operatorname{ord}_F\left(K_{Y/(X,B)}\right)\right)$$

with $K_{Y/(X,B)} := K_Y - \mu^*(K_X + B)$. This is well-defined (i.e. independent of the choice of $\mu$), by compatibility of canonical divisor classes under push-forward. By construction, $A_{(X,B)}$ is homogeneous with respect to the natural action of $\mathbb{R}_+^*$ on divisorial valuations by scaling, i.e. $A_{(X,B)}(c\,v) = cA_{(X,B)}(v)$ for all $c > 0$.

As a real valued function on $k(X)^*$, $c\,v$ converges pointwise to the trivial valuation $v_{\mathrm{triv}}$ as $c \to 0$. It is thus natural to set $A_{(X,B)}(v_{\mathrm{triv}}) := 0$.

## 2. Test configurations

2.1. **Terminology and notation.** In what follows, $(X, L)$ is a pair consisting of a proper scheme $X$ over $k$ and a line bundle $L$ on $X$. Given a scheme $S$ over $k$, we denote by $(X_S, L_S)$ the base change to $S$, i.e. $X_S := X \times_k S$ and $L_S := p^*L$, with $p : X \times_k S \to X$ the projection.

Most often, $(X, L)$ will be polarized, i.e. $L$ will be ample, but it is sometimes useful to consider the general case. Similarly, it will be convenient to allow some flexibility in the definition of test configurations; we shall use the following terminology.

**Definition 2.1.** *A test configuration $\mathcal{X}$ for $X$ consists of the following data:*

(i) *a flat and proper morphism of schemes $\pi : \mathcal{X} \to \mathbb{A}^1$;*
(ii) *a $\mathbb{G}_m$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{A}^1$;*
(iii) *an isomorphism $\mathcal{X}_1 \simeq X$.*

*A test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ comprises as additional data:*

(iv) *a $\mathbb{G}_m$-linearized $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$;*
(v) *an isomorphism $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L)$ extending the one in (iii).*

We say that $(\mathcal{X},\mathcal{L})$ is ample, semiample, ... (resp. normal, $S_1$, ... ) when $\mathcal{L}$ (resp. $\mathcal{X}$) has this property. By Proposition 2.7 below, $\mathcal{X}$ is in fact automatically a variety (i.e. reduced and irreducible) when $X$ is.

By a $\mathbb{G}_m$-linearized $\mathbb{Q}$-line bundle $\mathcal{L}$ as in (iv), we mean that $r\mathcal{L}$ is an actual $\mathbb{G}_m$-linearized line bundle for some $r\in\mathbb{Z}_{>0}$ that is not part of the data. The isomorphism in (v) then means $(\mathcal{X},r\mathcal{L}_1)\simeq(X,rL)$.

We denote the central fiber of $\mathcal{X}$ by $\mathcal{X}_0:=\pi^{-1}(0)$. This is an effective Cartier divisor on $\mathcal{X}$ by the flatness of $\pi$.

**Remark 2.2.** *For each $c\in\mathbb{Q}$, the $\mathbb{G}_m$-linearization of the $\mathbb{Q}$-line bundle $\mathcal{L}$ may be twisted by $t^c$, in the sense that the $\mathbb{G}_m$-linearization of $r\mathcal{L}$ is twisted by the character $t^{rc}$ with $r$ divisible enough. The resulting test configuration can be identified with $(\mathcal{X},\mathcal{L}+c\mathcal{X}_0)$,*

Given two test configurations $\mathcal{X}$, $\mathcal{X}'$ for $X$, the isomorphism $\mathcal{X}_1\simeq X\simeq\mathcal{X}'_1$ induces a canonical isomorphism $\mathcal{X}\setminus\mathcal{X}_0\simeq\mathcal{X}'\setminus\mathcal{X}'_0$, the unique $\mathbb{G}_m$-equivariant isomorphism compatible with the projections to $\mathbb{A}^1\setminus\{0\}$ and the identifications of $X$ with the fibers over $t=1$.

We say that $\mathcal{X}'$ *dominates* $\mathcal{X}$ if the above canonical isomorphism extends to a morphism $\mathcal{X}'\to\mathcal{X}$. When it is an isomorphism, we sometimes abuse notation slightly and write $\mathcal{X}'=\mathcal{X}$ (which is reasonable given that the isomorphism is canonical).

A *pull-back* of a test configuration $(\mathcal{X},\mathcal{L})$ for $(X,L)$ is a test configuration $(\mathcal{X}',\mathcal{L}')$ where $\mathcal{X}'$ dominates $\mathcal{X}$ and $\mathcal{L}'$ is the pull-back of $\mathcal{L}$.

Suppose $X$ is normal. We then define the *normalization* $(\widetilde{\mathcal{X}},\tilde{\mathcal{L}})$ of a test configuration $(\mathcal{X},\mathcal{L})$ to be the pull-back under the normalization $\nu:\widetilde{\mathcal{X}}\to\mathcal{X}$. Note that $(\widetilde{\mathcal{X}},\tilde{\mathcal{L}})$ is (semi)ample if $(\mathcal{X},\mathcal{L})$ is, since $\nu$ is finite.

## 2.2. **Examples.**

**Example 2.3.** *Every $\mathbb{G}_m$-action on $X$ induces a diagonal $\mathbb{G}_m$-action on $X_{\mathbb{A}^1}$, and hence a test configuration $\mathcal{X}$. Similarly, a $\mathbb{G}_m$-linearization of $rL$ for some $r\geq1$ induces a test configuration $(\mathcal{X},\mathcal{L})$ for $(X,L)$.*

Such test configurations are called *product* test configurations. A product test configuration is *trivial* if the $\mathbb{G}_m$-action on $X$ is trivial. By Remark 2.2, $(\mathcal{X},\mathcal{L})$ is trivial iff

$$(\mathcal{X},\mathcal{L}+c\mathcal{X}_0)=(X_{\mathbb{A}^1},L_{\mathbb{A}^1})\quad\text{for some } c\in\mathbb{Q}.$$

**Example 2.4.** *Assume $L$ is ample and fix $r\geq1$ such that $rL$ is very ample. Consider the corresponding closed embedding $X\hookrightarrow\mathbb{P}:=\mathbb{P}V^*$ with $V:=H^0(X,rL)$. Every 1-parameter subgroup $\lambda:\mathbb{G}_m\to\mathrm{GL}(V)$ induces an ample test configuration $(\mathcal{X}_\lambda,\mathcal{L}_\lambda)$ for $(X,L)$. By definition, $\mathcal{X}_\lambda$ is the schematic closure (*i.e. *the 'flat limit') in $\mathbb{P}\times\mathbb{A}^1$ of the image of the closed embedding $X\times\mathbb{G}_m\hookrightarrow\mathbb{P}\times\mathbb{G}_m$ mapping $(x,t)$ to $(\lambda(t)x,t)$. Note that $\lambda$ is trivial iff $(\mathcal{X}_\lambda,\mathcal{L}_\lambda)$ is, while $(\mathcal{X}_\lambda,\mathcal{L}_\lambda)$ is a product iff $\lambda$ preserves $X$. The flat limit uniquely exists (*e.g. *from* [Har, Proposition 9.8]*). By Proposition 2.7 below, the schematic closure is simply given by the Zariski closure when $X$ is a variety.*

*Conversely, every ample test configuration $(\mathcal{X},\mathcal{L})$ may be obtained as above. Indeed, for each $r\geq1$ such that $r\mathcal{L}$ is a relatively ample line bundle, $\mathcal{V}:=\pi_*\mathcal{O}(r\mathcal{L})$ is torsion free by flatness of $\pi$. It is therefore $\mathbb{G}_m$-equivariantly isomorphic to $\mathbb{A}^1\times V$ for a certain $\mathbb{G}_m$-action $\lambda:\mathbb{G}_m\to\mathrm{GL}(V)$ (see Corollary 1.3). We then get a $\mathbb{G}_m$-equivariant embedding $\mathcal{X}\hookrightarrow\mathbb{P}\times\mathbb{A}^1$, and hence $(\mathcal{X},\mathcal{L})=(\mathcal{X}_\lambda,\mathcal{L}_\lambda)$.*

**Example 2.5.** *The* deformation to the normal cone *of a closed subscheme $Z \subset X$ is the blow-up $\rho : \mathcal{X} \to X_{\mathbb{A}^1}$ along $Z \times \{0\}$. Thus $\mathcal{X}$ is a test configuration dominating $X_{\mathbb{A}^1}$. By* [Ful, Chapter 5]*, its central fiber splits as $\mathcal{X}_0 = E + F$, where $E = \rho^{-1}(Z \times \{0\})$ is the exceptional divisor and $F$ is the strict transform of $X \times \{0\}$, which is isomorphic to the blow-up of $X$ along $Z$.*

**Example 2.6.** *More generally we can blow up any $\mathbb{G}_m$-invariant ideal $\mathfrak{a}$ on $X \times \mathbb{A}^1$ supported on the central fiber. We discuss this point in §2.6.*

2.3. **Scheme theoretic features.** Recall that a scheme $Z$ satisfies *Serre's condition $S_k$* iff

$$\operatorname{depth} \mathcal{O}_{Z,\xi} \geq \min\{\operatorname{codim} \xi, k\} \quad \text{for every point } \xi \in Z.$$

In particular, $Z$ is $S_1$ iff it has no embedded points. While we will not use it, one can show that $Z$ is $S_2$ iff it has no embedded points and satisfies the Riemann extension property across closed subsets of codimension at least 2.

On the other hand, $Z$ is *regular in codimension $k$* ($R_k$ for short) iff $\mathcal{O}_{Z,\xi}$ is regular for every $\xi \in \mathcal{X}$ of codimension at most $k$. Equivalently $Z$ is $R_k$ iff its singular locus has codimension greater than $k$. Note that $Z$ is $R_0$ iff it is generically reduced.

Serre's criterion states that $Z$ is normal iff it is $R_1$ and $S_2$. Similarly, $Z$ is reduced iff it is $R_0$ and $S_1$ (in other words, iff $Z$ is generically reduced and without embedded points).

**Proposition 2.7.** *Let $\mathcal{X}$ be a test configuration for $X$.*

(i) *$\mathcal{X}$ is reduced (resp. irreducible, resp. a variety) iff so is $X$.*
(ii) *$\mathcal{X}$ is $S_2$ iff $\mathcal{X}_0$ has no embedded point and $X$ is $S_2$.*
(iii) *If $X$ is $R_1$ and $\mathcal{X}_0$ is generically reduced (that is, 'without multiple components'), then $\mathcal{X}$ is $R_1$.*
(iv) *If $X$ is normal and $\mathcal{X}_0$ is reduced, then $\mathcal{X}$ is normal.*

*Proof.* The isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times \mathbb{G}_m$ shows that $\mathcal{X} \setminus \mathcal{X}_0$ is irreducible (resp. $R_k$, $S_k$) iff $X$ is. On the other hand, the flatness of $\pi$ implies that $\mathcal{X}_0$ is a Cartier divisor and that every associated (i.e. generic or embedded) point of $\mathcal{X}$ belongs to $\mathcal{X} \setminus \mathcal{X}_0$ (cf. [Har, Proposition III.9.7]).

As a first consequence, $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times \mathbb{G}_m$ is dense in $\mathcal{X}$, so that $\mathcal{X}$ is irreducible iff $X$ is. Since $\mathcal{X}_0$ is a Cartier divisor, we also have

$$\operatorname{depth} \mathcal{O}_{\mathcal{X}_0,\xi} = \operatorname{depth} \mathcal{O}_{\mathcal{X},\xi} - 1$$

for each $\xi \in \mathcal{X}_0$, so that $\mathcal{X}$ is $S_k$ iff $X$ is $S_k$ and $\mathcal{X}_0$ is $S_{k-1}$.

It remains to show that $\mathcal{X}_0$ being generically reduced and $X$ being $R_1$ imply that $\mathcal{X}$ is $R_1$. But every codimension one point $\xi \in \mathcal{X}$ either lies the open subset $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times \mathbb{G}_m$, in which case $\mathcal{X}$ is regular at $\xi$, or is a generic point of the Cartier divisor $\mathcal{X}_0$. In the latter case, the closed point of $\operatorname{Spec} \mathcal{O}_{\mathcal{X},\xi}$ is a reduced Cartier divisor; hence $\mathcal{O}_{\mathcal{X},\xi}$ is regular. □

2.4. **Compactifications.** For some purposes it is convenient to compactify test configurations. The following notion provides a canonical way of doing so.

**Definition 2.8.** *The* compactification $\bar{\mathcal{X}}$ *of a test configuration $\mathcal{X}$ for $X$ is defined by gluing together $\mathcal{X}$ and $X_{\mathbb{P}^1 \setminus \{0\}}$ along their respective open subsets $\mathcal{X} \setminus \mathcal{X}_0$ and $X_{\mathbb{A}^1 \setminus \{0\}}$, which are identified using the canonical $\mathbb{G}_m$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X_{\mathbb{A}^1 \setminus \{0\}}$.*

The compactification comes with a $\mathbb{G}_m$-equivariant flat morphism $\bar{\mathcal{X}} \to \mathbb{P}^1$, still denoted by $\pi$. By construction, $\pi^{-1}(\mathbb{P}^1 \setminus \{0\})$ is $\mathbb{G}_m$-equivariantly isomorphic to $X_{\mathbb{P}^1\setminus\{0\}}$ over $\mathbb{P}^1 \setminus \{0\}$.

Similarly, a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ admits a compactification $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$, where $\bar{\mathcal{L}}$ is a $\mathbb{G}_m$-linearized $\mathbb{Q}$-line bundle on $\bar{\mathcal{X}}$. Note that $\bar{\mathcal{L}}$ is relatively (semi)ample iff $\mathcal{L}$ is.

**Example 2.9.** *When $\mathcal{X}$ is the product test configuration defined by a $\mathbb{G}_m$-action on $X$, the compactification $\bar{\mathcal{X}} \to \mathbb{P}^1$ may be alternatively described as the locally trivial fiber bundle with typical fiber $X$ associated to the principal $\mathbb{G}_m$-bundle $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$,* i.e.

$$\bar{\mathcal{X}} = \left( (\mathbb{A}^2 \setminus \{0\}) \times X \right) / \mathbb{G}_m$$

*with $\mathbb{G}_m$ acting diagonally. Note in particular that $\bar{\mathcal{X}}$ is not itself a product in general. For instance, the $\mathbb{G}_m$-action $t \cdot [x : y] = [t^d x : y]$ on $X = \mathbb{P}^1$ gives rise to the Hirzebruch surface $\bar{\mathcal{X}} \simeq \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)\right)$.*

2.5. **Test configurations and filtrations.** By the reverse Rees construction of §1.2, every test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ induces a $\mathbb{Z}$-filtration of the graded algebra

$$R(X, rL) = \bigoplus_{m \in \mathbb{N}} H^0(X, mrL)$$

for $r$ divisible enough. More precisely, for each $r$ such that $r\mathcal{L}$ is a line bundle, we define a filtration on $R(X, rL)$ by letting $F^\lambda H^0(X, mrL)$ be the (injective) image of the weight-$\lambda$ part $H^0(\mathcal{X}, rm\mathcal{L})_\lambda$ of $H^0(\mathcal{X}, mr\mathcal{L})$ under the restriction map

$$H^0(\mathcal{X}, mr\mathcal{L}) \to H^0(\mathcal{X}, mr\mathcal{L})_{t=1} = H^0(X, mrL).$$

Alternatively, we have

$$F^\lambda H^0(X, mrL) = \left\{ s \in H^0(X, mrL) \mid t^{-\lambda} \bar{s} \in H^0(\mathcal{X}, mr\mathcal{L}) \right\} \tag{2.1}$$

where $\bar{s} \in H^0(\mathcal{X} \setminus \mathcal{X}_0, mr\mathcal{L})$ denotes the $\mathbb{G}_m$-invariant section defined by $s \in H^0(X, mrL)$.

**Remark 2.10.** *When $(\mathcal{X}, \mathcal{L})$ is a normal test configuration, the previous construction extends to an $\mathbb{R}$-filtration on $R(X, L)$. Indeed, (2.1) shows that a section $s \in H^0(X, mrL)$ belongs to $F^\lambda H^0(X, mrL)$ iff $s^k \in F^{k\lambda} H^0(X, kmrL)$ for some $k$, and we may thus set for each $\lambda \in \mathbb{R}$*

$$F^\lambda H^0(X, mL) := \left\{ s \in H^0(X, mL) \mid t^{-\lceil r\lambda \rceil} \overline{s^r} \in H^0(\mathcal{X}, mr\mathcal{L}) \text{ for } r \text{ divisible enough} \right\}.$$

*This shows that the compatibility with the point of view of* [WN12].

**Proposition 2.11.** *Assume $L$ is ample. Then the above construction sets up a one-to-one correspondence between ample test configurations for $(X, L)$ and finitely generated $\mathbb{Z}$-filtrations of $R(X, rL)$ for $r$ divisible enough.*

*Proof.* When $(\mathcal{X}, \mathcal{L})$ is an ample test configuration, the $\mathbb{Z}$-filtration it defines on $R(X, rL)$ is finitely generated in the sense of Definition 1.1, since

$$\bigoplus_{m \in \mathbb{N}} \left( \bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} F^\lambda H^0(X, mrL) \right) = R(\mathcal{X}, r\mathcal{L})$$

is finitely generated over $k[t]$. Conversely, let $F^\bullet$ be a finitely generated $\mathbb{Z}$-filtration of $R(X, rL)$ for some $r$. Replacing $r$ with a multiple, we may assume that the graded $k[t]$-algebra

$$\bigoplus_{m\in\mathbb{N}} \left( \bigoplus_{\lambda\in\mathbb{Z}} t^{-\lambda} F^\lambda H^0(X, mrL) \right)$$

is generated in degree $m = 1$, and taking the Proj over $\mathbb{A}^1$ defines an ample test configuration for $(X, rL)$, hence also one for $(X, L)$. Using §1.2, it is straightforward to see that the two constructions are inverse to each other. □

For later use, we note:

**Lemma 2.12.** *Let $(\mathcal{X}, \mathcal{L})$ be an ample test configuration. For each $m$ divisible enough, the successive minima of the induced $\mathbb{Z}$-filtration $F^\bullet H^0(X, mL)$ coincide with the $\mathbb{G}_m$-weights of $H^0(\mathcal{X}_0, m\mathcal{L}_0)$.*

*Proof.* By construction, the successive minima of $F^\bullet H^0(X, mL)$ coincide with the $\mathbb{G}_m$-weights of $H^0(\mathcal{X}, m\mathcal{L})$. By ampleness, the restriction map $H^0(\mathcal{X}, m\mathcal{L}) \to H^0(\mathcal{X}_0, m\mathcal{L}_0)$ is surjective for $m \gg 1$, and it follows that $H^0(\mathcal{X}, m\mathcal{L})$ and $H^0(\mathcal{X}_0, m\mathcal{L}_0)$ have the same $\mathbb{G}_m$-weights. □

Still assuming $L$ is ample, let $(\mathcal{X}, \mathcal{L})$ be merely semiample. The $\mathbb{Z}$-filtration it defines on $R(X, rL)$ is still finitely generated, as

$$\bigoplus_{m\in\mathbb{N}} \left( \bigoplus_{\lambda\in\mathbb{Z}} t^{-\lambda} F^\lambda H^0(X, mrL) \right) = R(\mathcal{X}, r\mathcal{L})$$

is finitely generated over $k[t]$.

**Definition 2.13.** *The* ample model *of a semiample test configuration $(\mathcal{X}, \mathcal{L})$ is defined as the unique ample test configuration $(\mathcal{X}_{\mathrm{amp}}, \mathcal{L}_{\mathrm{amp}})$ corresponding to the finitely generated $\mathbb{Z}$-filtration defined by $(\mathcal{X}, \mathcal{L})$ on $R(X, rL)$ for $r$ divisible enough.*

Ample models admit the following alternative characterization.

**Proposition 2.14.** *The ample model $(\mathcal{X}_{\mathrm{amp}}, \mathcal{L}_{\mathrm{amp}})$ of a semiample test configuration $(\mathcal{X}, \mathcal{L})$ is the unique ample test configuration such that:*

(i) *$(\mathcal{X}, \mathcal{L})$ is a pull-back of $(\mathcal{X}_{\mathrm{amp}}, \mathcal{L}_{\mathrm{amp}})$;*
(ii) *the canonical morphism $\mu : \mathcal{X} \to \mathcal{X}_{\mathrm{amp}}$ satisfies $\mu_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{\mathrm{amp}}}$.*

Note that (ii) implies that $\mathcal{X}_{\mathrm{amp}}$ is normal whenever $\mathcal{X}$ is.

*Proof.* Choose $r \geq 1$ such that $r\mathcal{L}$ is a globally generated line bundle. By Corollary 1.3, the vector bundle $\pi_* \mathcal{O}(r\mathcal{L})$ is $\mathbb{G}_m$-equivariantly trivial over $\mathbb{A}^1$, and we thus get an induced $\mathbb{G}_m$-equivariant morphism $f : \mathcal{X} \to \mathbb{P}^N_{\mathbb{A}^1}$ over $\mathbb{A}^1$ for some $N$ with the property that $f^* \mathcal{O}(1) = r\mathcal{L}$. The Stein factorization of $f$ thus yields an ample test configuration $(\mathcal{X}', \mathcal{L}')$ satisfying (i) and (ii). By the projection formula, these properties guarantee that $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ induce the same $\mathbb{Z}$-filtration on $R(X, rL)$, and hence $(\mathcal{X}', \mathcal{L}') = (\mathcal{X}_{\mathrm{amp}}, \mathcal{L}_{\mathrm{amp}})$ by Proposition 2.11. □

2.6. **Flag ideals.** In this section, $X$ is normal. Let us briefly discuss (a small variant of) the *flag ideal* point of view of [Oda12, Oda13b]. We will use the following terminology.

**Definition 2.15.** *A* determination *of a test configuration $\mathcal{X}$ for $X$ is a normal test configuration $\mathcal{X}'$ dominating both $\mathcal{X}$ and $X_{\mathbb{A}^1}$.*

Note that a determination always exists: just pick $\mathcal{X}'$ to be the normalization of the graph of the canonical birational map $\mathcal{X} \dashrightarrow X_{\mathbb{A}^1}$.

Similarly, a determination of a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ is a normal test configuration $(\mathcal{X}', \mathcal{L}')$ such that $\mathcal{X}'$ is a determination of $\mathcal{X}$ and $\mathcal{L}'$ is the pull-back of $\mathcal{L}$ under the morphism $\mathcal{X}' \to \mathcal{X}$ (i.e. $(\mathcal{X}', \mathcal{L}')$ is a pull-back of $(\mathcal{X}, \mathcal{L})$). In this case, denoting by $\rho : \mathcal{X}' \to X_{\mathbb{A}^1}$ the canonical morphism, we have $\mathcal{L}' = \rho^* L_{\mathbb{A}^1} + D$ for a unique $\mathbb{Q}$-Cartier divisor $D$ supported on $\mathcal{X}'_0$, by the normality of $\mathcal{X}'$.

**Definition 2.16.** *For each $m$ such that $m\mathcal{L}$ is a line bundle, we define the* flag ideal *of $(\mathcal{X}, m\mathcal{L})$ as the $\mathbb{G}_m$-invariant, integrally closed fractional ideal*

$$\mathfrak{a}^{(m)} := \rho_* \mathcal{O}_{\mathcal{X}'}(mD).$$

By Lemma 2.17 below, $\mathfrak{a}^{(m)}$ is indeed independent of the choice of a determination. In particular, $\mathfrak{a}^{(m)}$ is also the flag ideal of $(\mathcal{X}', m\mathcal{L}')$ for every pull-back $(\mathcal{X}', \mathcal{L}')$ of $(\mathcal{X}, \mathcal{L})$.

Since $\mathfrak{a}^{(m)}$ is a $\mathbb{G}_m$-invariant fractional ideal on $X_{\mathbb{A}^1}$ that is trivial outside the central fiber, it is of the form

$$\mathfrak{a}^{(m)} = \sum_{\lambda \in \mathbb{Z}} t^{-\lambda} \mathfrak{a}_\lambda^{(m)} \tag{2.2}$$

where $\mathfrak{a}_\lambda^{(m)} \subset \mathcal{O}_X$ is a non-increasing sequence of integrally closed ideals on $X$ with $\mathfrak{a}_\lambda^{(m)} = 0$ for $\lambda \gg 0$ and $\mathfrak{a}_\lambda^{(m)} = \mathcal{O}_X$ for $\lambda \ll 0$ (see Proposition 2.18 below for the choice of sign).

**Lemma 2.17.** *The flag ideal $\mathfrak{a}^{(m)}$ is independent of the choice of a determination $(\mathcal{X}', \mathcal{L}')$.*

*Proof.* Let $(\mathcal{X}'', \mathcal{L}'')$ be another determination of $(\mathcal{X}, \mathcal{L})$ (and recall that $\mathcal{X}'$ and $\mathcal{X}''$ are normal, by definition). Since any two determinations of $(\mathcal{X}, \mathcal{L})$ are dominated by a third one, we may assume that $\mathcal{X}''$ dominates $\mathcal{X}'$. Denoting by $\mu' : \mathcal{X}'' \to \mathcal{X}'$ the corresponding morphism, the fractional ideal attached to $(\mathcal{X}''', \mathcal{L}'')$ is then given by

$$(\rho \circ \mu')_* \mathcal{O}_{\mathcal{X}''}(m\mu'^* D)$$

By the projection formula we have

$$\mu'_* \mathcal{O}_{\mathcal{X}''}(m\mu'^* D) = \mathcal{O}_{\mathcal{X}'}(mD) \otimes \mu'_* \mathcal{O}_{\mathcal{X}''},$$

and we get the desired result since $\mu'_* \mathcal{O}_{\mathcal{X}''} = \mathcal{O}_{\mathcal{X}'}$ by normality of $\mathcal{X}'$. ☐

**Proposition 2.18.** *Let $(\mathcal{X}, \mathcal{L})$ be a semiample test configuration for $(X, L)$, for each $m$ with $m\mathcal{L}$ a line bundle, let $F^\bullet H^0(X, mL)$ be the corresponding $\mathbb{Z}$-filtration and $\mathfrak{a}^{(m)}$ be the flag ideal of $(\mathcal{X}, m\mathcal{L})$. Then, for $m$ sufficiently divisible and all $\lambda \in \mathbb{Z}$, the sheaf $\mathcal{O}(mL) \otimes \mathfrak{a}_\lambda^{(m)}$ is globally generated, and*

$$F^\lambda H^0(X, mL) = H^0\left(X, \mathcal{O}(mL) \otimes \mathfrak{a}_\lambda^{(m)}\right)$$

*In particular, the successive minima of $F^\bullet H^0(X, mL)$ (see §1) are exactly the $\lambda \in \mathbb{Z}$ with $\mathfrak{a}_\lambda^{(m)} \neq \mathfrak{a}_{\lambda+1}^{(m)}$, with the largest one being $\lambda_{\max}^{(m)} = \max\left\{\lambda \in \mathbb{Z} \mid \mathfrak{a}_\lambda^{(m)} \neq 0\right\}$.*

*Proof.* Let $(\mathcal{X}', \mathcal{L}')$ be a determination of $(\mathcal{X}, \mathcal{L})$, i.e. a pull-back such that $\mathcal{X}'$ is normal and dominates $X_{\mathbb{A}^1}$. By normality of $\mathcal{X}$, the morphism $\mu : \mathcal{X}' \to \mathcal{X}$ satisfies $\mu_* \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}}$. By the projection formula, we $(\mathcal{X}', \mathcal{L}')$ and $(\mathcal{X}, \mathcal{L})$ define the same $\mathbb{Z}$-filtration of $R(X, rL)$ for $r$ divisible enough. Since $\mathfrak{a}^{(m)}$ is also the flag ideal of $(\mathcal{X}', m\mathcal{L}')$, we may assume to begin with that $\mathcal{X}$ dominates $X_{\mathbb{A}^1}$. Denoting by $\rho : \mathcal{X} \to X_{\mathbb{A}^1}$ the canonical morphism, we then have $\mathcal{L} = \rho^* L_{\mathbb{A}^1} + D$ and

$$\mathfrak{a}^{(m)} = \rho_* \mathcal{O}_{\mathcal{X}}(mD),$$

and hence

$$\rho_* \mathcal{O}(m\mathcal{L}) = \mathcal{O}(mL_{\mathbb{A}^1}) \otimes \mathfrak{a}^{(m)}$$

by the projection formula. As a consequence, $H^0(X_{\mathbb{A}^1}, \mathcal{O}(mL_{\mathbb{A}^1}) \otimes t^{-\lambda} \mathfrak{a}_\lambda^{(m)})$ is isomorphic to the weight-$\lambda$ part of $H^0(\mathcal{X}, m\mathcal{L})$, and the first point follows.

When $m\mathcal{L}$ is globally generated on $\mathcal{X}$, so is $\rho_* \mathcal{O}(m\mathcal{L})$ on $X_{\mathbb{A}^1}$. Decomposing into weight spaces thus shows that $\mathcal{O}(mL) \otimes \mathfrak{a}_\lambda^{(m)}$ is globally generated on $X$ for all $\lambda \in \mathbb{Z}$. We therefore have $\mathfrak{a}_\lambda^{(m)} \neq \mathfrak{a}_{\lambda+1}^{(m)}$ iff $F^\lambda H^0(X, mL) \neq F^{\lambda+1} H^0(X, mL)$, hence the second point. $\square$

## 3. Donaldson-Futaki invariants and K-stability

In this section, $(X, L)$ is a polarized scheme over $k$. Our goal is to provide an elementary, self-contained treatment of Donaldson-Futaki invariants. To this end, we use, as in [Don05, p.470], the compactified test configurations introduced in §2.4.

### 3.1. **Donaldson-Futaki invariants.**

Write $N_m = h^0(X, mL)$ for $m \geq 1$. The Donaldson-Futaki invariant of an ample test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ describes the subdominant term in the asymptotic expansion of $w_m / mN_m$ as $m \to \infty$, where $w_m \in \mathbb{Z}$ is the weight of the $\mathbb{G}_m$-action on $\det H^0(\mathcal{X}_0, m\mathcal{L}_0)$.

**Lemma 3.1.** *Let $(\mathcal{X}, \mathcal{L})$ be an ample test configuration for $(X, L)$, with compactification $\pi : (\bar{\mathcal{X}}, \bar{\mathcal{L}}) \to \mathbb{P}^1$. For every $m$ divisible enough, we have*

$$w_m = \chi(\bar{\mathcal{X}}, m\bar{\mathcal{L}}) - N_m,$$

*where $\chi$ stands for the Euler characteristic. In particular, $w_m$ is a polynomial of $m$ of degree at most $n+1$, for $m$ sufficiently divisible.*

*Proof.* By flatness of $\pi$, the direct image sheaf $\pi_* \mathcal{O}(m\bar{\mathcal{L}})$, being torsion free, is a vector bundle on $\mathbb{P}^1$ of rank $N_m = h^0(mL)$. As $\bar{\mathcal{L}}$ is $\pi$-ample, the restriction map $\pi_* \mathcal{O}(m\bar{\mathcal{L}}) \to H^0(\mathcal{X}_0, m\mathcal{L}_0)$ is surjective for $m$ divisible enough. This shows that $w_m$ is also the weight of the $\mathbb{G}_m$-action on the fiber at 0 of the line bundle $\det \pi_* \mathcal{O}(m\bar{\mathcal{L}})$, and hence

$$w_m = \deg \det \pi_* \mathcal{O}(m\bar{\mathcal{L}}) = \deg \pi_* \mathcal{O}(m\bar{\mathcal{L}}),$$

since $\pi_* \mathcal{O}(m\bar{\mathcal{L}})$ is $\mathbb{G}_m$-equivariantly trivial away from 0 by construction of the compactification. By the usual Riemann-Roch theorem on $\mathbb{P}^1$, we infer

$$\chi(\mathbb{P}^1, \pi_* \mathcal{O}(m\bar{\mathcal{L}})) = w_m + N_m.$$

Using relative ampleness again, Serre vanishing implies $R^p \pi_* \mathcal{O}(m\mathcal{L}) = 0$ for $p > 0$ and $m$ divisible enough. As a result, the relevant Leray spectral sequence degenerates; hence

$$H^q(\bar{\mathcal{X}}, m\bar{\mathcal{L}}) \simeq H^q(\mathbb{P}^1, \pi_* \mathcal{O}(m\bar{\mathcal{L}}))$$

for all $q \in \mathbb{N}$ and all $m$ divisible enough. This yields in particular

$$\chi(\bar{\mathcal{X}}, m\bar{\mathcal{L}}) = \chi(\mathbb{P}^1, \pi_* \mathcal{O}(m\bar{\mathcal{L}})),$$

which proves that $w_m = \chi(\bar{\mathcal{X}}, m\bar{\mathcal{L}}) - N_m$.

Now, for $m$ sufficiently divisible, $m \mapsto \chi(\bar{\mathcal{X}}, m\bar{\mathcal{L}})$ is a polynomial of degree at most $m+1$, by a version of the Hilbert-Serre theorem usually attributed to Snapper (cf. [Kle66, §1]). Similarly, $m \mapsto \chi(X, mL)$ is a polynomial of degree at most $m$. This completes the proof since $N_m = \chi(X, mL)$ for $m \gg 0$ by the ampleness of $L$. □

Lemma 3.1 yields an asymptotic expansion

$$\frac{w_m}{mN_m} = F_0 + m^{-1}F_1 + m^{-2}F_2 + \dots \tag{3.1}$$

with $F_i = F_i(\mathcal{X}, \mathcal{L}) \in \mathbb{Q}$ satisfying the following obvious 'scaling property':

**Lemma 3.2.** *For $c \in \mathbb{Q}$ we have $F_0(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0) = F_0(\mathcal{X}, \mathcal{L}) + c$ and $F_i(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0) = F_i(\mathcal{X}, \mathcal{L})$, for $i \geq 1$.*

**Definition 3.3.** *The* Donaldson-Futaki invariant *of an ample test configuration $(\mathcal{X}, \mathcal{L})$ is*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := -2F_1(\mathcal{X}, \mathcal{L}).$$

The factor 2 in the definition is here just for convenience, while the sign is chosen so that K-semistability will correspond to $\mathrm{DF} \geq 0$.

Using Lemma 3.1, we easily obtain an intersection theoretic formula for Donaldson-Futaki invariants (see [Wan12] and [LX14, Example 3]).

**Proposition 3.4.** *For each ample test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$, we have*

$$F_0(\mathcal{X}, \mathcal{L}) = \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)(L^n)}. \tag{3.2}$$

*If $\mathcal{X}$ (and hence $X$) is further normal, then*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = V^{-1}\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n\right) + \bar{S}F_0(\mathcal{X}, \mathcal{L}). \tag{3.3}$$

*with $V := (L^n)$ and*

$$\bar{S} := -n\frac{(K_X \cdot L^{n-1})}{(L^n)}.$$

Here $K_X$ and $K_{\bar{\mathcal{X}}/\mathbb{P}^1} = K_{\bar{\mathcal{X}}} - \pi^* K_{\mathbb{P}^1}$ are understood as Weil divisor classes on the normal varieties $X$ and $\bar{\mathcal{X}}$, respectively.

**Remark 3.5.** *Arithmetically, the expression for $F_0$ corresponds to the (normalized, logarithmic) height of $(X, L) \times_k k(t)$ with respect to the model $(\bar{\mathcal{X}}, \bar{\mathcal{L}}) \to \mathbb{P}^1$, cf.* [Wan12].

**Remark 3.6.** *When $X$ is smooth and $k = \mathbb{C}$, $\bar{S}$ coincides with the mean value of the scalar curvature $S(\omega)$ of any Kähler form $\omega \in c_1(L)$ (hence the chosen notation).*

*Proof of Proposition 3.4.* The usual asymptotic Riemann-Roch theorem [Kle66, §1] yields

$$N_m = \frac{m^n}{n!}V + O(m^{n-1})$$

and

$$\chi(\bar{\mathcal{X}}, m\bar{\mathcal{L}}) = \frac{m^{n+1}}{(n+1)!}\left(\bar{\mathcal{L}}^{n+1}\right) + O(m^n),$$

from which the formula for $F_0(\mathcal{X}, \mathcal{L})$ follows immediately. When $\mathcal{X}$ (and hence $X$ and $\bar{\mathcal{X}}$) is normal, the two-term asymptotic Riemann-Roch theorem on a normal variety (cf. Theorem A.1 in the appendix) yields

$$N_m = V\frac{m^n}{n!}\left[1 + \frac{\bar{S}}{2}m^{-1} + O\left(m^{-2}\right)\right],$$

and

$$\begin{aligned} w_m &= -N_m + \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)!}m^{n+1} - \frac{(K_{\bar{\mathcal{X}}} \cdot \bar{\mathcal{L}}^n)}{2n!}m^n + O\left(m^{n-1}\right) \\ &= \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)!}m^{n+1} - \frac{(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n)}{2n!}m^n + O\left(m^{n-1}\right), \end{aligned}$$

using that $(\pi^* K_{\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) = -2V$ since $\deg K_{\mathbb{P}^1} = -2$. The formula for $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ now follows from a straightforward computation. □

3.2. **Behavior under normalization.** The following result describes the behavior of the Donaldson-Futaki invariant under normalization, and can be viewed as an effective version of [RT07, Proposition 5.1] and [ADVLN11, Corollary 3.9].

**Proposition 3.7.** *Assume that $X$ is normal. Let $(\mathcal{X}, \mathcal{L})$ be an ample test configuration for $(X, L)$, and let $(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}})$ be its normalization, with morphism $\nu : \widetilde{\mathcal{X}} \to \mathcal{X}$. Then*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \mathrm{DF}(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}}) + 2V^{-1}\sum_E l_E\left(E \cdot \mathcal{L}^n\right),$$

*where $E$ ranges over the irreducible components of $\mathcal{X}_0$ contained in the singular locus of $\mathcal{X}$ and $l_E \in \mathbb{N}^*$ is the length of $\left(\nu_* \mathcal{O}_{\widetilde{\mathcal{X}}}\right)/\mathcal{O}_{\mathcal{X}}$ at the generic point of $E$.*

*In particular, $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \mathrm{DF}(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}})$, with equality iff $\mathcal{X}$ is regular in codimension 1.*

*Proof.* For ease of notation, we denote by $(\mathcal{Y}, \mathcal{M}) = (\widetilde{\mathcal{X}}, \tilde{\mathcal{L}})$ the normalization $(\mathcal{X}, \mathcal{L})$. The normalization $\nu : \mathcal{Y} \to \mathcal{X}$ extends to $\nu : \bar{\mathcal{Y}} \to \bar{\mathcal{X}}$. Denoting by $w_m$ and $\tilde{w}_m$ the $\mathbb{G}_m$-weights of $\det H^0(\mathcal{X}_0, m\mathcal{L}_0)$ and $\det H^0(\mathcal{Y}_0, m\mathcal{M}_0)$, Lemma 3.1 yields

$$\tilde{w}_m - w_m = \chi(\bar{\mathcal{Y}}, m\bar{\mathcal{M}}) - \chi(\bar{\mathcal{X}}, m\bar{\mathcal{L}}) = \chi(\bar{\mathcal{Y}}, m\nu^*\bar{\mathcal{L}}) - \chi(\bar{\mathcal{X}}, m\bar{\mathcal{L}}). \tag{3.4}$$

Since $\nu$ is finite, we have $R^q\nu_*\mathcal{O}_{\bar{\mathcal{Y}}} = 0$ for all $q \geq 1$, and the Leray spectral sequence gives

$$\chi(\bar{\mathcal{Y}}, m\nu^*\bar{\mathcal{L}}) = \chi\left(\bar{\mathcal{X}}, \mathcal{O}(m\bar{\mathcal{L}}) \otimes \nu_*\mathcal{O}_{\bar{\mathcal{Y}}}\right).$$

By normality of $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times \mathbb{G}_m$, the sheaf $\mathcal{F} := \left(\nu_*\mathcal{O}_{\bar{\mathcal{Y}}}\right)/\mathcal{O}_{\bar{\mathcal{X}}}$ is supported on $\mathcal{X}_0$, and may thus also be written as $\mathcal{F} = \nu_*(\mathcal{O}_{\mathcal{Y}}/\mathcal{O}_{\mathcal{X}}) = \nu_*(\mathcal{O}_{\widetilde{\mathcal{X}}}/\mathcal{O}_{\mathcal{X}})$.

The additivity of the Euler characteristic in exact sequences further yields

$$\chi\left(\bar{\mathcal{X}}, \mathcal{O}(m\bar{\mathcal{L}}) \otimes \nu_*\mathcal{O}_{\bar{\mathcal{Y}}}\right) = \chi\left(\bar{\mathcal{X}}, m\bar{\mathcal{L}}\right) + \chi\left(\bar{\mathcal{X}}, \mathcal{O}(m\bar{\mathcal{L}}) \otimes \mathcal{F}\right).$$

Since $X$ is normal, $Z := \operatorname{supp} \mathcal{F}$ is contained in $\mathcal{X}_0$. Set $d := \dim Z$, denote by $Z_i$ the $d$-dimensional irreducible components of $Z$, and let $l_i$ be the length of $\mathcal{F}$ at the generic point of $Z_i$. By [Kle66, §2], we have

$$\begin{aligned}\chi\left(\bar{\mathcal{X}}, \mathcal{O}(m\bar{\mathcal{L}}) \otimes \mathcal{F}\right) &= \frac{m^d}{d!} \sum_i l_i(\bar{\mathcal{L}}^d \cdot Z_i) + O\left(m^{d-1}\right) \\ &= \frac{m^n}{n!} \sum_E l_E \left(E \cdot \mathcal{L}^n\right) + O(m^{n-1})\end{aligned}$$

in the notation of the proposition. Since $N_m = \frac{m^n}{n!} V + O(m^{n-1})$, (3.4) gives

$$F_1(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}}) = F_1(\mathcal{X}, \mathcal{L}) + V^{-1} \sum_E l_E \left(E \cdot \mathcal{L}^n\right),$$

and hence the desired formula in view of Definition 3.3. The ampleness of $\mathcal{L}$ on $\mathcal{X}_0$ further gives $\sum_E l_E \left(E \cdot \mathcal{L}^n\right) \geq 0$, with equality iff $\mathcal{X}$ is regular at each generic point of $\mathcal{X}_0$, i.e. iff $\mathcal{X}$ is $R_1$. □

3.3. **K-stability.** As noted in [LX14, §3.1], a normal polarized variety $(X, L)$ may admit a non-trivial test configuration $(\mathcal{X}, \mathcal{L})$ whose normalization $(\widetilde{\mathcal{X}}, \tilde{\mathcal{L}})$ trivial. By Proposition 3.7, such test configurations have to satisfy $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$. Taking this into account, we arrive at the following precise definition of K-stability.

**Definition 3.8.** *A normal polarized variety* $(X, L)$ *is* K-semistable *if* $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ *for every normal, ample test configuration* $(\mathcal{X}, \mathcal{L})$*, and* K-stable *if equality holds iff* $(\mathcal{X}, \mathcal{L})$ *is trivial,* i.e. $(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0) = (X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$ *for some* $c \in \mathbb{Q}$.

Thanks to Proposition 3.7, K-semistability implies $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all (not necessarily normal) ample test configurations.

3.4. **The log case.** Now assume that $X$ is normal and let $B$ be a boundary on $X$ (see §1.5). We then introduce a log version of the 'mean scalar cuvature' $\bar{S}$ by setting

$$\bar{S}_B := -nV^{-1}\left((K_X + B) \cdot L^{n-1}\right).$$

The intersection theoretic formula for DF suggests the following generalization for pairs.

**Definition 3.9.** *Let* $B$ *be a boundary on* $X$*. For each normal test configuration* $(\mathcal{X}, \mathcal{L})$ *for* $(X, L)$*, we define the* $B$-twisted Donaldson-Futaki invariant *of* $(\mathcal{X}, \mathcal{L})$ *as*

$$\mathrm{DF}_B(\mathcal{X}, \mathcal{L}) := V^{-1}\left(\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1} + \bar{\mathcal{B}}\right) \cdot \bar{\mathcal{L}}^n\right) + \bar{S}_B F_0(\mathcal{X}, \mathcal{L}),$$

*where* $\bar{\mathcal{B}}$ *is the* $\mathbb{Q}$*-Weil divisor of* $\mathcal{X}$ *obtained by taking the component-wise Zariski closure in* $\bar{\mathcal{X}}$ *of the* $\mathbb{Q}$*-Weil divisor* $B_{\mathbb{G}_m}$ *on* $X_{\mathbb{G}_m} \hookrightarrow \bar{\mathcal{X}}$*, and is* $F_0(\mathcal{X}, \mathcal{L})$ *as in Proposition 3.4.*

We emphasize that the $\mathbb{Q}$-Weil divisor $K_{\bar{\mathcal{X}}/\mathbb{P}^1} + \bar{\mathcal{B}}$ may not be $\mathbb{Q}$-Cartier in general.

**Definition 3.10.** *Given a boundary* $B$ *on* $X$*, we say that the polarized pair* $((X, B), L)$ *is* K-semistable *if* $\mathrm{DF}_B(\mathcal{X}, \mathcal{L}) \geq 0$ *for all normal, ample test configurations* $(\mathcal{X}, \mathcal{L})$*, and* K-stable *if equality holds only when* $(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0) = (X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$ *for some* $c \in \mathbb{Q}$.

**Remark 3.11.** *Assume that $B$ is effective and non-zero, and set $V_B := (B \cdot L^{n-1}) > 0$. Then $\bar{S}_B = \bar{S} - nV^{-1}V_B$, and hence*

$$\mathrm{DF}_B(\mathcal{X}, \mathcal{L}) = \mathrm{DF}(\mathcal{X}, \mathcal{L}) + (\bar{S} - \bar{S}_B)\left(F_0(\mathcal{X}, \mathcal{L}) - F_0'\right)$$

*with*

$$F_0' := \frac{(\bar{\mathcal{B}} \cdot \bar{\mathcal{L}}^n)}{nV_B}.$$

*This shows the compatibility of the above definitions with those appearing in* [Don12, LS14].

## 4. Valuations and test configurations

In what follows, $X$ denotes a normal variety of dimension $n$, with function field $K = k(X)$. The function field of $X_{\mathbb{A}^1}$ is then given by $K(t)$. We shall relate valuations on $K$ and $K(t)$ from both an algebraic and geometric point of view.

### 4.1. **Restriction and Gauss extension.**

First consider a valuation $w$ on $K(t)$. We denote by $r(w)$ its restriction to $K$.

**Lemma 4.1.** *If $w$ is an Abhyankhar valuation, then so is $r(w)$. If $w$ is divisorial, then $r(w)$ is either divisorial or trivial.*

*Proof.* The first assertion follows from Abhyankar's inequality (1.4). Indeed, if $w$ is Abhyankhar, then $\operatorname{tr.deg}(w) + \operatorname{rat.rk}(w) = n+1$, so (1.4) gives $\operatorname{tr.deg}(r(w)) + \operatorname{rat.rk}(r(w)) \geq n$. As the opposite inequality always holds, we must have $\operatorname{tr.deg}(r(w)) + \operatorname{rat.rk}(r(w)) = n$, i.e. $r(w)$ is Abhyankar.

We also have $\operatorname{tr.deg}(r(w)) \leq \operatorname{tr.deg}(w)$, so if $w$ is divisorial, then $\operatorname{tr.deg}(r(w)) = n$ or $\operatorname{tr.deg}(r(w)) = n-1$, corresponding to $r(w)$ being trivial or divisorial, respectively. □

The restriction map $r$ is far from injective, but we can construct a natural one-sided inverse by exploiting the $k^*$-action (or $\mathbb{G}_m$-action) on $K(t) = k(X_{\mathbb{A}^1})$ defined by $(a \cdot f)(t) = f(a^{-1}t)$ for $a \in k^*$ and $f \in K(t)$. In terms of the Laurent polynomial expansion

$$f = \sum_{\lambda \in \mathbb{Z}} f_\lambda t^\lambda \tag{4.1}$$

with $f_\lambda \in K$, the $k^*$-action on $K(t)$ reads

$$a \cdot f = \sum_{\lambda \in \mathbb{Z}} a^{-\lambda} f_\lambda t^\lambda. \tag{4.2}$$

**Lemma 4.2.** *A valuation $w$ on $K(t)$ is $k^*$-invariant iff*

$$w(f) = \min_{\lambda \in \mathbb{Z}} \left(r(w)(f_\lambda) + \lambda w(t)\right). \tag{4.3}$$

*for all $f \in K(t)$ with Laurent polynomial expansion (4.1). In particular, $r(w)$ is trivial iff $w$ is the multiple of the $t$-adic valuation.*

*Proof.* In view of (4.2), it is clear that (4.3) implies $k^*$-invariance. Conversely let $w$ be a $k^*$-invariant valuation on $K(t)$. The valuation property of $w$ shows that

$$w(f) \geq \min_{\lambda \in \Lambda} \left(r(w)(f_\lambda) + \lambda w(t)\right)$$

Set $\Lambda := \{\lambda \in \mathbb{Z} \mid f_\lambda \neq 0\}$ and pick distinct elements $a_\mu \in k^*$, $\mu \in \Lambda$ (recall that $k$ is algebraically closed, and hence infinite). The Vandermonde matrix $(a_\mu^\lambda)_{\lambda,\mu\in\Lambda}$ is then invertible, and each term $f_\lambda t^\lambda$ with $\lambda \in \Lambda$ may thus be expressed as $k$-linear combination of $(a_\mu \cdot f)_{\mu\in\Lambda}$. Using the valuation property of $w$ again, we get for each $\lambda \in \Lambda$

$$r(w)(f_\lambda) + \lambda w(t) = w\left(f_\lambda t^\lambda\right) \geq \min_{\mu\in\Lambda} w(a_\mu \cdot f) = w(f),$$

where the right-hand equality holds by $k^*$-invariance of $w$. The result follows. $\square$

**Definition 4.3.** *The* Gauss extension *of a valuation $v$ on $K$ is the valuation $G(v)$ on $K(t)$ defined by*

$$G(v)(f) = \min_{\lambda\in\mathbb{Z}} \left(v(f_\lambda) + \lambda\right)$$

*for all $f$ with Laurent polynomial expansion (4.1).*

Note that $r(G(v)) = v$ for all valuations $v$ on $K$, while a valuation $w$ on $K(t)$ satisfies $w = G(r(w))$ iff it is $k^*$-invariant and $w(t) = 1$, by Lemma 4.2. Further, the Gauss extension of $v$ is the smallest extension $w$ with $w(t) = 1$.

4.2. **Geometric interpretation.** We now relate the previous algebraic considerations to test configurations. For each test configuration $\mathcal{X}$ for $X$, the canonical birational map $\mathcal{X} \dashrightarrow X_{\mathbb{A}^1}$ yields an isomorphism $k(\mathcal{X}) \simeq K(t)$. When $\mathcal{X}$ is normal, every irreducible component $E$ of $\mathcal{X}_0$ therefore defines a divisorial valuation $\mathrm{ord}_E$ on $K(t)$.

**Definition 4.4.** *Let $\mathcal{X}$ be a normal test configuration for $X$. For each component $E$ of $\mathcal{X}_0$, we set $v_E := b_E^{-1} r(\mathrm{ord}_E)$ with $b_E = \mathrm{ord}_E(\mathcal{X}_0) = \mathrm{ord}_E(t)$. We say that $E$ is* non-trivial *if it is not the strict transform of $X \times \{0\}$.*

Since $E$ is preserved under the $\mathbb{G}_m$-action on $\mathcal{X}$, $\mathrm{ord}_E$ is $k^*$-invariant, and we infer from Lemma 4.1 and Lemma 4.2:

**Lemma 4.5.** *For each component $E$ of $\mathcal{X}_0$, we have* $\mathrm{ord}_E = b_E G(v_E)$, i.e.

$$\mathrm{ord}_E(f) = b_E \min_\lambda \left(v_E(f_\lambda) + \lambda\right).$$

*in terms of the Laurent polynomial expansion (4.1). Further, $E$ is non-trivial iff $v_E$ is nontrivial, and hence a divisorial valuation on $X$.*

By construction, divisorial valuations on $X$ of the form $v_E$ have a value group $\Gamma_v = v(K^*)$ contained in $\mathbb{Q}$. Thus they are of the form $v_E = c\, \mathrm{ord}_F$ with $c \in \mathbb{Q}_{>0}$ and $F$ a prime divisor on a normal variety $Y$ mapping birationally to $X$. Conversely, we prove:

**Theorem 4.6.** *A divisorial valuation $v$ on $X$ is of the form $v = v_E$ for a non-trivial component $E$ of a normal test configuration iff $\Gamma_v$ is contained in $\mathbb{Q}$. In this case, we may recover $b_E$ as the denominator of the generator of $\Gamma_v$.*

**Lemma 4.7.** *A divisorial valuation $w$ on $K(t)$ satisfying $w(t) > 0$ is $k^*$-invariant iff $w = c\, \mathrm{ord}_E$ with $c > 0$ and $E$ an irreducible component of the central fiber $\mathcal{X}_0$ of a normal test configuration $\mathcal{X}$ of $X$.*

*Proof.* If $E$ is an irreducible component of $\mathcal{X}_0$, then $\operatorname{ord}_E(t) > 0$, and the $\mathbb{G}_m$-invariance of $E$ easily implies that $\operatorname{ord}_E$ is $k^*$-invariant. Conversely, let $w$ be a $k^*$-invariant divisorial valuation on $K(t)$ satisfying $w(t) > 0$. The center $\xi$ on $X \times \mathbb{A}^1$ is then $\mathbb{G}_m$-invariant and contained in $X \times \{0\}$. If we let $\mathcal{Y}_1$ be the test configuration obtained by blowing-up the closure of $\xi$ in $X \times \mathbb{A}^1$, then the center $\xi_1$ of $w$ on $\mathcal{Y}_1$ is again $\mathbb{G}_m$-invariant by $k^*$-invariance of $w$, and the blow-up $\mathcal{Y}_2$ of the closure of $\xi_1$ is thus a test configuration. Continuing this way, we get a tower of test configurations

$$X \times \mathbb{A}^1 \leftarrow \mathcal{Y}_1 \leftarrow \mathcal{Y}_2 \leftarrow \cdots \leftarrow \mathcal{Y}_i \leftarrow \ldots$$

Since $w$ is divisorial, a result of Zariski (cf. [KM98, Lemma 2.45]) guarantees that the closure of the center $\xi_i$ of $w$ on $\mathcal{Y}_i$ has codimension 1 for $i \gg 1$. We then have $w = c \operatorname{ord}_E$ with $E$ the closure of the center of $w$ on the normalization $\mathcal{X}$ of $\mathcal{Y}_i$. □

*Proof of Theorem 4.6.* Let $E$ be a non-trivial irreducible component of $\mathcal{X}_0$ for a normal test configuration $\mathcal{X}$ of $X$. Since the value group of $\operatorname{ord}_E$ on $k(\mathcal{X}) = K(t)$ is $\mathbb{Z}$, the value group of $v_E$ on $k(X) = K$ is of the form $\frac{c}{b_E}\mathbb{Z}$ for some positive integer $c$. Lemma 4.5 yields $\mathbb{Z} = c\mathbb{Z} + b_E\mathbb{Z}$, so that $c$ and $b_E$ are coprime.

Conversely, let $v$ be a divisorial valuation on $X$ with $\Gamma_v = \frac{c}{b}\mathbb{Z}$ for some coprime positive integers $b, c$. Then $w := bG(v)$ is a $k^*$-invariant divisorial valuation on $K(t)$ with value group $c\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$. By Lemma 4.7, we may thus find a normal test configuration $\mathcal{X}$ for $X$ and a non-trivial component $E$ of $\mathcal{X}_0$ such that $\operatorname{ord}_E = w$. We then have $b_E = w(t) = b$, and hence $v = v_E$. □

4.3. **Rees valuations and deformation to the normal cone.** Our goal in this section is to relate the Rees valuations of a closed subscheme $Z \subset X$ to the valuations associated to the normalization of the deformation to the normal cone of $Z$, see Example 2.5.

**Theorem 4.8.** *Let $Z \subset X$ be a closed subcheme, $\mathcal{X}$ the deformation to the normal cone of $Z$, and $\widetilde{\mathcal{X}}$ its normalization, so that $\mu : \widetilde{\mathcal{X}} \to X_{\mathbb{A}^1}$ is the normalized blow-up of $Z \times \{0\}$. Then the Rees valuations of $Z$ coincide with the valuations $v_E$, where $E$ runs over the non-trivial irreducible components of $\widetilde{\mathcal{X}}_0$.*

If we denote by $E_0$ the strict transform of $X \times \{0\}$ in $\mathcal{X}$, one can show that $\mathcal{X} \setminus E_0$ is isomorphic to the Spec over $X$ of the *extended Rees algebra* $\mathcal{O}_X[t^{-1}\mathfrak{a}, t]$, where $\mathfrak{a}$ is the ideal of $Z$, cf. [Ful, pp.87–88]. We thus see that Theorem 4.8 is equivalent to the well-known fact that the Rees valuations of $\mathfrak{a}$ coincide with the restrictions to $X$ of the Rees valuations of the principal ideal $(t)$ of the extended Rees algebra (see for instance [HS, Exercise 10.5]). We nevertheless provide a proof for the benefit of the reader.

**Lemma 4.9.** *Let $\mathfrak{b} = \sum_{\lambda \in \mathbb{N}} \mathfrak{b}_\lambda t^\lambda$ be a $\mathbb{G}_m$-invariant ideal of $X \times \mathbb{A}^1$, and let*

$$\overline{\mathfrak{b}} = \sum_{\lambda \in \mathbb{N}} (\overline{\mathfrak{b}})_\lambda t^\lambda$$

*be its integral closure. For each $\lambda$ we then have $\overline{\mathfrak{b}_\lambda} \subset (\overline{\mathfrak{b}})_\lambda$, with equality for $\lambda = 0$.*

*Proof.* Each $f \in \overline{\mathfrak{b}_\lambda}$ satisfies a monic equation $f^d + \sum_{j=1}^d b_j f^{d-j} = 0$ with $b_j \in \mathfrak{b}_\lambda^j$. Then

$$(t^\lambda f)^d + \sum_{j=1}^d (t^{\lambda j} b_j)(t^\lambda f)^{d-j} = 0$$

with $t^{\lambda j}b_j \in (t^\lambda \mathfrak{b}_\lambda)^j \subset \mathfrak{b}^j$. It follows that $t^\lambda f \in \overline{\mathfrak{b}}$, which proves the first assertion.

Conversely, we may choose $l \gg 1$ such that the $\mathbb{G}_m$-invariant ideal $\mathfrak{c} := \overline{\mathfrak{b}^l}$ satisfies $\overline{\mathfrak{b}}\cdot\mathfrak{c} = \mathfrak{b}\cdot\mathfrak{c}$ (cf. proof of Lemma 1.7). Write $\mathfrak{c} = \sum_{\lambda\geq\lambda_0}\mathfrak{c}_\lambda t^\lambda$ with $\mathfrak{c}_{\lambda_0} \neq 0$. Then $(\overline{\mathfrak{b}})_0 \cdot \mathfrak{c}_{\lambda_0} t^{\lambda_0}$ is contained in the weight $\lambda_0$ part of $\mathfrak{b}\cdot\mathfrak{c}$, which is equal to $(\mathfrak{b}_0 \cdot \mathfrak{c}_{\lambda_0})t^{\lambda_0}$. We thus have $(\overline{\mathfrak{b}})_0 \cdot \mathfrak{c}_{\lambda_0} \subset \mathfrak{b}_0 \cdot \mathfrak{c}_{\lambda_0}$, and hence $(\overline{\mathfrak{b}})_0 \subset \overline{\mathfrak{b}_0}$ by the determinant trick. □

*Proof of Theorem 4.8.* Let $\mathfrak{a}$ be the ideal defining $Z$. By Theorem 1.9, we are to check that:

(i) $\overline{\mathfrak{a}^m} = \bigcap_E \{f \in \mathcal{O}_X \mid v_E(f) \geq m\}$ for all $m \in \mathbb{N}$;
(ii) no $E$ can be omitted in (i).

Set $D := \mu^{-1}(Z \times \{0\})$. Since $\operatorname{ord}_E$ is $k^*$-invariant, Lemma 4.2 yields

$$\operatorname{ord}_E(D) = \operatorname{ord}_E(\mathfrak{a} + (t)) = \min\{r(\operatorname{ord}_E)(\mathfrak{a}), b_E\}.$$

We claim that we have in fact $\operatorname{ord}_E(D) = b_E$. As recalled in Example 2.5, the blow-up $\rho : \mathcal{X} \to X \times \mathbb{A}^1$ along $Z \times \{0\}$ satisfies $\mathcal{X}_0 = \mu^{-1}(Z \times \{0\}) + F$, with $F$ the strict transform of $X \times \{0\}$. Denoting by $\nu : \widetilde{\mathcal{X}} \to \mathcal{X}$ the normalization morphism, we infer $\widetilde{\mathcal{X}}_0 = D + \nu^*F$, and hence $b_E = \operatorname{ord}_E(\widetilde{\mathcal{X}}_0) = \operatorname{ord}_E(D)$.

This shows in particular that the valuations $b_E^{-1}\operatorname{ord}_E$ are the Rees valuations of $\mathfrak{a} + (t)$. We also get that $v_E(\mathfrak{a}) = b_E^{-1} r(\operatorname{ord}_E)(\mathfrak{a}) \geq 1$, and hence $\overline{\mathfrak{a}^m} \subset \bigcap_E \{f \in \mathcal{O}_X \mid v_E(f) \geq m\}$. Conversely, assume $f \in \mathcal{O}_X$ satisfies $v_E(f) \geq m$ for all $E$. Since the $b_E^{-1}\operatorname{ord}_E$ are the Rees valuations of $\mathfrak{a} + (t)$, applying Theorem 1.9 on $X \times \mathbb{A}^1$ yields $f \in \overline{(\mathfrak{a} + (t))^m}$. Since $\mathfrak{a}^m$ is the weight 0 part of $(\mathfrak{a} + (t))^m$, Lemma 4.9 yields $f \in \overline{\mathfrak{a}^m}$, and we have thus established (i).

Finally, let $S$ be any finite set of $k^*$-invariant valuations $w$ on $K(t)$ such that

$$\overline{\mathfrak{a}^m} = \bigcap_{w\in S} \{f \in \mathcal{O}_X \mid r(w)(f) \geq m\}$$

for all $m \in \mathbb{N}$. We claim that we then have

$$\overline{(\mathfrak{a} + (t))^m} = \bigcap_{w\in S} \{f \in \mathcal{O}_{\mathcal{X}} \mid w(f) \geq m\}$$

for all $m \geq \mathbb{N}$. This will prove (ii), by the minimality of the set of Rees valuations of $\mathfrak{a} + (t)$. So assume that $f \in \mathcal{O}_{\mathcal{X}}$ satisfies $w(f) \geq m$ for all $w \in S$. In terms of the Laurent expansion (4.1), we get $r(w)(f_\lambda) + \lambda \geq m$ for all $\lambda$, $w$, and hence $f_\lambda \in \overline{\mathfrak{a}^{m-\lambda}}$ by assumption. By Lemma 4.9, we conclude as desired that $f \in \overline{(\mathfrak{a} + (t))^m}$. □

**Corollary 4.10.** *Let $(X, L)$ be a polarized scheme and $Z \subset X$ a closed subscheme. Then there exists a normal, ample test configuration $(\mathcal{X}, \mathcal{L})$ such that the Rees valuations of $Z$ are exactly the divisorial valuations $v_E$ on $X$ associated to the non-trivial irreducible components of $\mathcal{X}_0$.*

*Proof.* Let $\mu : \mathcal{X} \to X \times \mathbb{A}^1$ be the normalized blow-up of $Z \times \{0\}$, so that $\mathcal{X}$ is the normalization of the deformation to the normal cone of $Z$. As recalled in Lemma 1.7, $D := \mu^{-1}(Z)$ is a Cartier divisor with $-D$ ample. We may thus choose $0 < c \ll 1$ such that $\mathcal{L} := \mu^*L_{\mathbb{A}^1} - cD$ is ample, and $(\mathcal{X}, \mathcal{L})$ is then a normal, ample test configuration. The rest follows from Theorem 4.8. □

4.4. **Log discrepancies.** In this section we assume that $k$ has characteristic 0

**Proposition 4.11.** *Let $B$ be a boundary on $X$. For every component $E$ of $\mathcal{X}_0$, the log discrepancies of $v_E$ and $\mathrm{ord}_E$ (with respect to $(X,B)$ and $(X_{\mathbb{A}^1}, B_{\mathbb{A}^1})$, respectively) are then related by*

$$A_{(X_{\mathbb{A}^1},B_{\mathbb{A}^1})}(\mathrm{ord}_E) = b_E\left(1 + A_{(X,B)}(v_E)\right)$$

Recall that $A_{(X,B)}(v_{\mathrm{triv}})$ is defined to be 0.

*Proof.* If $E$ is the strict transform of $X \times \{0\}$, then $A_{(X,B)}(v_E) = A_{(X,B)}(v_{\mathrm{triv}}) = 0$, while $A_{(X_{\mathbb{A}^1},B_{\mathbb{A}^1})}(\mathrm{ord}_E) = b_E = 1$.

Assume now that $E$ is non-trivial. Since $v_E$ is a divisorial valuation on $X$, we may find a proper birational morphism $\mu : X' \to X$ with $X'$ smooth and a smooth irreducible divisor $F \subset X'$ such that $v_E = c\,\mathrm{ord}_F$ for some rational $c > 0$. By Lemma 4.5, the divisorial valuation $\mathrm{ord}_E$ is monomial on $X'_{\mathbb{A}^1}$ with respect to the SNC divisor $X' \times \{0\} + F_{\mathbb{A}^1}$, with weights $\mathrm{ord}_E(X' \times \{0\}) = b_E$ and $\mathrm{ord}_E(F_{\mathbb{A}^1}) = b_E v_E(F) = b_E c$. It follows (see e.g. [JM12, Prop. 5.1]) that

$$\begin{aligned} A_{(X_{\mathbb{A}^1},B_{\mathbb{A}^1})}(\mathrm{ord}_E) &= b_E A_{(X_{\mathbb{A}^1},B_{\mathbb{A}^1})}\left(\mathrm{ord}_{X\times\{0\}}\right) + b_E c A_{(X_{\mathbb{A}^1},B_{\mathbb{A}^1})}\left(\mathrm{ord}_{F_{\mathbb{A}^1}}\right) \\ &= b_E + b_E c A_{(X,B)}\left(\mathrm{ord}_F\right) = b_E\left(1 + A_{(X,B)}(v_E)\right), \end{aligned}$$

which completes the proof. □

## 5. Non-Archimedean metrics

Let $X$ be a proper scheme and $L$ a line bundle on $X$. Motivated by Berkovich space considerations (see §5.3 below) we introduce the following notion.

**Definition 5.1.** *Two test configurations $(\mathcal{X}_1, \mathcal{L}_1)$, $(\mathcal{X}_2, \mathcal{L}_2)$ for $(X, L)$ are* equivalent *if there exists a test configuration $(\mathcal{X}_3, \mathcal{L}_3)$ that is a pull-back of both $(\mathcal{X}_1, \mathcal{L}_1)$ and $(\mathcal{X}_2, \mathcal{L}_2)$.*

*An equivalence class is called a* non-Archimedean metric *on $L$, and is denoted by $\phi$. We denote by $\phi_{\mathrm{triv}}$ the equivalence class of $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$.*

**Definition 5.2.** *Assume $L$ is ample. Then a non-Archimedean metric $\phi$ on $L$ is called* semipositive *if some (or, equivalently, any) representative $(\mathcal{X}, \mathcal{L})$ of $\phi$ is semiample.*

*We denote by $\mathcal{H}^{\mathrm{NA}}(L)$ the set of all non-Archimedean semipositive metrics on $L$,* i.e. *the quotient of the set of semiample test configurations by the above equivalence relation.*

We also write $\mathcal{H}^{\mathrm{NA}}$ when no confusion is possible. The notation mimics $\mathcal{H} = \mathcal{H}(L)$ for the space of smooth positive (Archimedean) metrics on $L$ when working over $\mathbb{C}$.

**Lemma 5.3.** *Assume that $X$ is normal. Then every metric $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$ is represented by a unique normal, ample test configuration $(\mathcal{X}, \mathcal{L})$. Every normal representative of $\phi$ is a pull-back of $(\mathcal{X}, \mathcal{L})$.*

*Proof.* We first prove uniqueness. Let $(\mathcal{X}_i, \mathcal{L}_i)$, $i = 1, 2$, be equivalent normal, ample test configurations, so that there exists $(\mathcal{X}_3, \mathcal{L}_3)$ as in Definition 5.1. For $i = 1, 2$, the birational morphism $\mu_i : \mathcal{X}_3 \to \mathcal{X}_i$ satisfies $(\mu_i)_*\mathcal{O}_{\mathcal{X}_3} = \mathcal{O}_{\mathcal{X}_i}$, by normality of $\mathcal{X}_i$. Using the projection formula, we get

$$R(\mathcal{X}_1, m\mathcal{L}_1) \simeq R(\mathcal{X}_2, m\mathcal{L}_2)$$

for any $m \geq 1$ such that $m\mathcal{L}_1$ and $m\mathcal{L}_2$ are both line bundles. By taking the Proj over $\mathbb{A}^1$, it follows that $(\mathcal{X}_1, \mathcal{L}_1) = (\mathcal{X}_2, \mathcal{L}_2)$.

Now pick a normal representative $(\mathcal{X}, \mathcal{L})$ of $\phi$. By Proposition 2.14, its ample model $(\mathcal{X}_{\text{amp}}, \mathcal{L}_{\text{amp}})$ is a normal, ample representative, and $(\mathcal{X}, \mathcal{L})$ is a pull-back of $(\mathcal{X}_{\text{amp}}, \mathcal{L}_{\text{amp}})$. This proves the existence part, as well as the final assertion. □

5.1. **Intersection numbers.** Various operations on test configurations descend to non-Archimedean metrics. As a first example, we have intersection numbers.

Every finite set of test configurations $\mathcal{X}_i$ for $X$ is dominated by some test configuration $\mathcal{X}$. Given finitely many non-Archimedean metrics $\phi_i$ on line bundles $L_i$, we may thus find representatives $(\mathcal{X}_i, \mathcal{L}_i)$ with $\mathcal{X}_i = \mathcal{X}$ independent of $i$.

**Definition 5.4.** *Let $\phi_i$ be a non-Archimedean metric on $L_i$ for $0 \leq i \leq n$. We define the* intersection number *of the $\phi_i$ as*

$$(\phi_0 \cdot \ldots \cdot \phi_n) := (\bar{\mathcal{L}}_0 \cdot \ldots \cdot \bar{\mathcal{L}}_n), \tag{5.1}$$

*where $(\mathcal{X}_i, \mathcal{L}_i)$ is any representative of $\phi_i$ with $\mathcal{X}_i = \mathcal{X}$ independent of $i$, and where $(\bar{\mathcal{X}}, \bar{\mathcal{L}}_i)$ is the compactification of $(\mathcal{X}, \mathcal{L}_i)$.*

By the projection formula, the right hand side of (5.1) is independent of the choice of representatives. Note that the intersection number $(\phi_0 \cdot \ldots \cdot \phi_n)$ may be negative even when the $\phi_i$ are semipositive, since $\bar{\mathcal{L}}_i$ is only *relatively* semiample with respect to $\bar{\mathcal{X}} \to \mathbb{P}^1$.

**Lemma 5.5.** *Assume that $X$ is normal. Let $L_2, \ldots, L_n$ be ample line bundles on $X$ with $L_i$ ample for $i > 1$. Let $\phi$ be a non-Archimedean metrics on $\mathcal{O}_X$, and $\phi_i \in \mathcal{H}^{\mathrm{NA}}(L_i)$ semipositive non-Archimedean metrics on $L_i$ for $2 \leq i \leq n$. Then*

$$(\phi \cdot \phi \cdot \phi_2 \cdot \ldots \cdot \phi_n) \leq 0.$$

*Proof.* Choose normal representatives $(\mathcal{X}, \mathcal{L})$, $(\mathcal{X}, \mathcal{L}_i)$ with the same test configuration $\mathcal{X}$ for $X$, so that $\mathcal{L} = D$ is a $\mathbb{Q}$-Cartier divisor supported on $\mathcal{X}_0$. Then the claim amounts to $(D \cdot D \cdot \mathcal{L}_2 \cdot \ldots \cdot \mathcal{L}_n) \leq 0$, which follows from a standard Hodge index argument (see for instance [BFJ12, Proof of Theorem 4.3], and also [YZ13a, YZ13b]). □

5.2. **Translation and scaling.** The space of test configurations admits two natural operations. First, there is a *translation action* of $\mathbb{Q}$, with $c \in \mathbb{Q}$ sending $(\mathcal{X}, \mathcal{L})$ to $(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0)$. This clearly induces an action on non-Archimedean metrics, denoted by $\phi \mapsto \phi + c$.

Second, we can define a *scaling action* of the semigroup $\mathbb{N}^*$ of positive integers $d$, mapping $(\mathcal{X}, \mathcal{L})$ to its base change by $t \mapsto t^d$. Again, this action descends to an action on non-Archimedean metrics, denoted by $\phi \mapsto \phi_d$.

Note that $\phi_{\text{triv}}$ is fixed under the scaling action.

**Lemma 5.6.** *For non-Archimedean metrics $\phi_0, \ldots, \phi_n$ as in Definition 5.4, we have*

$$((\phi_0)_d \cdot \ldots \cdot (\phi_n)_d) = d(\phi_0 \cdot \ldots \cdot \phi_n)$$

*and*

$$((\phi_0 + c) \cdot \phi_1 \cdot \ldots \cdot \phi_n) = (\phi_0 \cdot \ldots \cdot \phi_n) + c(L_1 \cdot \ldots \cdot L_n)$$

*for all $d \in \mathbb{N}^*$ and $c \in \mathbb{Q}$.*

*Proof.* The first equality again follows from the projection formula. By flatness, we have

$$(\mathcal{X}_0 \cdot \bar{\mathcal{L}}_1 \cdot \ldots \cdot \bar{\mathcal{L}}_n) = (L_1 \cdot \ldots \cdot L_n),$$

which implies the second equality. $\square$

Both the translation and scaling action preserve the subset $\mathcal{H}^{\mathrm{NA}}(L)$ of semipositive metrics, for $L$ ample. For later use, we record the following obvious fact.

**Lemma 5.7.** *Suppose $X$ is normal, let $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$, and let $(\mathcal{X}, \mathcal{L})$ be its normal, ample representative. Then the normal, ample representative of $\phi_d$ is the test configuration $(\mathcal{X}_d, \mathcal{L}_d)$ with $\mathcal{X}_d$ the normalization of the base change of $\mathcal{X}$ by $t \mapsto t^d$ and $\mathcal{L}_d$ the pull-back of $\mathcal{L}$ by the induced finite morphism $\mathcal{X}_d \to \mathcal{X}$.*

5.3. **Berkovich space interpretation.** Let us now briefly explain the term "non-Archimedean metric". See [BJ15] for more details.

Equip the base field $k$ with the trivial norm $|\cdot|_0$, i.e. $|a|_0 = 1$ for $a \in k^*$. Also equip the field $K := k((t))$ of Laurent series with the non-Archimedean norm in which $|t| = e^{-1}$.

To any proper scheme $X$ over $k$ is then associated a Berkovich analytification $X^{\mathrm{an}}$, a compact Hausdorff space equipped with a structure sheaf [Berk90, Berk93]. Similarly, any line bundle $L$ on $X$ has an analytification $L^{\mathrm{an}}$. The valued field extension $K/k$ further gives rise to analytifications $X_K^{\mathrm{an}}$ and $L_K^{\mathrm{an}}$, together with a natural morphism $X_K^{\mathrm{an}} \to X^{\mathrm{an}}$ under which $L^{\mathrm{an}}$ pulls pack to $L_K^{\mathrm{an}}$. The Gauss extension in §4 gives a section $X^{\mathrm{an}} \to X_K^{\mathrm{an}}$, whose image exactly consists of the $\mathbb{G}_m$-invariant points.

After the base change $k[t] \to k[\![t]\!]$, any test configuration $(\mathcal{X}, \mathcal{L})$ defines a normal model of $(X_K, L_K)$ over the valuation ring $k[\![t]\!]$ of $K = k((t))$. When $\mathcal{X}$ (and hence $X$) is normal, this further induces a continuous metric on $L_K^{\mathrm{an}}$, i.e. a function on the total space satisfying certain natural conditions. Using the Gauss extension, we obtain a metric also on $L^{\mathrm{an}}$.

Replacing a normal test configuration $(\mathcal{X}, \mathcal{L})$ by a pullback does not change the induced metric on $L^{\mathrm{an}}$, and one may in fact show that two normal test configurations induce the same metric iff they are equivalent in the Definition 5.1. This justifies the name non-Archimedean metric for an equivalence class of test configurations. Further, in the analysis of [BFJ12, BFJ15a], semipositive metrics play the role of Kähler metrics in complex geometry.

However, we abuse terminology a little since there are non-Archimedean metrics on $L^{\mathrm{an}}$ that do not come from test configurations. For example, any filtration on $R(X, L)$ defines a metric on $L^{\mathrm{an}}$. For many purposes [BFJ12, BFJ15a] it is crucial to work with a more flexible notion of metrics.

## 6. Duistermaat-Heckman measures

6.1. **The Duistermaat-Heckman measure of a polarized $T$-scheme.** Let $T \simeq \mathbb{G}_m^r$ be a torus, with character lattice $M = \mathrm{Hom}(T, \mathbb{G}_m) \simeq \mathbb{Z}^r$. Every finite dimensional $T$-module $V$ admits a weight decomposition $V = \bigoplus_{\lambda \in M} V_\lambda$, where $t \in T$ acts on $V_\lambda$ by multiplication by $\lambda(t)$. The *weight measure* of $V$ is the probability measure $\mu_V$ on $M_{\mathbb{R}}$ defined by

$$\mu_V := \frac{1}{\dim V} \sum_{\lambda \in M} (\dim V_\lambda) \delta_\lambda,$$

As a straightforward consequence of the equivariant Riemann-Roch theorem for schemes, due to Edidin and Graham [EG97] and discussed in Appendix B, we prove:

**Theorem 6.1.** *Let $(X, L)$ be a polarized $T$-scheme,* i.e. *a projective $T$-scheme with an ample $T$-linearized line bundle $L$. Then the rescaled weight measures*

$$\mu_m := (1/m)_* \mu_{H^0(X,mL)}$$

*have uniformly bounded support, and converge weakly to a probability measure $\mathrm{DH}_{(X,L)}$ on $M_\mathbb{R}$ as $m \to \infty$. Further, for each polynomial function $P$ on $M_\mathbb{R}$, $\int_{M_\mathbb{R}} P(\lambda)\mu_m(d\lambda)$ admits a full asymptotic expansion in $m$.*

We call $\mathrm{DH}_{(X,L)}$ the *Duistermaat-Heckman measure* of the polarized $T$-scheme $(X, L)$.

**Remark 6.2.** *In order to explain the terminology, consider the case where $X$ is a smooth complex variety, and choose a maximal compact torus $T_c \simeq (S^1)^r$ of $T$ and a $T_c$-invariant smooth metric $\phi$ on $L$ with strictly positive curvature $\omega = dd^c\phi$. We then get a moment map $\mu : X \to M_\mathbb{R}$ for the $T_c$-action on the symplectic manifold $(X, \omega)$. The Duistermaat-Heckman measure as originally defined in* [DH82] *is $\mu_*(\omega^n)$, but this is known to coincide (up to normalization of the mass) with $\mathrm{DH}_{(X,L)}$ as defined above (see for instance* [WN12, Theorem 9.1] *and* [BWN14, Proposition 4.1]*).*

*Proof of Theorem 6.1.* Since $L$ is ample, $R(X, L)$ is finitely generated. It easily follows that the weights of $H^0(X, mL)$ grow at most linearly with $m$, which proves the first assertion. Since $\mu_m$ has both uniformly bounded support and fixed total mass, it will converge in the weak topology iff $\int_\mathbb{R} P\, d\mu_m$ converges for each polynomial function. We are in fact going to show that $\int_\mathbb{R} P\, d\mu_m$ admits a full asymptotic expansion for each homogeneous polynomial $P \in \mathrm{S}^d M^*$, which will conclude the proof of the Theorem.

Let $\pi : X \to \operatorname{Spec} k$ be the structure morphism. Since the higher cohomology of $mL$ vanishes for $m \gg 1$, (B.1) of Appendix B yields

$$\sum_{\lambda \in M} \dim H^0(X, mL)_\lambda \frac{\lambda^d}{d!} = \left(\pi_* \left(e^{mc_1^T(L)} \cdot \tau^T(\mathcal{O}_X)\right)\right)_d$$
$$= \frac{m^{d+n}}{(d+n)!} \pi_* \left(c_1^T(L)^{d+n} \cdot [X]_T\right) + \text{l.o.t.},$$

where $[X]_T \in \mathrm{CH}_n^T(X)$ is the $T$-equivariant fundamental class of $X$ and the equality takes place in $\mathrm{CH}_T^d(\operatorname{Spec} k)_\mathbb{Q} \simeq \mathrm{S}^d M_\mathbb{Q}$. Since

$$\dim H^0(X, mL) = \frac{m^n}{n!}(L^n) + O(m^{n-1})$$

is itself a polynomial of degree $n$, we conclude that

$$\int_{M_\mathbb{R}} P(\lambda)\mu_m(d\lambda) = \frac{\sum_{\lambda \in M} \dim H^0(X, mL)_\lambda (P \cdot \lambda^d)}{m^d \dim H^0(X, mL)}$$

admits a full asymptotic expansion, with dominant term

$$\int_{M_\mathbb{R}} P(\lambda)\, \mathrm{DH}_{(X,L)}(d\lambda) = \binom{d+n}{n}^{-1} \frac{P \cdot \pi_* \left(c_1^T(L)^{d+n} \cdot [X]_T\right)}{(L^n)}. \tag{6.1}$$

□

Using this last identity, we next show that the study of Duistermaat-Heckman measures reduces to the case of polarized (normal) varieties.

**Proposition 6.3.** *Let $(X, L)$ be a polarized $T$-scheme of dimension $n$. Let $X_i$ be its $n$-dimensional irreducible components (with reduced scheme structure) and set $L_i := L|_{X_i}$. Then*

$$\mathrm{DH}_{(X,L)} = \sum_i c_i \mathrm{DH}_{(X_i,L_i)}$$

*where $c_i = m_i(L_i^n)/(L^n)$ and $m_i$ is the multiplicity of $X_i$ in $X$,* i.e. *the length of $\mathcal{O}_X$ at the generic point of $X_i$ (and hence $\sum_i c_i = 1$).*

*If $X$ is a variety with normalization morphism $\nu : \widetilde{X} \to X$, then*

$$\mathrm{DH}_{(X,L)} = \mathrm{DH}_{(\widetilde{X},\nu^* L)}.$$

*Proof.* Recall that $X$ and $X_{\mathrm{red}}$ have isomorphic equivariant Chow (co)homology. We may thus view $[X_T]$ as an element of $\mathrm{CH}^T_n(X_{\mathrm{red}})$. Since $T$ is connected, each $X_i$ is $T$-stable, hence defines a class $[X_i]_T \in \mathrm{CH}^T_n(X_{\mathrm{red}})$, and we then have

$$[X]_T = \sum_i m_i [X_i]_T.$$

The first assertion is now a direct consequence of (6.1), which similarly implies the second assertion thanks to the projection formula

$$\nu_*\left(c_1^T(\nu^* L)^{d+n} \cdot [\widetilde{X}]\right) = c_1^T(L)^{d+n} \cdot \nu_*[\widetilde{X}] = c_1^T(L)^{d+n} \cdot [X].$$

□

Relying on [Ok96], we prove:

**Proposition 6.4.** *Let $(X, L)$ be a polarized $T$-variety of dimension $n$. The support of $\mathrm{DH}_{(X,L)}$ is then given by the closed convex hull $P$ in $M_{\mathbb{R}}$ of the set*

$$\bigcup_{m \geq 1} \left\{ \frac{\lambda}{m} \mid H^0(X, mL)_\lambda \neq 0 \right\}$$

*and is a rational polytope. In particular, $\mathrm{DH}_{(X,L)}$ is a Dirac mass iff the $T$-action on $X$ is trivial. Further, $\mathrm{DH}_{(X,L)} = f\, d\lambda$ with $f \in C^0(P)$ such that $f^{1/(n-\dim P)}$ is concave.*

*Proof.* Since $R(X, L)$ is a finitely generated domain,

$$\Gamma := \left\{ (m, \lambda) \in \mathbb{N} \times M \mid H^0(X, mL)_\lambda \neq 0 \right\}$$

is a finitely generated sub-semigroup of $\mathbb{N} \times M$ (cf. [Bri87, Proposition 2.1]); hence $P$ is a rational polytope. In [Ok96], Okounkov constructs a convex body[2] $\Delta \subset \mathbb{R}^n$ and a linear map $p : \mathbb{R}^n \to M_{\mathbb{R}}$ such that $p(\Delta) = P$ and $p_*\mu = \mathrm{DH}_{(X,L)}$, with $\mu$ the Lebesgue measure on $\Delta$, normalized to mass 1. In particular, $\operatorname{supp} \mathrm{DH}_{(X,L)} = P$, so $\mathrm{DH}_{(X,L)}$ is a Dirac mass iff the $T$-action on $H^0(X, mL)$ has only one weight for all $m$ divisible enough. By the ampleness of $L$, this implies that the $T$-action on $X = \operatorname{Proj} R(X, L)$ is trivial.

The last assertion is a consequence of the Brunn-Minkowski inequality. □

[2]The convex body $\Delta$ is the Okounkov body of $(X, L)$ with respect to a suitable valuation of maximal rational rank, cf. [Bou14].

**Remark 6.5.** *Okounkov also mentions on* [Ok96, p.1] *that the density $f$ is piecewise polynomial on $P$, but while this is a classical result of Duistermaat and Heckman when $X$ is a smooth complex variety, we were not able to locate a proof in the general case. In particular, the proof of* [BP90, Proposition 3.4] *is unfortunately incomplete. The results of §6.3 below imply piecewise polynomiality when $T \simeq \mathbb{G}_m$ has rank* 1.

6.2. **The limit measure of a filtration.** Let $(X, L)$ be a polarized variety of dimension $n$, and set $R = R(X, L)$. In this section, we review and complement the study in [BC11] of a natural measure on $\mathbb{R}$ associated to a general $\mathbb{R}$-filtration $F^\bullet R$ on $R$.

Recall that the *volume* of a graded subalgebra $S \subset R$ is defined as

$$\mathrm{vol}(S) := \limsup_{m\to\infty} \frac{n!}{m^n} \dim S_m \in \mathbb{R}_{\geq 0}. \tag{6.2}$$

The following result is proved using Okounkov bodies [LM09, KK12] (see also the first author's appendix in [Szé14]).

**Lemma 6.6.** *Let $S \subset R$ be a graded subalgebra containing an ample series,* i.e.

(i) *$S_m \neq 0$ for all $m \gg 1$;*
(ii) *there exists $\mathbb{Q}$-divisors $A$ and $E$, ample and effective respectively, such that $L = A+E$ and $H^0(X, mA) \subset S_m \subset H^0(X, mL)$ for all $m$ divisible enough.*

*Then $\mathrm{vol}(S) > 0$, and the limsup in (6.2) is a limit. For each $m \gg 1$, let $\mathfrak{a}_m \subset \mathcal{O}_X$ be the base ideal of $S_m$,* i.e. *the image of the evaluation map $S_m \otimes \mathcal{O}_X(-mL) \to \mathcal{O}_X$, and let $\mu_m : X_m \to X$ be the normalized blow-up of $X$ along $\mathfrak{a}_m$, so that $\mathcal{O}_{X_m} \cdot \mathfrak{a}_m = \mathcal{O}_{X_m}(-F_m)$ with $F_m$ an effective Cartier divisor. Then we also have*

$$\mathrm{vol}(S) = \lim_{m\to\infty} \left(\mu_m^* L - \tfrac{1}{m} F_m\right)^n .$$

Now let $F^\bullet R$ be a filtration of the graded ring $R$, as defined in §1.1. We denote by

$$\lambda_{\max}^{(m)} = \lambda_1^{(m)} \geq \cdots \geq \lambda_{N_m}^{(m)} = \lambda_{\min}^{(m)}$$

the successive minima of $F^\bullet H^0(X, mL)$. As $R$ is an integral domain, the sequence $(\lambda_{\max}^{(m)})_{m\in\mathbb{N}}$ is superadditive in the sense that $\lambda_{\max}^{(m+m')} \geq \lambda_{\max}^{(m)} + \lambda_{\max}^{(m')}$, and this implies that

$$\lambda_{\max} = \lambda_{\max}(F^\bullet R) := \lim_{m\to\infty} \frac{\lambda_{\max}^{(m)}}{m} = \sup_{m\geq 1} \frac{\lambda_{\max}^{(m)}}{m} \in (-\infty, +\infty].$$

By definition, we have $\lambda_{\max} < +\infty$ iff there exists $C > 0$ such that $F^\lambda H^0(X, mL) = 0$ for any $\lambda, m$ such that $\lambda \geq Cm$, and we then say that $F^\bullet R$ has *linear growth.*

**Remark 6.7.** *In contrast, there always exists $C > 0$ such that $F^\lambda H^0(X, mL) = H^0(X, mL)$ for any $\lambda, m$ such that $\lambda \leq -Cm$. This is a simple consequence of the finite generation of $R$, cf.* [BC11, Lemma 1.5].

For each $\lambda \in \mathbb{R}$, we define a graded subalgebra of $R$ by setting

$$R^{(\lambda)} := \bigoplus_{m\in\mathbb{N}} F^{m\lambda} H^0(X, mL).$$

The main result of [BC11] may be summarized as follows.

**Theorem 6.8.** *Let $F^\bullet R$ be a filtration with linear growth.*

(i) *For each* $\lambda < \lambda_{\max}$, $R^{(\lambda)}$ *contains an ample series.*

(ii) *The function* $\lambda \mapsto \operatorname{vol}\left(R^{(\lambda)}\right)^{1/n}$ *is concave on* $(-\infty, \lambda_{\max})$, *and vanishes on* $(\lambda_{\max}, +\infty)$.

(iii) *If we introduce for each* $m$ *the probability measure on* $\mathbb{R}$

$$\nu_m := \frac{1}{N_m} \sum_j \delta_{m^{-1}\lambda_j^{(m)}} = -\frac{d}{d\lambda} \frac{\dim F^{m\lambda} H^0(X, mL)}{N_m}, \tag{6.3}$$

*then* $\nu_m$ *has uniformly bounded support and converges weakly as* $m \to \infty$ *to the probability measure*

$$\nu := -\frac{d}{d\lambda} V^{-1} \operatorname{vol}\left(R^{(\lambda)}\right).$$

We call $\nu$ the *limit measure* of the filtration $F^\bullet R$. The log concavity property of $\operatorname{vol}\left(R^{(\lambda)}\right)$ immediately yields:

**Lemma 6.9.** *The support of the limit measure* $\nu$ *is given by* $\operatorname{supp} \nu = [\lambda_{\min}, \lambda_{\max}]$ *with*

$$\lambda_{\min} := \inf \left\{ \lambda \in \mathbb{R} \mid \operatorname{vol}\left(R^{(\lambda)}\right) < V \right\}.$$

*Further,* $\nu$ *is absolutely continuous with respect to the Lebesgue measure, except perhaps for a Dirac mass at* $\lambda_{\max}$.

More precisely, the mass of $\nu$ on $\{\lambda_{\max}\}$ is equal to $\lim_{\lambda \to (\lambda_{\max})_-} \operatorname{vol}\left(R^{(\lambda)}\right)$.

**Remark 6.10.** *While we trivially have* $\lambda_{\min} \ge \limsup_{m\to\infty} k^{-1}\lambda_{\min}^{(m)}$, *the inequality can be strict in general, but it will be an equality for the filtrations considered in §6.3 and §6.4.*

6.3. **Finitely generated filtrations.** Let $(X, L)$ be a polarized variety. We shall show that the limit measure of a finitely generated $\mathbb{Z}$-filtration of $R = R(X, L)$ is well behaved. Recall from §2.5 that we can associate an ample test configuration to such a filtration.

**Theorem 6.11.** *Let* $F^\bullet R$ *be a finitely generated* $\mathbb{Z}$*-filtration of* $R = R(X, L)$, *with limit measure* $\nu$ *and associated ample test configuration* $(\mathcal{X}, \mathcal{L})$. *Then:*

(i) $F^\bullet R$ *has linear growth;*

(ii) *the density on* $(-\infty, \lambda_{\max})$ *is a piecewise polynomial function, of degree at most* $n-1$;

(iii) *the barycenter of* $\nu$ *is equal to*

$$\int_{\mathbb{R}} \lambda \, \nu(d\lambda) = \frac{\left(\bar{\mathcal{L}}^{n+1}\right)}{(n+1)(L^n)},$$

*with* $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ *the compactification of* $(\mathcal{X}, \mathcal{L})$.

*Proof.* The following argument is inspired by the proof of [ELMNP06, Proposition 4.13].[3] For each $\tau = (m, \lambda) \in \mathbb{N} \times \mathbb{Z}$, let $\mathfrak{a}_\tau$ be the base ideal of $F^\lambda H^0(X, mL)$, i.e. the image of the evaluation map $F^\lambda H^0(X, mL) \otimes \mathcal{O}_X(-mL) \to \mathcal{O}_X$. Let $\mu_\tau : X_\tau \to X$ be the normalized blow-up of $\mathfrak{a}_\tau$, which is also the normalized blow-up of its integral closure $\overline{\mathfrak{a}_\tau}$. We have

$$\mathcal{O}_{X_\tau} \cdot \mathfrak{a}_\tau = \mathcal{O}_{X_\tau} \cdot \overline{\mathfrak{a}_\tau} = \mathcal{O}_{X_\tau}(-F_\tau)$$

[3]While the base field in *loc.cit.* is $\mathbb{C}$, the results we use are valid over an arbitrary algebraically closed field.

with $F_\tau$ a Cartier divisor, and we set

$$V_\tau := \left(\mu_\tau^* L - \tfrac{1}{m} F_\tau\right)^n .$$

Since $R^{(\lambda)}$ contains an ample series for $\lambda \in (-\infty, \lambda_{\max})$, Lemma 6.6 yields

$$\mathrm{vol}\left(R^{(\lambda)}\right) = \lim_{m\to\infty} V_{(m,\lceil m\lambda\rceil)}.$$

Now, we use the finite generation of $F^\bullet R$, which implies that the $\mathbb{N}\times\mathbb{Z}$-graded $\mathcal{O}_X$-algebra $\bigoplus_{\tau\in\mathbb{N}\times\mathbb{Z}} \mathfrak{a}_\tau$ is finitely generated. By [ELMNP06, Proposition 4.7], we may thus find a positive integer $d$ and finitely many vectors $e_i = (m_i, \lambda_i) \in \mathbb{N}\times\mathbb{Z}$, $1 \le i \le r$, with the following properties:

(i) $e_1 = (0,-1)$, $e_r = (0,1)$, and the slopes $a_i := \lambda_i/m_i$ are strictly increasing with $i$;
(ii) Every $\tau \in \mathbb{N}\times\mathbb{Z}$ may be written as $\tau = p_i e_i + p_{i+1} e_{i+1}$ with $i$, $p_i, p_{i+1} \in \mathbb{N}$ uniquely determined, and the integral closures of $\mathfrak{a}_{d\tau}$ and $\mathfrak{a}_{de_i}^{p_i} \cdot \mathfrak{a}_{de_{i+1}}^{p_{i+1}}$ coincide.

Choose a projective birational morphism $\mu : X' \to X$ with $X'$ normal and dominating the blow-up of each $\mathfrak{a}_{de_i}$, so that there is a Cartier divisor $E_i$ with $\mathcal{O}_{X'} \cdot \mathfrak{a}_{de_i} = \mathcal{O}_{X'}(-E_i)$. For all $\tau = (m,\lambda) \in \mathbb{N}\times\mathbb{Z}$ written as in (ii) as $\tau = p_i e_i + p_{i+1} e_{i+1}$, we get

$$\mathcal{O}_{X'} \cdot \mathfrak{a}_{de_i}^{p_i} \cdot \mathfrak{a}_{de_{i+1}}^{p_{i+1}} = \mathcal{O}_{X'}(-(p_i E_i + p_{i+1} E_{i+1})),$$

and the universal property of normalized blow-ups therefore shows that $\mu$ factors through the normalized blow-up of $\mathfrak{a}_{de_i}^{p_i} \cdot \mathfrak{a}_{de_{i+1}}^{p_{i+1}}$. By Lemma 1.7, the latter is also the normalized blow-up of

$$\overline{\mathfrak{a}_{de_i}^{p_i} \cdot \mathfrak{a}_{de_{i+1}}^{p_{i+1}}} = \overline{\mathfrak{a}_{d\tau}},$$

so we infer that

$$\mathfrak{a}_{d\tau} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}\left(-(p_i E_i + p_{i+1} E_{i+1})\right),$$

with $p_i E_i + p_{i+1} E_{i+1}$ the pull-back of $F_{d\tau}$. As a result, we get

$$V_{d\tau} = \left(\mu^* L - \tfrac{1}{dm}\left(p_i E_i + p_{i+1} E_{i+1}\right)\right)^n .$$

Now pick $\lambda \in (0, \lambda_{\max})$, so that $\lambda \in [a_i, a_{i+1})$ for some $i$. We infer from the previous discussion that

$$\mathrm{vol}\left(R^{(\lambda)}\right) = \lim_{m\to\infty} V_{(m,\lceil m\lambda\rceil)} = \left(\mu^* L - (f_i(\lambda) E_i + f_{i+1}(\lambda) E_{i+1})\right)^n$$

for some affine functions $f_i, f_{i+1}$, and we conclude as desired that $\mathrm{vol}\left(R^{(\lambda)}\right)$ is a piecewise polynomial function of $\lambda \in (-\infty, \lambda_{\max})$, of degree at most $n$.

Finally we prove (iii). By Lemma 2.12, the successive minima of $F^\bullet H^0(X, mL)$ coincide with the $\mathbb{G}_m$-weights of $H^0(\mathcal{X}_0, m\mathcal{L}_0)$, for all $m$ divisible enough. In particular, the $\mathbb{G}_m$-weight $w_m$ of $\det H^0(\mathcal{X}_0, m\mathcal{L}_0)$, equals the sum of the $\mathbb{G}_m$-weights of $H^0(\mathcal{X}_0, m\mathcal{L}_0)$. By Theorem 6.8 we then have

$$\int_{\mathbb{R}} \lambda\, \nu(d\lambda) = \lim_{m\to\infty} \frac{w_m}{m N_m},$$

and the result thus follows from Proposition 3.4. □

**Remark 6.12.** *For a finitely generated $\mathbb{Z}$-filtration $F^\bullet R$, the graded subalgebra $R^{(\lambda)}$ is finitely generated for each $\lambda \in \mathbb{Q}$* [ELMNP06, Lemma 4.8]. *In particular,* $\mathrm{vol}\left(R^{(\lambda)}\right) \in \mathbb{Q}$ *for all $\lambda \in \mathbb{Q} \cap (-\infty, \lambda_{\max})$.*

6.4. **The filtration defined by a divisorial valuation.** Any valuation $v$ on $X$ defines a filtration $F_v^\bullet R$ on $R$ by setting

$$F_v^\lambda H^0(X, mL) := \left\{ s \in H^0(X, mL) \mid v(s) \geq \lambda \right\}.$$

The corresponding norm on $R$ is thus given by $\|s\|_m = e^{-v(s)}$ for all $s \in H^0(X, mL)$. As a special case of [BKMS14, Proposition 2.12], $F_v^\bullet R$ has linear growth for any divisorial valuation $v$. The following result will be needed later on.

**Lemma 6.13.** *Let $v$ be a divisorial valuation on $X$, and let $\nu$ be the limit measure of $F_v^\bullet R$. Then* $\operatorname{supp} \nu = [0, \lambda_{\max}]$. *In other words, we have*

$$\lim_{m \to \infty} \frac{\dim \left\{ s \in H^0(X, mL) \mid v(s) \geq \lambda m \right\}}{N_m} < 1$$

*for any $\lambda > 0$.*

The equivalence between the two statements follows from Lemma 6.9 above.

*Proof.* Let $Z \subset X$ be the closure of the center $c_X(v)$ of $v$ on $X$, and $w$ a Rees valuation of $Z$. Since the center of $w$ on $X$ belongs to $Z = \overline{c_X(v)}$, the general version of Izumi's theorem in [HS01] yields a constant $C > 0$ such that $v(f) \leq C w(f)$ for all $f \in \mathcal{O}_{X, c_X(w)}$.

Let $\mu \colon X' \to X$ be the normalized blow-up of $Z$ and set $E := \mu^{-1}(Z)$. By definition, the Rees valuations of $Z$ are given up to scaling by vanishing order along the irreducible components of $E$. Given $\lambda > 0$, we infer that

$$\{v \geq \lambda m\} \subset \mu_* \mathcal{O}_{X'}(-m\delta E)$$

for all $0 < \delta \ll 1$ and all $m \geq 1$. It follows that

$$\left\{ s \in H^0(X, mL) \mid v(s) \geq \varepsilon \lambda m \right\} \hookrightarrow H^0\left(X', m\left(\mu^* L - \delta E\right)\right),$$

so that $\operatorname{vol}\left(R^{(\lambda)}\right) \leq (\mu^* L - \delta E)^n$. But since $-E$ is $\mu$-ample, $\mu^* L - \delta E$ is ample on $X'$ for $0 < \delta \ll 1$, so that

$$\frac{d}{d\lambda} (\mu^* L - \delta E)^n = -n \left( E \cdot (\mu^* L - \delta E)^{n-1} \right) < 0.$$

It follows that

$$\operatorname{vol}\left(R^{(\lambda)}\right) \leq (\mu^* L - \delta E)^n < (\mu^* L)^n = V$$

for $0 < \delta \ll 1$, hence the result. $\square$

**Remark 6.14.** *At least in characteristic zero, the continuity of the volume function shows that* $\operatorname{vol}\left(R^{(\lambda)}\right) \to 0$ *as $\lambda \to \lambda_{\max}$ from below, so that $\nu$ has no atom at $\lambda_{\max}$, and is thus absolutely continuous on $\mathbb{R}$ (cf.* [BKMS14, Proposition 2.25]*).*

*On the other hand, $F^\bullet R$ is not finitely generated in general. Indeed, well-known examples of irrational volume show that* $\operatorname{vol}\left(R^{(1)}\right)$ *can sometimes be irrational (compare Remark 6.12).*

6.5. **The Duistermaat-Heckman measure of a non-Archimedean metric.** In this section, $L$ is ample. For any semiample test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, L)$, define the Duistermaat-Heckman measure $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ as the limit measure of the finitely generated filtration on $R = R(X, L)$ induced by $(\mathcal{X}, \mathcal{L})$. The terminology is justified by the second assertion in the following result:

**Proposition 6.15.** *The measure $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ only depends on the non-Archimedean metric $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$ defined by $(\mathcal{X}, \mathcal{L})$. Further, if $(\mathcal{X}, \mathcal{L})$ is ample, then $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ equals the Duistermaat-Heckman measure of the polarized $\mathbb{G}_m$-scheme $(\mathcal{X}_0, \mathcal{L}_0)$.*

This allows us to define the Duistermaat-Heckman measure $\mathrm{DH}_\phi$ of any metric $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$. Let us write down some of its main properties.

**Theorem 6.16.** *Let $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$ be a semipositive non-Archimedean metric on $L$.*

(i) *The measure $\mathrm{DH}_\phi$ equals the Duistermaat-Heckman measure of the polarized $\mathbb{G}_m$-scheme $(\mathcal{X}_0, \mathcal{L}_0)$, for any ample representative $(\mathcal{X}, \mathcal{L})$ of $\phi$.*

(ii) *The measure $\mathrm{DH}_\phi$ has piecewise polynomial density on $(-\infty, \lambda_{\max})$.*

(iii) *The barycenter of $\mathrm{DH}_\phi$ is equal to*

$$\int_{\mathbb{R}} \lambda \, \mathrm{DH}_\phi(d\lambda) = \frac{(\phi^{n+1})}{(n+1)(L^n)}.$$

(iv) *For $c \in \mathbb{Q}$ and $d \in \mathbb{N}^*$, $\mathrm{DH}_{\phi+c}$ and $\mathrm{DH}_{\phi_d}$ are the pushforwards of $\mathrm{DH}_\phi$ by $\lambda \mapsto \lambda + c$ and $\lambda \mapsto d\lambda$, respectively.*

*Proof of Proposition 6.15.* First assume that $(\mathcal{X}, \mathcal{L})$ is ample. By Lemma 2.12, the successive minima of $F^\bullet H^0(X, mL)$ coincide with the weights of the $\mathbb{G}_m$-action on $H^0(\mathcal{X}_0, m\mathcal{L}_0)$ for $m$ sufficiently divisible. This implies the second assertion of the proposition.

To prove the first assertion, it suffices to prove that $\mathrm{DH}_{(\mathcal{X},\mathcal{L})} = \mathrm{DH}_{(\mathcal{X}',\mathcal{L}')}$, whenever $(\mathcal{X}, \mathcal{L})$ is semiample and $(\mathcal{X}', \mathcal{L}')$ is a pullback of $(\mathcal{X}, \mathcal{L})$ via $\mu : \mathcal{X}' \to \mathcal{X}$.

Let $F^\bullet R$ (resp. $F'^\bullet R$) be the filtration defined by $(\mathcal{X}, \mathcal{L})$ (resp. $(\mathcal{X}', \mathcal{L}')$), and let $\nu$ (resp. $\nu'$) be the corresponding limit measure. Replacing replacing $\mathcal{L}$ with $\mathcal{L} + c\mathcal{X}_0$ translates the two measures by $c$, so we may assume $\nu$ and $\nu'$ are supported in $\mathbb{R}_+$. We claim that

$$F^\lambda H^0(X, mL) \subset F'^\lambda H^0(X, mL) \tag{6.4}$$

for all $\lambda$. Indeed, since $\mathcal{X}$ and $\mathcal{X}'$ are reduced by Proposition 2.7, $\mathcal{O}_\mathcal{X}$ injects into $\mu_* \mathcal{O}_{\mathcal{X}'}$, and the projection formula thus yields a $\mathbb{G}_m$-equivariant inclusion $H^0(\mathcal{X}, m\mathcal{L}) \hookrightarrow H^0(\mathcal{X}', m\mathcal{L}')$. In particular, we get an inclusion of the weight-$\lambda$ parts, and the claim follows by restricting to $H^0(X, mL)$. As a consequence of (6.4), we get

$$\nu'\{x > \lambda\} \geq \nu\{x > \lambda\} \tag{6.5}$$

for all $\lambda \in \mathbb{R}$, thanks to Theorem 6.8.

On the other hand, we claim that $\nu$ and $\nu'$ have the same barycenter $\bar{\lambda}$. Granted this claim, we get

$$\int_{\mathbb{R}_+} \nu\{x > \lambda\} d\lambda = \int_{\mathbb{R}_+} \lambda \, d\nu = \bar{\lambda} = \int_{\mathbb{R}_+} \lambda \, d\nu' = \int_{\mathbb{R}_+} \nu'\{x > \lambda\} d\lambda \tag{6.6}$$

since $\nu$ and $\nu'$ are supported in $\mathbb{R}_+$. Since $\lambda \mapsto \nu'\{x > \lambda\}$ and $\lambda \mapsto \nu\{x > \lambda\}$ are right-continuous, it follows from (6.5) and (6.6) that $\nu'\{x > \lambda\} = \nu\{x > \lambda\}$ for all $\lambda \in \mathbb{R}_+$, and hence $\nu = \nu'$, completing the proof.

It remains to prove the claim. Let $(\mathcal{X}_{\mathrm{amp}}, \mathcal{L}_{\mathrm{amp}})$ and $(\mathcal{X}'_{\mathrm{amp}}, \mathcal{L}'_{\mathrm{amp}})$ be the ample models of $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$, respectively; see Proposition 2.14. Then $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}_{\mathrm{amp}}, \mathcal{L}_{\mathrm{amp}})$ define the same filtration on $R$, and hence the same limit measure $\nu$. By Theorem 6.11, the barycenter $\bar{\lambda}$ of $\nu$ satisfies $(n+1)(L^n)\bar{\lambda} = (\bar{\mathcal{L}}_{\mathrm{amp}}^{n+1}) = (\bar{\mathcal{L}}^{n+1})$. Similarly, the barycenter $\bar{\lambda}'$ of $\nu'$ satisfies $(n+1)(L^n)\bar{\lambda}' = (\bar{\mathcal{L}}^{n+1})$. Since $\bar{\mathcal{L}}' = \mu^*\bar{\mathcal{L}}$ with $\mu$ birational, we get $(\bar{\mathcal{L}}^{n+1}) = (\bar{\mathcal{L}}'^{n+1})$, which completes the proof of the claim. □

*Proof of Theorem 6.16.* The assertion (i) is immediate from Proposition 6.15. Taking any ample representative $(\mathcal{X}, \mathcal{L})$ of $\phi$, Theorem 6.11 implies (ii) and (iii). The first point of (iv) follows from the fact that the $\mathbb{G}_m$-weights of $H^0(\mathcal{X}_0, m(\mathcal{L} + c\mathcal{X}_0)|_{\mathcal{X}_0})$ are obtained by translating those of $H^0(\mathcal{X}_0, m\mathcal{L}_0)$ by $mc$ (cf. Remark 2.2). Similarly, denoting by $(\mathcal{X}', \mathcal{L}')$ the base change of $(\mathcal{X}, \mathcal{L})$ by $t \mapsto t^d$, then $(\mathcal{X}'_0, \mathcal{L}'_0) \simeq (\mathcal{X}_0, \mathcal{L}_0)$, but with the $\mathbb{G}_m$-action composed with $t \mapsto t^d$. As a result, the $\mathbb{G}_m$-weights of $H^0(\mathcal{X}'_0, m\mathcal{L}'_0)$ are obtained by multiplying those of $H^0(\mathcal{X}_0, m\mathcal{L}_0)$ by $d$, and the second point follows.

□

6.6. **Test configurations with zero norm.** The $L^p$-norm of a test configuration was introduced in [Don05, p.458], at least for $p$ an even integer. As in [His12, Definition 4.11], we introduce more systematically:

**Definition 6.17.** *Let $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$ be a semipositive non-Archimedean metric, with associated Duistermaat-Heckman measure $\nu = \mathrm{DH}_\phi$. For each $p \in [1, \infty]$, the $L^p$-norm $\|\phi\|_p$ is defined as the $L^p(\nu)$-norm of $\lambda - \bar{\lambda}$, with $\bar{\lambda} := \int_{\mathbb{R}} \lambda \, d\nu$ the barycenter of $\nu$.*

The invariance properties of the Duistermaat-Heckman measures imply:

**Lemma 6.18.** *Let $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$ be a semipositive non-Archimedean metric. Then we have $\|\phi + c\|_p = \|\phi\|_p$ for all $c \in \mathbb{Q}$, and $\|\phi_d\|_p = d\|\phi\|_p$ for all $d \in \mathbb{N}^*$.*

The main result of this section is the following characterization of non-Archimedean metrics with trivial norm. It implies Theorem A in the introduction.

**Theorem 6.19.** *Assume $X$ is normal and $L$ ample. Let $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$ be a semipositive non-Archimedean metric on $L$. Then the following are equivalent:*

(i) *the Duistermaat-Heckman measure $\mathrm{DH}_\phi$ is a Dirac mass;*
(ii) *for some (or, equivalently, any) $p \in [1, \infty]$, $\|\phi\|_p = 0$;*
(iii) *the normal ample representative of $\phi$ is a trivial test configuration.*

**Remark 6.20.** *For $p = 2$, the equivalence between (ii) and (iii) was independently established in* [Der14a, Theorem 4.7].

The proof of Theorem 6.19 relies on the following precise description of the support of the Duistermaat-Heckman measure.

**Theorem 6.21.** *Let $(\mathcal{X}, \mathcal{L})$ be a normal, semiample test configuration dominating $X_{\mathbb{A}^1}$, and write $\mathcal{L} = \rho^* L_{\mathbb{A}^1} + D$ with $\rho : \mathcal{X} \to X_{\mathbb{A}^1}$ the canonical morphism. Then the support $[\lambda_{\min}, \lambda_{\max}]$ of its Duistermaat-Heckman measure satisfies*

$$\lambda_{\min} = \min_E b_E^{-1} \operatorname{ord}_E(D) \quad \textit{and} \quad \lambda_{\max} = \max_E b_E^{-1} \operatorname{ord}_E(D) = \operatorname{ord}_{E_0}(D),$$

*where $E$ runs over the irreducible components of $\mathcal{X}_0$, $b_E := \operatorname{ord}_E(\mathcal{X}_0) = \operatorname{ord}_E(t)$, and $E_0$ is the strict transform of $X \times \{0\}$ (which has $b_{E_0} = 1$).*

*Proof of Theorem 6.19.* The equivalence between (i) and (ii) holds by the definition of the norm (Definition 6.17). Further, (iii) trivially implies (i), so we only need to show that (i) implies (iii). To this end, let $(\mathcal{X}, \mathcal{L})$ be a normal representative of $\phi$ dominating $X_{\mathbb{A}^1}$ via $\rho : \mathcal{X} \to X_{\mathbb{A}^1}$. Since $\mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ is a Dirac mass, Theorem 6.21 shows that $D$ is proportional to $\mathcal{X}_0$. Hence $\mathcal{L} = \rho^*(L_{\mathbb{A}^1} + cX \times \{0\})$ for some $c \in \mathbb{Q}$, which proves as desired that the normal ample representative of $(\mathcal{X}, \mathcal{L})$ is trivial. □

We now prepare for the proof of Theorem 6.21.

**Lemma 6.22.** *In the notation of Theorem 6.21, the induced filtration of $R$ satisfies for all $m$ divisible enough and all $\lambda \in \mathbb{Z}$*

$$F^\lambda H^0(X, mL) = \bigcap_E \left\{ s \in H^0(X, mL) \mid v_E(s) + m\, b_E^{-1} \operatorname{ord}_E(D) \geq \lambda \right\}$$

*where $E$ runs over the irreducible components of $\mathcal{X}_0$.*

According to Lemma 4.5, $v_E$ is a divisorial valuation on $X$ for $E \neq E_0$, while $v_{E_0}$ is the trivial valuation (so that $v_{E_0}(s)$ is either 0 for $s \neq 0$, or $+\infty$ for $s = 0$).

*Proof.* Pick any $m$ such that $m\mathcal{L}$ is a line bundle. By (2.1), a section $s \in H^0(X, mL)$ is in $F^\lambda_{(\mathcal{X},\mathcal{L})} H^0(X, mL)$ iff $\bar{s}t^{-\lambda} \in H^0(\mathcal{X}, m\mathcal{L})$, with $\bar{s}$ the $\mathbb{G}_m$-invariant rational section of $m\mathcal{L}$ induced by $s$. By normality of $\mathcal{X}$, this amounts in turn to $\operatorname{ord}_E\left(\bar{s}t^{-\lambda}\right) \geq 0$ for all $E$, i.e. $\operatorname{ord}_E(\bar{s}) \geq \lambda b_E$ for all $E$. The result follows since $m\mathcal{L} = \rho^*(mL_{\mathbb{A}^1}) + mD$ implies that

$$\operatorname{ord}_E(\bar{s}) = r(\operatorname{ord}_E)(s) + m \operatorname{ord}_E(D) = b_E v_E(s) + m \operatorname{ord}_E(D).$$

□

**Lemma 6.23.** *In the notation of Theorem 6.21, the filtration $F^\bullet H^0(X, mL)$ satisfies*

$$\frac{\lambda^{(m)}_{\min}}{m} = \min_E b_E^{-1} \operatorname{ord}_E(D) \quad \text{and} \quad \frac{\lambda^{(m)}_{\max}}{m} = \operatorname{ord}_{E_0}(D) = \max_E b_E^{-1} \operatorname{ord}_E(D)$$

*for all $m$ divisible enough.*

*Proof.* Set $c := \min_E b_E^{-1} \operatorname{ord}_E(D)$, and pick $m$ divisible enough (so that $mc$ is in particular an integer). The condition $v_E(s) + m \operatorname{ord}_E(D) \geq mcb_E$ automatically holds for all $s \in H^0(X, mL)$, since $v_E(s) \geq 0$. By Lemma 6.22, we thus have $F^{mc} H^0(X, mL) = H^0(X, mL)$, and hence $mc \leq \lambda^{(m)}_{\min}$.

We may assume $mL$ is globally generated, so for every $E$ we may find a section

$$s \in H^0(X, mL) = F^{\lambda^{(m)}_{\min}} H^0(X, mL)$$

that does not vanish at the center of $v_E$ on $X$, i.e. $v_E(s) = 0$. By Lemma 6.22, it follows that $m \operatorname{ord}_E(D) \geq \lambda^{(m)}_{\min} b_E$. Since this holds for every $E$, we conclude that $mc \geq \lambda^{(m)}_{\min}$.

We next use that $m\mathcal{L} = \rho^*(mL_{\mathbb{A}^1}) + mD$ is globally generated. This implies in particular that $\mathcal{O}_{\mathcal{X}}(mD)$ is $\rho$-globally generated, which reads

$$\mathcal{O}_{\mathcal{X}}(mD) = \rho_* \mathcal{O}_{\mathcal{X}}(mD) \cdot \mathcal{O}_{\mathcal{X}}$$

as fractional ideals. But we trivially have $\rho_* \mathcal{O}_{\mathcal{X}}(mD) \subset \mathcal{O}_{X_{\mathbb{A}^1}}(m\rho_* D)$, and we infer

$$D \leq \rho^* \rho_* D.$$

Now $\rho_* D = \operatorname{ord}_{E_0}(D) X \times \{0\}$, hence $\rho^* \rho_* D = \operatorname{ord}_{E_0} \mathcal{X}'_0$, which yields $\operatorname{ord}_E(D) \le \operatorname{ord}_{E_0}(D) b_E$, hence $\operatorname{ord}_{E_0}(D) = \max_E b_E^{-1} \operatorname{ord}_E(D)$.

Since $\rho_* \mathcal{O}(mD)$ is the flag ideal $\mathfrak{a}^{(m)}$ of Definition 2.16, we also see that

$$m \max_E b_E^{-1} \operatorname{ord}_E(D) = \min \{\lambda \in \mathbb{Z} \mid mD \le \lambda \mathcal{X}_0\}$$

$$= \min \left\{\lambda \in \mathbb{Z} \mid t^{-\lambda} \in \mathfrak{a}^{(m)}\right\} = \max \left\{\lambda \in \mathbb{Z} \mid \mathfrak{a}_\lambda^{(m)} \neq 0\right\},$$

and we conclude thanks to Proposition 2.18. □

*Proof of Theorem 6.21.* In view of Lemma 6.9, the description of the supremum of the support of $\nu = \mathrm{DH}_{(\mathcal{X},\mathcal{L})}$ follows directly from Lemma 6.23.

We now turn to the infimum. The subtle point of the argument is that it is not a priori obvious that the stationary value

$$\frac{\lambda_{\min}^{(m)}}{m} = \min_E b_E^{-1} \operatorname{ord}_E(D)$$

given by Lemma 6.23, which is of course the infimum of the support of $\nu^{(m)}$ as in (6.3), should also be the infimum of the support of their weak limit $\nu = \lim_m \nu^{(m)}$. What is trivially true is the inequality

$$\min_E b_E^{-1} \operatorname{ord}_E(D) = \inf \operatorname{supp} \nu^{(m)} \le \inf \operatorname{supp} \nu.$$

Now pick $\lambda > \min_E b_E^{-1} \operatorname{ord}_E(D)$. According to Lemma 6.9, it remains to show that

$$\lim_{m\to\infty} \frac{\dim F^{m\lambda} H^0(X, mL)}{N_m} < 1. \tag{6.7}$$

Note that $\varepsilon := \lambda b_E - \operatorname{ord}_E(D) > 0$ for at least one component $E$. By Lemma 6.22, it follows that

$$F^{m\lambda} H^0(X, mL) \subset \left\{s \in H^0(X, mL) \mid v_E(s) \ge m\varepsilon\right\}. \tag{6.8}$$

By Lemma 4.5, $v_E$ is either the trivial valuation or a divisorial valuation. In the former case, the right-hand side of (6.8) consists of the zero section only, while in the latter case we get (6.7) thanks to Lemma 6.13. □

## 7. Non-Archimedean functionals

The aim of this section is to introduce non-Archimedean analogues of several classical functionals in Kähler geometry. Using these, we formulate and study a uniform notion of K-stability. Througout this section, $(X, L)$ is a normal polarized variety. Write $V = (L^n)$ and $\mathcal{H}^{\mathrm{NA}} = \mathcal{H}^{\mathrm{NA}}(L)$.

**Definition 7.1.** *A functional $F$ on $\mathcal{H}^{\mathrm{NA}}$ is* homogeneous *if $F(\phi_d) = dF(\phi)$ for all $\phi \in \mathcal{H}^{\mathrm{NA}}$ and $d \in \mathbb{N}^*$, and* translation invariant *if $F(\phi + c) = F(\phi)$ for all $\phi \in \mathcal{H}^{\mathrm{NA}}$ and $c \in \mathbb{Q}$.*

For example, Lemma 6.18 shows that the $L^p$-norm is homogeneous and translation invariant. In the next few sections we shall define several more functionals on $\mathcal{H}^{\mathrm{NA}}(L)$, all modeled upon classical functionals on the space of Kähler metrics.

7.1. **The non-Archimedean Monge-Ampère energy.**

**Definition 7.2.** *The* non-Archimedean Monge-Ampère energy functional $E^{\mathrm{NA}} : \mathcal{H}^{\mathrm{NA}} \to \mathbb{R}$ *is defined by*

$$E^{\mathrm{NA}}(\phi) := \frac{\left(\phi^{n+1}\right)}{(n+1)V}.$$

Note that $E(\phi_{\mathrm{triv}}) = 0$ since $(\phi_{\mathrm{triv}}^{n+1}) = (L_{\mathbb{P}^1}^{n+1}) = 0$. Lemma 5.6 and Theorem 6.16 imply:

**Lemma 7.3.** *The functional $E^{\mathrm{NA}}$ is homogeneous and satisfies*

$$E^{\mathrm{NA}}(\phi + c) = E^{\mathrm{NA}}(\phi) + c \tag{7.1}$$

*for all $\phi \in \mathcal{H}^{\mathrm{NA}}$ and $c \in \mathbb{Q}$. We further have*

$$E^{\mathrm{NA}}(\phi) = \int_{\mathbb{R}} \lambda \, \mathrm{DH}_\phi(d\lambda).$$

**Lemma 7.4.** *For each $\phi \in \mathcal{H}^{\mathrm{NA}}$ we have*

$$E^{\mathrm{NA}}(\phi) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \left( (\phi - \phi_{\mathrm{triv}}) \cdot \phi^j \cdot \phi_{\mathrm{triv}}^{n-j} \right).$$

*Further, for $j = 0, \ldots, n-1$, we have the inequality*

$$\left( (\phi - \phi_{\mathrm{triv}}) \cdot \phi^j \cdot \phi_{\mathrm{triv}}^{n-j} \right) \geq \left( (\phi - \phi_{\mathrm{triv}}) \cdot \phi^{j+1} \cdot \phi_{\mathrm{triv}}^{n-j-1} \right). \tag{7.2}$$

*Proof.* Since $(\phi_{\mathrm{triv}}^{n+1}) = 0$, we get

$$(n+1)V E^{\mathrm{NA}}(\phi) = (\phi^{n+1}) - (\phi_{\mathrm{triv}}^{n+1}) = \sum_{j=0}^{n} \left( (\phi - \phi_{\mathrm{triv}}) \cdot \phi^j \cdot \phi_{\mathrm{triv}}^{n-j} \right).$$

The inequality (7.2) is now a consequence of Lemma 5.5. □

7.2. **The non-Archimedean $I$ and $J$-functionals.**

**Definition 7.5.** *The* non-Archimedean $I$ and $J$-functionals *on $\mathcal{H}^{\mathrm{NA}}$ are defined by*

$$I^{\mathrm{NA}}(\phi) := V^{-1} \left( \phi \cdot \phi_{\mathrm{triv}}^n \right) - V^{-1} \left( (\phi - \phi_{\mathrm{triv}}) \cdot \phi^n \right)$$

*and*

$$J^{\mathrm{NA}}(\phi) := V^{-1}(\phi \cdot \phi_{\mathrm{triv}}^n) - E^{\mathrm{NA}}(\phi).$$

**Remark 7.6.** *In our notation, the expression for the* minimum norm $\|(\mathcal{X}, \mathcal{L})\|_m$ *given in* [Der14a, Remark 3.19] *reads*

$$\|(\mathcal{X}, \mathcal{L})\|_m = \frac{1}{n+1}(\phi^{n+1}) - \left( (\phi - \phi_{\mathrm{triv}}) \cdot \phi^n \right),$$

i.e. $\|(\mathcal{X}, \mathcal{L})\|_m = V J^{\mathrm{NA}}(\phi)$*, where $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$ is the metric induced by $(\mathcal{X}, \mathcal{L})$.*

**Lemma 7.7.** *For each $\phi \in \mathcal{H}^{\mathrm{NA}}$, we have*

$$V^{-1}(\phi \cdot \phi_{\mathrm{triv}}^n) = \lambda_{\max}(F_\phi^\bullet R) = \sup \operatorname{supp} \mathrm{DH}_\phi$$

*Proof.* Choose a normal, semiample test configuration $(\mathcal{X},\mathcal{L})$ representing $\phi$ and such that $\mathcal{X}$ dominates $X_{\mathbb{A}^1}$. Denote by $\rho:\mathcal{X}\to X_{\mathbb{A}^1}$ the canonical morphism, so that $\mathcal{L}=\rho^*L_{\mathbb{A}^1}+D$ for a unique $\mathbb{Q}$-Cartier divisor $D$ supported on $\mathcal{X}_0$. Then

$$(\phi\cdot\phi_{\mathrm{triv}}^n)=((\phi-\phi_{\mathrm{triv}})\cdot\phi_{\mathrm{triv}}^n)=\left(D\cdot\rho^*L_{\mathbb{A}^1}^n\right)=(\rho_*D\cdot L_{\mathbb{A}^1}^n)=V\operatorname{ord}_{E_0}(D)$$

with $E_0$ the strict transform of $X\times\{0\}$ on $\mathcal{X}$. Theorem 6.21 yields the desired conclusion. □

**Proposition 7.8.** *The non-Archimedean functionals $I^{\mathrm{NA}}$ and $J^{\mathrm{NA}}$ are non-negative, translation invariant, homogeneous, and satisfy*

$$\tfrac{1}{n}J^{\mathrm{NA}}\le I^{\mathrm{NA}}-J^{\mathrm{NA}}\le nJ^{\mathrm{NA}}.$$

*We further have*

$$J^{\mathrm{NA}}(\phi)=\sup\operatorname{supp}\mathrm{DH}_\phi-\int_{\mathbb{R}}\lambda\,\mathrm{DH}_\phi(d\lambda)$$

*for all $\phi\in\mathcal{H}^{\mathrm{NA}}$.*

*Proof.* Translation invariance and homogeneity follow directly from Lemma 5.6, while non-negativity is a consequence of (7.2). The latter also shows that

$$\begin{aligned}V^{-1}\left((\phi-\phi_{\mathrm{triv}})\cdot\phi_{\mathrm{triv}}^n\right)+nV^{-1}\left((\phi-\phi_{\mathrm{triv}})\cdot\phi^n\right)&\le(n+1)E^{\mathrm{NA}}(\phi)\\&\le nV^{-1}\left((\phi-\phi_{\mathrm{triv}})\cdot\phi_{\mathrm{triv}}^n\right)+V^{-1}\left((\phi-\phi_{\mathrm{triv}})\cdot\phi^n\right).\end{aligned}$$

This implies

$$\begin{aligned}n\left(I^{\mathrm{NA}}(\phi)-J^{\mathrm{NA}}(\phi)\right)&=n\left(E^{\mathrm{NA}}(\phi)-V^{-1}\left((\phi-\phi_{\mathrm{triv}})\cdot\phi^n\right)\right)\\&\ge V^{-1}\left((\phi-\phi_{\mathrm{triv}})\cdot\phi_{\mathrm{triv}}^n\right)-E^{\mathrm{NA}}(\phi)=J^{\mathrm{NA}}(\phi),\end{aligned}$$

and similarly for the other inequality.

The final assertion is a consequence of Lemma 7.3 and Lemma 7.7. □

**Corollary 7.9.** *For all $\phi\in\mathcal{H}^{\mathrm{NA}}$ we have $J^{\mathrm{NA}}(\phi)\ge\frac{1}{2}\|\phi\|_1$, and $J^{\mathrm{NA}}(\phi)=0$ iff $\phi=\phi_{\mathrm{triv}}+c$ for some $c\in\mathbb{Q}$.*

This result was also obtained by Dervan, see [Der14a, Theorem 1.2].

*Proof.* With $\bar\lambda:=\int_{\mathbb{R}}\lambda\mathrm{DH}_\phi(d\lambda)$, Proposition 7.8 and Definition 6.17 yield

$$J^{\mathrm{NA}}(\phi)\ge\int_{\{\lambda>\bar\lambda\}}(\lambda-\bar\lambda)\mathrm{DH}_\phi(d\lambda)=\frac{1}{2}\int|\lambda-\bar\lambda|\mathrm{DH}_\phi(d\lambda)=\frac{1}{2}\|\phi\|_1.$$

The last point of the corollary now follows from Theorem 6.19. □

7.3. **The non-Archimedean Mabuchi functional.** We assume from now on that the base field $k$ has characteristic 0. Let $B$ be a boundary on $X$. In view of Definition 7.2, we may rewrite the Donaldson-Futaki invariant with respect to the pair $(X,B)$ of a normal test configuration $(\mathcal{X},\mathcal{L})$ (see §3.4) as

$$\mathrm{DF}_B(\mathcal{X},\mathcal{L})=V^{-1}\left(\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1}+\bar{\mathcal{B}}\right)\cdot\bar{\mathcal{L}}^n\right)+\bar S_BE^{\mathrm{NA}}(\mathcal{X},\mathcal{L}),\tag{7.3}$$

with $\bar{\mathcal{B}}$ the (component-wise) Zariski closure of $B\times\mathbb{G}_m$ in $\bar{\mathcal{X}}$.

Since canonical divisor classes are compatible under push-forward, the projection formula shows that $\mathrm{DF}_B$ is invariant under pull-back, hence descends to a functional $\mathrm{DF}_B:\mathcal{H}^{\mathrm{NA}}\to\mathbb{R}$. While it is straightforward to see that $\mathrm{DF}_B$ is translation invariant, it is, however, *not* homegenous, and we therefore introduce an ‘error term’ to recover this property.

**Definition 7.10.** *The* non-Archimedean Mabuchi functional *of the polarized pair* $((X,B),L)$ *is the functional* $M_B^{\mathrm{NA}}:\mathcal{H}^{\mathrm{NA}}(L)\to\mathbb{R}$ *defined, for* $\phi\in\mathcal{H}^{\mathrm{NA}}(L)$ *by*

$$M_B^{\mathrm{NA}}(\phi):=\mathrm{DF}_B(\phi)+V^{-1}\left((\mathcal{X}_{0,\mathrm{red}}-\mathcal{X}_0)\cdot\bar{\mathcal{L}}^n\right)$$

*for each normal, semiample test configuration* $(\mathcal{X},\mathcal{L})$ *representing* $\phi$.

It is clear that $M_B^{\mathrm{NA}}\le\mathrm{DF}_B$ on $\mathcal{H}^{\mathrm{NA}}$. We can make this more precise as follows.

**Definition 7.11.** *A non-Archimedean metric* $\phi\in\mathcal{H}^{\mathrm{NA}}$ *is* reduced *if the central fiber* $\mathcal{X}_0$ *of its unique normal ample representative* $(\mathcal{X},\mathcal{L})$ *is reduced.*

**Proposition 7.12.** *We have* $M_B^{\mathrm{NA}}(\phi)\le\mathrm{DF}_B(\phi)$ *for* $\phi\in\mathcal{H}^{\mathrm{NA}}$*, with equality iff* $\phi$ *is reduced.*

**Proposition 7.13.** *The non-Archimedean Mabuchi functional* $M_B^{\mathrm{NA}}:\mathcal{H}^{\mathrm{NA}}(L)\to\mathbb{R}$ *is translation invariant and homogeneous.*

*Proof.* Translation invariance is straightforward to verify. As for homogeneity, it is enough to prove it for

$$(\mathcal{X},\mathcal{L})\mapsto\left(\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1}+(\mathcal{X}_{0,\mathrm{red}}-\mathcal{X}_0)\right)\cdot\bar{\mathcal{L}}^n\right).$$

As in [LX14, §4], this, in turn, is a consequence of the pull-back formula for log canonical divisors. More precisely, let $(\mathcal{X}_d,\mathcal{L}_d)$ be the normalized base change of $(\mathcal{X},\mathcal{L})$ as in Lemma 5.7, and denote by $f_d:\mathbb{P}^1\to\mathbb{P}^1$ and $g_d:\bar{\mathcal{X}}_d\to\bar{\mathcal{X}}$ the induced finite morphisms, which both have degree $d$. The pull-back formula for log canonical divisors (see for instance [BdFF12, Lemma 3.3]) yields

$$K_{\bar{\mathcal{X}}_d}+(\mathcal{X}_{d,0})_{\mathrm{red}}+(\mathcal{X}_{d,\infty})_{\mathrm{red}}=g_d^*\left(K_{\bar{\mathcal{X}}}+\mathcal{X}_{0,\mathrm{red}}+\mathcal{X}_{\infty,\mathrm{red}}\right).$$

and

$$K_{\mathbb{P}^1}+[0]+[\infty]=f_d^*(K_{\mathbb{P}^1}+[0]+[\infty]),$$

The fibers over $\infty$ being reduced, it follows that

$$K_{\bar{\mathcal{X}}_d/\mathbb{P}^1}+\left((\mathcal{X}_{d,0})_{\mathrm{red}}-\mathcal{X}_{d,0}\right)=g_d^*\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1}+(\mathcal{X}_{0,\mathrm{red}}-\mathcal{X}_0)\right).$$

Since $g_d$ has degree $d$, we get as desired

$$\left(\left(K_{\bar{\mathcal{X}}_d/\mathbb{P}^1}+\left((\mathcal{X}_{d,0})_{\mathrm{red}}-\mathcal{X}_{d,0}\right)\right)\cdot\bar{\mathcal{L}}_d^n\right)=d\left(\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1}+(\mathcal{X}_{0,\mathrm{red}}-\mathcal{X}_0))\right)\cdot\bar{\mathcal{L}}^n\right)$$

by the projection formula. □

The homogeneity property of $M_B^{\mathrm{NA}}$ will turn out to be particularly useful in conjunction with the following weak form of semistable reduction.

**Proposition 7.14.** *For every* $\phi\in\mathcal{H}^{\mathrm{NA}}$*,* $\phi_d$ *is reduced for all* $d$ *divisible enough. For such* $d$ *we have* $M_B^{\mathrm{NA}}(\phi)=d^{-1}\,\mathrm{DF}_B(\phi_d)$.

*Proof.* Let $(\mathcal{X},\mathcal{L})$ be the normal ample representative of $\phi$, so that the normal ample representative of $\phi_d$ is given by $(\mathcal{X}_d,\mathcal{L}_d)$ as in Lemma 5.7. The result follows from the well-known elementary fact (compare for instance [KKMS, pp.100–101]) that the central fiber of the normalized base change $\mathcal{X}_d$ of $\mathcal{X}$ by $t\mapsto t^d$ becomes reduced as soon as $d$ is divisible by all coefficients of $\mathcal{X}_0$. □

7.4. **Entropy and Ricci energy.**

**Definition 7.15.** *Given a boundary $B$ on $X$, we define*

(i) *the* non-Archimedean entropy functional $H_B^{\mathrm{NA}} : \mathcal{H}^{\mathrm{NA}} \to \mathbb{R}$ *by setting*

$$H_B^{\mathrm{NA}}(\phi) := V^{-1} \sum_E A_{(X,B)}(v_E) b_E (E \cdot \mathcal{L}^n),$$

*where $E$ runs over the non-trivial irreducible components of the central fiber of any normal representative $(\mathcal{X}, \mathcal{L})$, $b_E := \mathrm{ord}_E(\mathcal{X}_0)$, $v_E = b_E^{-1} r(\mathrm{ord}_E)$, and $A_{(X,B)}(v_E)$ is the log discrepancy of $v_E$ with respect to the pair $(X, B)$;*

(ii) *the* non-Archimedean Ricci energy $R_B^{\mathrm{NA}} : \mathcal{H}^{\mathrm{NA}} \to \mathbb{R}$ *by*

$$R_B^{\mathrm{NA}}(\phi) := V^{-1} \left( \psi_{\mathrm{triv}} \cdot \phi^n \right)$$

*with $\psi_{\mathrm{triv}}$ the trivial non-Archimedean metric on $K_X + B$.*

More concretely, we have

$$R_B^{\mathrm{NA}}(\phi) = V^{-1} \left( p^*(K_X + B) \cdot \bar{\mathcal{L}}^n \right)$$

for any representative $(\mathcal{X}, \mathcal{L})$ dominating $X_{\mathbb{A}^1}$, with $p : \bar{\mathcal{X}} \to X$ the induced morphism.

We have the following non-Archimedean analogue of the Chen-Tian formula.

**Proposition 7.16.** *For each boundary $B$, we have*

$$M_B^{\mathrm{NA}} = H_B^{\mathrm{NA}} + \bar{S}_B E^{\mathrm{NA}} + R_B^{\mathrm{NA}} \quad \text{on } \mathcal{H}^{\mathrm{NA}}(L).$$

**Corollary 7.17.** *The functionals $H_B^{\mathrm{NA}}$ and $R_B^{\mathrm{NA}}$ are translation invariant and homogeneous.*

*Proof.* It is straightforward to see that the Ricci energy functional is translation invariant and homogeneous. These properties hold also for the Mabuchi functional and the Monge-Ampère energy functionals, so we conclude using Proposition 7.16 that they must hold for the entropy functional. (One may also verify this directly.) □

*Proof of Proposition 7.16.* Let $(\mathcal{X}, \mathcal{L})$ be a representative of $\phi \in \mathcal{H}^{\mathrm{NA}}$ dominating $X_{\mathbb{A}^1}$, and let $\rho : \bar{\mathcal{X}} \to X_{\mathbb{P}^1}$ be the corresponding morphism. For each non-trivial component $E$ of $\mathcal{X}_0$, set $A_E := A_{(X_{\mathbb{A}^1}, B_{\mathbb{A}^1})}(\mathrm{ord}_E)$. This satisfies

$$A_E = A_{(X,B)}(v_E) + \mathrm{ord}_E(\mathcal{X}_0) \tag{7.4}$$

by Proposition 4.11. Since $\bar{\mathcal{B}}$ is the the strict transform of $B_{\mathbb{P}^1}$ on $\bar{\mathcal{X}}$ and the non-trivial components of $\mathcal{X}_0$ are exactly the $\rho$-exceptional prime divisors, we have

$$K_{\bar{\mathcal{X}}} + \bar{\mathcal{B}} = \rho^* \left( K_{X_{\mathbb{P}^1}} + B_{\mathbb{P}^1} \right) + \sum_E (A_E - 1) E,$$

Combining this with (7.4) and denoting as above by $p : \bar{\mathcal{X}} \to X$ the composition of $\rho$ with the projection $X_{\mathbb{P}^1} \to X$, we get

$$\left( \left( K_{\bar{\mathcal{X}}/\mathbb{P}^1} + \bar{\mathcal{B}} \right) \cdot \mathcal{L}^n \right) = \left( p^* \left( K_X + B \right) \right) \cdot \mathcal{L}^n ) + \sum_E \left( A_{(X,B)}(v_E) + \mathrm{ord}_E(\mathcal{X}_0) - 1 \right) (E \cdot \mathcal{L}^n)$$

$$= V\, R_B^{\mathrm{NA}}(\phi) + V\, H_B^{\mathrm{NA}}(\phi) + \left( (\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}}) \cdot \mathcal{L}^n \right).$$

The result follows in view of (7.3). □

**Remark 7.18.** *In the terminology of* [Oda12], $H_B^{\mathrm{NA}}(\phi) + V^{-1}\left((\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}}) \cdot \mathcal{L}^n\right)$ *coincides (up to a multiplicative constant) with the 'discrepancy term' of the Donaldson-Futaki invariant, while* $\bar{S}_B\, E^{\mathrm{NA}}(\phi) + R_B^{\mathrm{NA}}(\phi)$ *corresponds to the 'canonical divisor part'.*

In the *Kähler-Einstein case*, i.e. when $K_X + B$ is numerically proportional to $L$, the formula for $M_B^{\mathrm{NA}}$ takes the following alternative form.

**Lemma 7.19.** *Assume that* $K_X + B \equiv \lambda L$ *for some* $\lambda \in \mathbb{Q}$. *Then*

$$M_B^{\mathrm{NA}} = H_B^{\mathrm{NA}} + \lambda\left(I^{\mathrm{NA}} - J^{\mathrm{NA}}\right).$$

*Proof.* The trivial non-Archimedean metric $\psi_{\mathrm{triv}}$ on $K_X + B$ corresponds $\lambda\phi_{\mathrm{triv}}$, with $\phi_{\mathrm{triv}}$ the trivial metric on $L$, and hence

$$R^{\mathrm{NA}}(\phi) = V^{-1}(\psi_{\mathrm{triv}} \cdot \phi^n) = \lambda V^{-1}\left(\phi_{\mathrm{triv}} \cdot \phi^n\right).$$

Since $\bar{S}_B = -n\lambda$, we infer

$$\begin{aligned}\bar{S}_B E^{\mathrm{NA}}(\phi) + R^{\mathrm{NA}}(\phi) &= \lambda V^{-1}\left[-\frac{n}{n+1}(\phi^{n+1}) + (\phi_{\mathrm{triv}} \cdot \phi^n)\right] \\ &= \lambda V^{-1}\left[\frac{1}{n+1}(\phi^{n+1}) - ((\phi - \phi_{\mathrm{triv}}) \cdot \phi^n)\right] \\ &= \lambda\left[E^{\mathrm{NA}}(\phi) - V^{-1}\left((\phi - \phi_{\mathrm{triv}}) \cdot \phi^n\right)\right] = \lambda\left(I^{\mathrm{NA}}(\phi) - J^{\mathrm{NA}}(\phi)\right).\end{aligned}$$

□

7.5. **Uniform K-stability.**

**Definition 7.20.** *A* norm $N$ *on* $\mathcal{H}^{\mathrm{NA}}$ *is a non-negative, homogeneous and translation invariant functional* $N : \mathcal{H}^{\mathrm{NA}} \to \mathbb{R}_+$ *such that* $N(\phi) = 0$ *iff* $\phi = \phi_{\mathrm{triv}} + c$ *for some* $c \in \mathbb{Q}$.

*Given a boundary* $B$ *on* $X$, *we say that the polarized pair* $((X,B),L)$ *is* uniformly K-stable with respect to the norm $N$ *if there exists* $\delta > 0$ *such that* $M_B^{\mathrm{NA}} \ge \delta N$ *on* $\mathcal{H}^{\mathrm{NA}}$.

The two cases we shall actually deal with are $N = J^{\mathrm{NA}}$ and $N = \|\cdot\|_p$. We shall then speak of *J-uniform K-stability* and *$L^p$-uniform K-stability*, respectively.

**Remark 7.21.** *By Remark 7.6, Dervan's notion of* uniform K-stability with respect to the minimum norm *introduced in* [Der14a] *is equivalent to J-uniform K-stability.*

In order to relate this with the usual notion of K-stability involving Donaldson-Futaki invariants (see Definition 3.8), we prove:

**Lemma 7.22.** *For any polarized pair* $((X,B),L)$, *any norm* $N$ *on* $\mathcal{H}^{\mathrm{NA}}$, *and any* $\delta > 0$, *the following assertions are equivalent:*

(i) $M_B^{\mathrm{NA}} \ge \delta N$ *on* $\mathcal{H}^{\mathrm{NA}}(L)$;
(ii) $\mathrm{DF}_B \ge \delta N$ *on* $\mathcal{H}^{\mathrm{NA}}(L)$.

In particular, the (trivial) implication (i)$\Longrightarrow$(ii) and the condition imposed on $N$ show that uniform K-stability with respect to $N$ indeed implies K-stability.

*Proof.* The implication (i)$\Longrightarrow$(ii) is trivial since $M_B^{\mathrm{NA}} \le \mathrm{DF}_B$. For the reverse implication, let $\phi \in \mathcal{H}^{\mathrm{NA}}$. By Proposition 7.12 we can pick $d \ge 1$ such that $M_B^{\mathrm{NA}}(\phi_d) = \mathrm{DF}_B(\phi_d)$. By assumption, $\mathrm{DF}_B(\phi_d) \ge \delta N(\phi_d)$, so we conclude using the homogeneity of $M_B^{\mathrm{NA}}$ and $N$. □

The same argument yields:

**Lemma 7.23.** *A polarized pair $((X,B),L)$ is K-stable iff $M_B^{\mathrm{NA}}(\phi)>0$ for every $\phi\in\mathcal{H}^{\mathrm{NA}}(L)$ such that $\phi-\phi_{\mathrm{triv}}$ is nonconstant.*

By Corollary 7.9, we have:

**Corollary 7.24.** *J-uniform K-stability implies $L^1$-uniform K-stability.*

The next result confirms G. Székelyhidi's expectation that $p=n/n-1$ is a threshold value for $L^p$-uniform K-stability. (cf. [Szé06, §3.1.1]).

**Proposition 7.25.** *A polarized pair $((X,B),L)$ cannot be $L^p$-uniformly K-stable unless $p\le\frac{n}{n-1}$. More precisely, there exists a sequence $\phi_\varepsilon\in\mathcal{H}^{\mathrm{NA}}(L)$, parametrized by $0<\varepsilon\ll 1$ rational, such that $M_B^{\mathrm{NA}}(\phi_\varepsilon)\sim\varepsilon^n$, $\|\phi_\varepsilon\|_p\sim\varepsilon^{1+\frac{n}{p}}$ for each $p\ge 1$, and $J^{\mathrm{NA}}(\phi_\varepsilon)=O\left(\varepsilon^{n+1}\right)$.*

*Proof.* Let $x\in X\setminus\operatorname{supp}B$ be a regular closed point, and $\mu:\mathcal{X}\to X_{\mathbb{A}^1}$ be the blow-up of $(x,0)$ (i.e. the deformation to the normal cone), with exceptional divisor $E$. For each rational $\varepsilon>0$ small enough, $\mathcal{L}_\varepsilon:=\mu^*L_{\mathbb{A}^1}-\varepsilon E$ is relatively ample, hence defines a normal, ample test configuration $(\mathcal{X},\mathcal{L}_\varepsilon)$ for $(X,L)$, with associated non-Archimedean metric $\phi_\varepsilon\in\mathcal{H}^{\mathrm{NA}}(L)$.

Lemma 6.22 gives the following description of the filtration $F_\varepsilon^\bullet R$ attached to $(\mathcal{X},\mathcal{L}_\varepsilon)$:

$$F_\varepsilon^{m\lambda}H^0(X,mL)=\left\{s\in H^0(X,mL)\mid v_E(s)\ge m(\lambda+\varepsilon)\right\}$$

for $\lambda\le 0$, and $F_\varepsilon^{m\lambda}H^0(X,mL)=0$ fir $\lambda>0$ otherwise. If we denote by $F$ the exceptional divisor of the blow-up $\mu:X'\to X$ at $x$, then $v_E=\operatorname{ord}_F$, and the Duistermaat-Heckman measure $\mathrm{DH}_\varepsilon$ is thus given by

$$\mathrm{DH}_\varepsilon(x>\lambda)=V^{-1}\left(\mu^*L-(\lambda+\varepsilon)F\right)^n=1-V^{-1}(\lambda+\varepsilon)^n$$

for $\lambda\in(-\varepsilon,0)$, $\mathrm{DH}_\varepsilon(x>\lambda)=1$ for $\lambda\le-\varepsilon$, and $\mathrm{DH}_\varepsilon(x>\lambda)=0$ for $\lambda>0$. Hence

$$\mathrm{DH}_\varepsilon=nV^{-1}\mathbf{1}_{[-\varepsilon,0]}(\lambda+\varepsilon)^{n-1}d\lambda+(1-V^{-1}\varepsilon^n)\delta_0.$$

We first see from this that

$$J^{\mathrm{NA}}(\phi_\varepsilon)=-E^{\mathrm{NA}}(\phi_e)=-\int_{\mathbb{R}}\lambda\,\mathrm{DH}_\varepsilon(d\lambda)=-\frac{n}{V}\int_{-\varepsilon}^0\lambda(\lambda+\varepsilon)^{n-1}\,d\lambda=O(\varepsilon^{n+1}),$$

and

$$\begin{aligned}\|\phi_\varepsilon\|_p^p&=\int_{\mathbb{R}}\left|\lambda-E^{\mathrm{NA}}(\phi_\varepsilon)\right|^p\,\mathrm{DH}_\varepsilon(\lambda)\\&=nV^{-1}\int_{-\varepsilon}^0\left|\lambda+O\left(\varepsilon^{n+1}\right)\right|^p(\lambda+\varepsilon)^{n-1}d\lambda+(1-V^{-1}\varepsilon^n)O\left(\varepsilon^{p(n+1)}\right)\\&=\varepsilon^{p+n}\left[nV^{-1}\int_0^1|t+O\left(\varepsilon^n\right)|^p(1-t)^{n-1}dt+O\left(\varepsilon^{n(p-1)}\right)+o(1)\right]\\&=\varepsilon^{p+n}(c+o(1))\end{aligned}$$

for some $c>0$. Finally, the estimate for $M_B^{\mathrm{NA}}(\phi_\varepsilon)$ is a special case of Proposition 8.12 below. □

## 8. K-stability and singularities of pairs

In this section, the base field $k$ is assumed to have characteristic 0.

8.1. **Odaka-type results for pairs.** Let $B$ be an effective boundary on $X$. Recall that the pair $(X,B)$ is lc (log canonical) if $A_{(X,B)}(v) \geq 0$ for all divisorial valuations $v$ on $X$, while $(X,B)$ is klt (Kawamata log terminal) if $A_{(X,B)}(v) > 0$ for all such $v$.

**Theorem 8.1.** *Let $(X,L)$ be a normal polarized variety, and $B$ an effective boundary on $X$. Then*

$$(X,B) \text{ lc} \Longleftrightarrow H_B^{\mathrm{NA}} \geq 0 \text{ on } \mathcal{H}^{\mathrm{NA}}$$

*and*

$$((X,B),L) \text{ K-semistable} \Longrightarrow (X,B) \text{ lc}.$$

The proof of this result, given in §8.3, follows rather closely the line of argument of [Oda13b]. The second implication is also observed in [OSu11, Theorem 6.1].

**Theorem 8.2.** *Let $(X,L)$ be a normal polarized variety and $B$ an effective boundary on $X$. Then the following assertions are equivalent:*

(i) *$(X,B)$ is klt;*
(ii) *there exists $\delta > 0$ such that $H_B^{\mathrm{NA}} \geq \delta J^{\mathrm{NA}}$ on $\mathcal{H}^{\mathrm{NA}}$;*
(iii) *$H_B^{\mathrm{NA}}(\phi) > 0$ for every $\phi \in \mathcal{H}^{\mathrm{NA}}$ that is not a translate of $\phi_{\mathrm{triv}}$.*

We prove this in §8.4. The proof of (iii)$\Longrightarrow$(i) is similar to that of [Oda13b, Theorem 1.3] (which deals with the Fano case), while that of (i)$\Longrightarrow$(ii) relies on an Izumi-type estimate (Theorem 8.13). As we shall see, (ii) holds with $\delta$ equal to the global log canonical threshold of $((X,B),L)$ (cf. Proposition 8.15 below).

The above results have the following consequences in the 'log Kähler-Einstein case', i.e. when $K_X + B \equiv \lambda L$ for some $\lambda \in \mathbb{R}$. First, we have a uniform version of [OSu11, Theorem 4.1, (i)]. Closely related results were independently obtained in [Der14a, §3.4].

**Corollary 8.3.** *Let $(X,L)$ be a normal polarized variety, $B$ an effective boundary on $X$, and assume that $K_X + B \equiv \lambda L$ with $\lambda > 0$. The following are then equivalent:*

(i) *$(X,B)$ is lc;*
(ii) *$((X,B),L)$ is J-uniformly K-stable, with $M_B^{\mathrm{NA}} \geq \frac{\lambda}{n} J^{\mathrm{NA}}$ on $\mathcal{H}^{\mathrm{NA}}$;*
(iii) *$((X,B),L)$ is K-semistable.*

Next, in the log Calabi-Yau case we get a uniform version of [OSu11, Theorem 4.1, (ii)]:

**Corollary 8.4.** *Let $(X,L)$ be normal polarized variety, $B$ an effective boundary on $X$, and assume that $K_X + B \equiv 0$. Then $((X,B),L)$ is K-semistable iff $(X,B)$ is lc. Further, the following assertions are equivalent:*

(i) *$(X,B)$ is klt;*
(ii) *$((X,B),L)$ is J-uniformly K-stable;*
(iii) *$((X,B),L)$ is K-stable.*

**Remark 8.5.** *By* [Oda12, Corollary 3.3], *there exist polarized $K$-stable Calabi-Yau orbifolds (which have log terminal singularities) $(X,L)$ that are not asymptotically Chow (or, equivalently, Hilbert) semistable. In view of Corollary 8.4, it follows that J-uniform K-stability does not imply asymptotic Chow stability in general.*

Finally, in the log Fano case we obtain:

**Corollary 8.6.** *Let $(X, L)$ be normal polarized variety, $B$ an effective boundary on $X$, and assume that $K_X + B \equiv -\lambda L$ with $\lambda > 0$. If $((X, B), L)$ is K-semistable, then $H_B^{\mathrm{NA}} \geq \frac{\lambda}{n} J^{\mathrm{NA}}$ on $\mathcal{H}^{\mathrm{NA}}$; in particular, $(X, B)$ is klt.*

A partial result in the reverse direction can be found in Proposition 8.16. See also [OSu11, Theorem 6.1] and [Der14a, Theorem 3.39] for closely related results. Corollaries 8.3, 8.4 and 8.6 are proved in §8.5.

8.2. **Lc and klt blow-ups.** The following result, due to Y. Odaka and C. Xu, deals with lc blow-ups. The proof is based on an ingeneous application of the MMP.

**Theorem 8.7.** [OX12, Theorem 1.1] *Let $B$ be an effective boundary on $X$ with coefficients at most 1. Then there exists a unique projective birational morphism $\mu : X' \to X$ such that the strict transform $B'$ of $B$ on $Y$ satisfies:*

(i) *the exceptional locus of $\mu$ is a (reduced) divisor $E$;*
(ii) *$(X', E + B')$ is lc and $K_{X'} + E + B'$ is $\mu$-ample.*

**Corollary 8.8.** *Let $B$ be an effective boundary on $X$, and assume that $(X, B)$ is not lc. Then there exists a closed subscheme $Z \subset X$ whose Rees valuations $v$ all satisfy $A_{(X,B)}(v) < 0$.*

*Proof.* If $B$ has a component $F$ with coefficient greater than 1, then $A_{(X,B)}(\mathrm{ord}_F) < 0$, and $Z := F$ has the desired property, since $\mathrm{ord}_F$ is its unique Rees valuation (cf. Example 1.10).

If not, Theorem 8.7 applies. Denoting by $A_i := A_{(X,B)}(\mathrm{ord}_{E_i})$ the log discrepancies of the component $E_i$ of $E$, we have

$$K_{X'} + E + B' = \pi^*(K_X + B) + \sum_i A_i E_i, \tag{8.1}$$

which proves that $\sum_i A_i E_i$ is $\mu$-ample, and hence $A_i < 0$ by the negativity lemma (or Lemma 1.12). Proposition 1.11 now yields the desired subscheme. □

We next prove an analogous result for klt pairs, using a well-known and easy consequence of the MMP as in [BCHM].

**Proposition 8.9.** *Let $B$ be an effective boundary, and assume that $(X, B)$ is not klt. Then there exists a closed subscheme $Z \subset X$ whose Rees valuations $v$ all satisfy $A_{(X,B)}(v) \leq 0$.*

*Proof.* If $B$ has a component $F$ with coefficient at least 1, then $A_{(X,B)}(\mathrm{ord}_F) \leq 0$, and we may again take $Z = F$.

Assume now that $B$ has coefficients less than 1. Let $\pi : X' \to X$ be a log resolution of $(X, B)$, which means $X'$ is smooth, the exceptional locus $E$ of $\mu$ is a (reduced) divisor, and $E + B'$ has SNC support, with $B'$ the strict transform of $B$. If we denote by $A_i := A_{(X,B)}(\mathrm{ord}_{E_i})$ the log discrepancies of the component $E_i$ of $E$, then (8.1) holds, and hence

$$K_{X'} + (1 - \varepsilon)E + B' = \pi^*(K_X + B) + \sum_i (A_i - \varepsilon)E_i \tag{8.2}$$

for any $0 < \varepsilon < 1$. If we pick $\varepsilon$ smaller than $\min_{A_i > 0} A_i$, the $\mathbb{Q}$-divisor $D := \sum_i (A_i - \varepsilon)F_i$ is then $\pi$-big (since the generic fiber of $\pi$ is a point), and is $\pi$-numerically equivalent to the log canonical divisor of the klt pair $(X', (1 - \varepsilon)E + B')$ by (8.2).

Picking any $m_0 \geq 1$ such that $m_0 D$ is a Cartier divisor, [BCHM, Theorem 1.2] shows that the $\mathcal{O}_X$-algebra of relative sections

$$R(X'/X, m_0 D) := \bigoplus_{m \in \mathbb{N}} \mu_* \mathcal{O}_{X'}(m m_0 D)$$

is finitely generated. Its relative Proj over $X$ yields a projective birational morphism $\mu : Y \to X$ with $Y$ normal, such that the induced birational map $\phi : X' \dashrightarrow Y$ is surjective in codimension one (i.e. $\phi^{-1}$ does not contract any divisor) and $\phi_* D = \sum_i (A_i - \varepsilon)\phi_* E_i$ is $\mu$-ample.

Since $D$ is $\pi$-exceptional and $\phi$ is surjective in codimension 1, $\phi_* D$ is also $\mu$-exceptional. By Lemma 1.12, $-\phi_* D$ is effective, its support coincides the exceptional locus of $\mu$. If follows that the $\mu$-exceptional prime divisors are exactly the strict transforms of those $E_i$'s with $A_i - \varepsilon < 0$, i.e. $A_i \leq 0$ by definition of $\varepsilon$. As before, we conclude by Proposition 1.11. □

### 8.3. **Estimates near the trivial valuation and proof of Theorem 8.1.**

**Definition 8.10.** *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration representing a metric $\phi \in \mathcal{H}^{\mathrm{NA}}$. For each irreducible component $E$ of $\mathcal{X}_0$, let $Z_E \subset X$ be the closure of the center of $v_E$ on $X$, and set $r_E := \operatorname{codim}_X Z_E$. Then the canonical birational map $\mathcal{X} \dashrightarrow X_{\mathbb{A}^1}$ maps $E$ onto $Z_E \times \{0\}$. Let $F_E$ be the generic fiber and define the* local degree $\deg_E(\phi)$ *of $\phi$ at $E$ as*

$$\deg_E(\phi) := (F_E \cdot \mathcal{L}^{r_E}).$$

Since $\mathcal{L}$ is semiample on $E \subset \mathcal{X}_0$, we have $\deg_E(\phi) \geq 0$, and $\deg_E(\phi) > 0$ iff $E$ is not contracted on the ample model of $(\mathcal{X}, \mathcal{L})$. The significance of these invariants is illustrated by the following estimate, whose proof is straightforward.

**Lemma 8.11.** *With the above notation, assume that $\mathcal{X}$ dominates $X_{\mathbb{A}^1}$ via $\rho$. Given $0 \leq j \leq n$ and line bundles $M_1, \dots, M_{n-j}$ on $X$, we have, for $0 < \varepsilon \ll 1$ rational.*

$$\begin{aligned} &\left(E \cdot (\rho^* L_{\mathbb{A}^1} + \varepsilon D)^j \cdot \rho^*\left(M_{1,\mathbb{A}^1} \cdot \ldots \cdot M_{n-j,\mathbb{A}^1}\right)\right) \\ &\qquad = \begin{cases} \varepsilon^{r_E}\left[\deg_E(\phi)\binom{j}{r_E}\left(Z_E \cdot L^{j-r_E} \cdot M_1 \cdot \ldots \cdot M_{n-j}\right)\right] + O\left(\varepsilon^{r_E+1}\right) & \text{for } j \geq r_E \\ 0 & \text{for } j < r_E. \end{cases} \end{aligned}$$

**Proposition 8.12.** *Pick $\phi \in \mathcal{H}^{\mathrm{NA}}$ that is not a translate of $\phi_{\mathrm{triv}}$, and let $(\mathcal{X}, \mathcal{L})$ be its unique normal ample representative. Set $r := \min_E r_E$, with $r_E = \operatorname{codim}_X Z_E$ and $E$ running over all non-trivial components of the ample model $(\mathcal{X}, \mathcal{L})$ of $\phi$ (and hence $r \geq 1$).*

*Let further $B$ be a boundary on $X$. Then $\phi_\varepsilon := \varepsilon\phi + (1-\varepsilon)\phi_{\mathrm{triv}}$ satisfies*

$$J^{\mathrm{NA}}(\phi_\varepsilon) = O\left(\varepsilon^{r+1}\right), \quad R_B^{\mathrm{NA}}(\phi_\varepsilon) = O\left(\varepsilon^{r+1}\right),$$

*and*

$$\begin{aligned} M_B^{\mathrm{NA}}(\phi_\varepsilon) &= H_B^{\mathrm{NA}}(\phi_\varepsilon) + O\left(\varepsilon^{r+1}\right) \\ &= \varepsilon^r\left[V^{-1}\sum_{r_E = r}\deg_E(\phi) b_E\left(Z_E \cdot L^{n-r}\right) A_{(X,B)}(v_E)\right] + O\left(\varepsilon^{r+1}\right). \end{aligned}$$

*Proof.* Let $(\mathcal{X}', \mathcal{L}')$ be a determination of $(\mathcal{X}, \mathcal{L})$, and write $\mathcal{L}' = \rho^* L_{\mathbb{A}^1} + D$. Note that $(\mathcal{X}', \rho^* L_{\mathbb{A}^1} + \varepsilon D)$ is a representative of $\phi_\varepsilon$. By translation invariance of $J^{\mathrm{NA}}$ and $M^{\mathrm{NA}}$, we may assume $\sup(\phi - \phi_{\mathrm{triv}}) = 0$. Then $\sup(\phi_\varepsilon - \phi_{\mathrm{triv}}) = 0$, and hence $J^{\mathrm{NA}}(\phi_\varepsilon) = -E^{\mathrm{NA}}(\phi_\varepsilon)$. Now, by Lemma 7.4 we have

$$(n+1)V E^{\mathrm{NA}}(\phi_\varepsilon) = \sum_{j=0}^{n} \left( \varepsilon D \cdot (\rho^* L_{\mathbb{A}^1} + \varepsilon D)^j \cdot \rho^* L_{\mathbb{A}^1}^{n-j} \right).$$

Since we have normalized $D$ by $\sup(\phi - \phi_{\mathrm{triv}}) = \mathrm{ord}_{E_0}(D) = 0$ for the strict transform $E_0$ of $X \times \{0\}$ on $\mathcal{X}'$, Lemma 8.11 implies $E^{\mathrm{NA}}(\phi_\varepsilon) = O(\varepsilon^{r+1})$, and hence $J^{\mathrm{NA}}(\phi_\varepsilon) = O(\varepsilon^{r+1})$.

Similarly,

$$\begin{aligned} V R_B^{\mathrm{NA}}(\phi_\varepsilon) &= \left( \rho^* K_{X_{\mathbb{P}^1}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n \right) = \left( \rho^* K_{X_{\mathbb{P}^1}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n \right) - \left( \rho^* K_{X_{\mathbb{P}^1}/\mathbb{P}^1} \cdot L_{\mathbb{P}^1}^n \right) \\ &= \sum_{j=0}^{n-1} \left( \varepsilon D \cdot (\rho^* L_{\mathbb{A}^1} + \varepsilon D)^j \cdot \rho^* L_{\mathbb{A}^1}^{n-j-1} \cdot \rho^* K_{X_{\mathbb{P}^1}/\mathbb{P}^1} \right) = O(\varepsilon^{r+1}). \end{aligned}$$

The expression for $M_B^{\mathrm{NA}}$ now follows from the Chen-Tian formula and Lemma 8.11 applied to

$$H_B^{\mathrm{NA}}(\phi_\varepsilon) = V^{-1} \sum_E A_{(X,B)}(v_E) b_E \left( E \cdot (\rho^* L_{\mathbb{A}^1} + \varepsilon D)^n \right)$$

where $E$ runs over the non-trivial components of $\mathcal{X}'_0$. □

*Proof of Theorem 8.1.* If $(X, B)$ is lc, the definition of $H_B^{\mathrm{NA}}$ shows that it is non-negative on $\mathcal{H}^{\mathrm{NA}}$.

Now assume that $(X, B)$ is not lc. By Corollary 8.8, there exists a closed subscheme $Z \subset X$ whose Rees valuations $v$ all satisfy $A_{(X,B)}(v) < 0$.

By Corollary 4.10, we can then find a normal, ample test configuration $(\mathcal{X}, \mathcal{L})$ such that $\{v_E \mid E \text{ a non-trivial component of } \mathcal{X}_0\}$ coincides up to scaling with the Rees valuations of $Z$, and hence $A_X(v_E) < 0$ for all $E$.

If we denote by $\phi \in \mathcal{H}^{\mathrm{NA}}$ the non-Archimedean metric defined by $(\mathcal{X}, \mathcal{L})$, then we directly get $H_B^{\mathrm{NA}}(\phi) < 0$, which proves the last asserion of the theorem.

On the other hand, Proposition 8.12 implies that $\phi_\varepsilon := \varepsilon\phi + (1-\varepsilon)\phi_{\mathrm{triv}}$ satisfies $M_B^{\mathrm{NA}}(\phi_\varepsilon) < 0$ for $0 < \varepsilon \ll 1$, which shows that $((X, B), L)$ cannot be K-semistable. □

8.4. **The global log canonical threshold and proof of Theorem 8.2.** Let $B$ be a (not necessarily effective) boundary on $X$. Recall that the pair $(X, B)$ is *sub-klt* if $A_{(X,B)}(v) > 0$ for all $v$ (while klt also requires $B \geq 0$).

The main interest of this notion is that if $\mu : X' \to X$ is a proper birational morphism (e.g. a log resolution) and $B'$ is the unique $\mathbb{Q}$-Weil divisor such that $\mu^*(K_X + B) = K_{X'} + B'$ and $\mu_* B' = B$, then $A_{(X,B)} = A_{(X',B')}$ shows that $(X', B')$ is also sub-klt, but $B'$ is not effective in general even when $B$ is.

The *log canonical threshold* of an effective $\mathbb{Q}$-Cartier divisor $D$ with respect to $(X, B)$ is defined as

$$\mathrm{lct}_{(X,B)}(D) := \sup \{ t \geq 0 \mid (X, B + tD) \text{ is klt} \},$$

and is positive since being klt is an open condition. Since $(X, B+tD)$ is klt iff $A_{(X,B+tD)}(v) = A_{(X,B)}(v) - tv(D) > 0$ for all divisorial valuations $v$ on $X$, we have

$$\mathrm{lct}_{(X,B)}(D) = \inf_v \frac{A_{(X,B)}(v)}{v(D)}.$$

Similarly, given an ideal $\mathfrak{a}$ and $c \in \mathbb{Q}_+$, we set

$$\mathrm{lct}_{(X,B)}(\mathfrak{a}^c) := \inf_v \frac{A_{(X,B)}(v)}{v(\mathfrak{a}^c)},$$

with $v(\mathfrak{a}^c) := cv(\mathfrak{a})$.

The main ingredient in the proof of (i)$\Longrightarrow$(ii) of Theorem 8.2 is the following.

**Theorem 8.13.** *If $((X, B), L)$ is a polarized sub-klt pair, then*

$$\inf_D \mathrm{lct}_{(X,B)}(D) = \inf_{\mathfrak{a},c} \mathrm{lct}_{(X,B)}(\mathfrak{a}^c), \tag{8.3}$$

*where the left-hand infimum is taken over all effective $\mathbb{Q}$-Cartier divisors $D$ $\mathbb{Q}$-linearly equivalent to $L$, and the right-hand one is over all non-zero ideals $\mathfrak{a} \subset \mathcal{O}_X$ and all $c \in \mathbb{Q}_+$ such that $L \otimes \mathfrak{a}^c$ is nef. Further, these two infima are strictly positive.*

Here we say that $L \otimes \mathfrak{a}^c$ is nef if $\mu^*L - cE$ is nef on the normalized blow-up $\mu : X' \to X$ of $\mathfrak{a}$, with $E$ the effective Cartier divisor such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$.

**Definition 8.14.** *The* global log canonical threshold $\mathrm{lct}((X, B), L)$ *of a polarized sub-klt pair $((X, B), L)$ is the common value of the two infima in Theorem 8.13.*

*Proof of Theorem 8.13.* Let us first prove that the two infima coincide. Let $D$ be an effective $\mathbb{Q}$-Cartier divisor $\mathbb{Q}$-linearly equivalent to $L$. Pick $m \geq 1$ such that $mD$ is Cartier, and set $\mathfrak{a} := \mathcal{O}_X(-mD)$ and $c := 1/m$. Then $v(\mathfrak{a}^c) = v(D)$ for all $v$, and $L \otimes \mathfrak{a}^c$ is nef since $L - cmD$ is even numerically trivial. Hence $\inf \mathrm{lct}_{(X,B)}(D) \leq \inf \mathrm{lct}_{(X,B)}(\mathfrak{a}^c)$.

Conversely, assume that $L \otimes \mathfrak{a}^c$ is nef, and let $\mu : X' \to X$ be the normalized blow-up of $X$ along $\mathfrak{a}$ and $E$ the effective Cartier divisor on $X'$ such that $\mathcal{O}_{X'}(-E) = \mathfrak{a} \cdot \mathcal{O}_{X'}$, so that $\mu^*L - cE$ is nef. Since $-E$ is $\mu$-ample, we can find $0 < c' \ll 1$ such that $\mu^*L - c'E$ is ample. Setting $c_\varepsilon := (1-\varepsilon)c + \varepsilon c'$, we then have $\mu^*L - c_\varepsilon E$ is ample for all $0 < \varepsilon < 1$.

Let also $B'$ the unique $\mathbb{Q}$-Weil divisor on $X'$ such that $\mu^*(K_X + B) = K_{X'} + B'$ and $\mu_*B' = B$, so that $(X', B')$ is a pair with $A_{(X,B)} = A_{(X',B')}$.

If we choose a log resolution $\pi : X'' \to X'$ of $(X', B' + E)$ and let $F = \sum_i F_i$ be the sum of all $\pi$-exceptional primes and of the strict transform of $B'_{\mathrm{red}} + E_{\mathrm{red}}$, then

$$\mathrm{lct}_{(X,B)}(\mathfrak{a}^{c_\varepsilon}) = \mathrm{lct}_{(X',B')}(c_\varepsilon E) = \min_i \frac{A_{(X',B')}(\mathrm{ord}_{F_i})}{\mathrm{ord}_{F_i}(c_\varepsilon D)}$$

Given $0 < \varepsilon < 1$, pick $m \gg 1$ such that

(i) $mc_\varepsilon \in \mathbb{N}$;
(ii) $m(\mu^*L - c_\varepsilon E)$ is very ample;
(iii) $m \geq \mathrm{lct}_{(X,B)}(\mathfrak{a}^{c_\varepsilon})$.

Let $H \in |m(\mu^*L - c_\varepsilon E)|$ be a general element, and set $D := \mu_*(c_\varepsilon E + m^{-1}H)$, so that $D$ is $\mathbb{Q}$-linearly equivalent to $L$ and $\mu^*D = c_\varepsilon E + m^{-1}H$.

By Bertini's theorem, $\pi$ is also a log resolution of $(X', B' + E + H)$, and hence

$$\operatorname{lct}_{(X,B)}(D) = \operatorname{lct}_{(X',B')}(c_\varepsilon E + m^{-1}H) = \min\left\{ \frac{A_{(X',B')}(v)}{v(c_\varepsilon E + m^{-1}H)} \;\middle|\; v = v_i \text{ or } v = \operatorname{ord}_H \right\}.$$

But $H$, being general, doesn't contain the center of $\operatorname{ord}_{D_i}$ on $X'$ and is not contained in $\operatorname{supp} E$, i.e. $\operatorname{ord}_{D_i}(H) = 0$ and $\operatorname{ord}_H(E) = 0$, and (iii) above shows that

$$\operatorname{lct}_{(X,B)}(D) = \min\left\{\operatorname{lct}_{(X,B)}(\mathfrak{a}^{c_\varepsilon}), m\right\} = \min_{(X,B)}(\mathfrak{a}^{c_\varepsilon}).$$

Since we have $\operatorname{lct}_{(X,B)}(\mathfrak{a}^{c_\varepsilon}) = \frac{c}{c_\varepsilon}\operatorname{lct}_{(X,B)}(\mathfrak{a}^c)$ with $c_\varepsilon/c$ arbitrarily close to 1, we conclude that the two infima in (8.3) are indeed equal.

We next show that the left-hand infimum in (8.3) is strictly positive, in two steps.

**Step 1**. We first treat the case where $X$ is smooth and $B = 0$. By Skoda's theorem (see for instance [JM12, Proposition 5.10]), we then have

$$v(D) \le \operatorname{ord}_p(D) A_X(v)$$

for every effective $\mathbb{Q}$-Cartier divisor $D$ on $X$, every divisorial valuation $v$, and every closed point $p$ in the closure of the center of $v$ on $X$. It is thus enough to show that $\operatorname{ord}_p(D)$ is uniformly bounded when $D \sim_{\mathbb{Q}} L$.

Let $\mu : X' \to X$ be the blow-up at $p$, with exceptional divisor $E$. Since $L$ is ample, there exists $\varepsilon > 0$ independent of $p$ such that $L_\varepsilon := \mu^* L - \varepsilon E$ is ample, by Seshadri's theorem.

Since $D$ is effective, we have $\mu^* D \ge \operatorname{ord}_p(D) E$, and hence

$$(L^n) = \left(\mu^* L \cdot L_\varepsilon^{n-1}\right) \ge \operatorname{ord}_p(D)(E \cdot L_\varepsilon^{n-1}) = \varepsilon^{n-1} \operatorname{ord}_p(D),$$

which yields the desired bound on $\operatorname{ord}_p(D)$.

**Step 2**. Suppose now that $(X, B)$ is a sub-klt pair. Pick a log resolution $\mu : X' \to X$, and let $B'$ be the unique $\mathbb{Q}$-divisor such that $\mu^*(K_X + B) = K_{X'} + B'$ and $\mu_* B' = B$, so that

$$A_{(X,B)}(v) = A_{(X',B')}(v) = A_{X'}(v) - v(B')$$

for all divisorial valuations $v$. Since $(X, B)$ is sub-klt, $B'$ has coefficients less than 1, so that there exists $0 < \varepsilon \ll 1$ such that $B' \le (1 - \varepsilon) B'_{\mathrm{red}}$. Since $B'_{\mathrm{red}}$ is a reduced SNC divisor, the pair $(X', B'_{\mathrm{red}})$ is lc, and hence $v(B') \le A_{X'}(v)$ for all divisorial valuations $v$. It follows that $v(B) \le (1 - \varepsilon) A_{X'}(v)$, i.e.

$$\varepsilon A_{X'}(v) \le A_{(X,B)}(v)$$

for all $v$. Pick any very ample effective divisor $H$ on $X'$ such that $L' := \mu^* L + H$ is ample. For each effective $\mathbb{Q}$-Cartier divisor $D \sim_{\mathbb{Q}} L$, $D' := \mu^* D + H$ is an effective $\mathbb{Q}$-Cartier divisor on $X'$ with $D' \sim_{\mathbb{Q}} L'$. By Step 1, we conclude, as desired, that

$$v(D) \le v(D') \le C A_{X'}(v) \le C \varepsilon^{-1} A_{(X,B)}(v).$$

□

**Proposition 8.15.** *For each polarized sub-klt pair $((X, B), L)$, we have*

$$H^{\mathrm{NA}} \ge \delta I^{\mathrm{NA}} \ge \frac{\delta}{n} J^{\mathrm{NA}}$$

*on $\mathcal{H}^{\mathrm{NA}}$ with $\delta := \operatorname{lct}((X, B), L) > 0$.*

*Proof.* Pick $\phi \in \mathcal{H}^{\mathrm{NA}}$, and let $(\mathcal{X}, \mathcal{L})$ be a representative such that the canonical birational map $\rho : \mathcal{X} \dashrightarrow X_{\mathbb{A}^1}$ is a morphism, so that $\mathcal{L} = \rho^* L_{\mathbb{A}^1} + D$.

Choose $m \geq 1$ such that $m\mathcal{L}$ is a globally generated line bundle, and let

$$\rho_* \mathcal{O}(mD) = \mathfrak{a}^{(m)} = \sum_{\lambda \in \mathbb{Z}} \mathfrak{a}_\lambda^{(m)} t^{-\lambda}$$

be the corresponding flag ideal. By Proposition 2.18, $\mathcal{O}(mL) \otimes \mathfrak{a}_\lambda^{(m)}$ is globally generated on $X$ for all $\lambda \in \mathbb{Z}$. In particular, $L \otimes (\mathfrak{a}_\lambda^{(m)})^{1/m}$ is nef, and hence

$$v(\mathfrak{a}_\lambda^{(m)}) \leq m\delta^{-1} A_{(X,B)}(v)$$

whenever $\mathfrak{a}_\lambda^{(m)}$ is non-zero.

Now let $E$ be a non-trivial component of $\mathcal{X}_0$. By Lemma 4.5, we have

$$\mathrm{ord}_E(\mathfrak{a}^{(m)}) = \min_\lambda \left( v_E(\mathfrak{a}_\lambda^{(m)}) - \lambda b_E \right)$$

with $b_E = \mathrm{ord}_E(\mathcal{X}_0)$, and hence

$$\mathrm{ord}_E(\mathfrak{a}^{(m)}) \leq m\delta^{-1} A_{(X,B)}(v_E) - b_E \max \left\{ \lambda \in \mathbb{Z} \mid \mathfrak{a}_\lambda^{(m)} \neq 0 \right\}$$

By Lemma **??**, we have

$$\max \left\{ \lambda \in \mathbb{Z} \mid \mathfrak{a}_\lambda^{(m)} \neq 0 \right\} = \lambda_{\max}^{(m)},$$

which is bounded above by

$$m\lambda_{\max}(F_\phi^\bullet R) = m(\phi \cdot \phi_{\mathrm{triv}}^n),$$

by Lemma 7.7. We have thus proved that

$$m^{-1} \mathrm{ord}_E(\mathfrak{a}^{(m)}) \leq \delta^{-1} A_{(X,B)}(v_E) - b_E V^{-1}(\phi \cdot \phi_{\mathrm{triv}}^n). \tag{8.4}$$

But since $mD$ is $\rho$-globally generated, we have $\mathcal{O}_{\mathcal{X}}(mD) = \mathcal{O}_{\mathcal{X}} \cdot \mathfrak{a}^{(m)}$, and hence

$$m^{-1} \mathrm{ord}_E(\mathfrak{a}^{(m)}) = -\mathrm{ord}_E(D).$$

Using (8.4) and $\sum_E b_E(E \cdot \mathcal{L}^n) = (\mathcal{X}_0 \cdot \mathcal{L}^n) = V$, we infer

$$-V^{-1}((\phi - \phi_{\mathrm{triv}}) \cdot \phi^n) = -V^{-1} (D \cdot \mathcal{L}^n) \leq \delta^{-1} H^{\mathrm{NA}}(\phi) - V^{-1}(\phi \cdot \phi_{\mathrm{triv}}^n)$$

and the result follows by the definition of $I^{\mathrm{NA}}$ and by Proposition 7.8. □

*Proof of Theorem 8.2.* The implication (i)$\Longrightarrow$(ii) follows from Proposition 8.15, whereas (ii)$\Longrightarrow$(iii) is trivial. Now assume that (iii) holds. If $(X, B)$ is not klt, Proposition 8.9 yields a closed subscheme $Z \subset X$ with $A_{(X,B)}(v) \leq 0$ for all Rees valuations $v$ of $Z$. By Corollary 4.10, we can thus find a normal, ample test configuration $(\mathcal{X}, \mathcal{L})$ such that $A_{(X,B)}(v_E) \leq 0$ for each non-trivial component $E$ of $\mathcal{X}_0$. The corresponding non-Archimedean metric $\phi \in \mathcal{H}^{\mathrm{NA}}$ therefore satisfies $H_B^{\mathrm{NA}}(\phi) \leq 0$, which contradicts (iii). □

8.5. **The Kähler-Einstein case.**

*Proof of Corollary 8.3.* The implication (iii)$\Longrightarrow$(i) follows from Theorem 8.1, and (ii)$\Longrightarrow$(iii) is trivial. Now assume (i), so that $H_B^{\mathrm{NA}} \geq 0$ on $\mathcal{H}^{\mathrm{NA}}$ by Theorem 8.1. By Lemma 7.19, we have $M_B^{\mathrm{NA}} = H_B^{\mathrm{NA}} + \lambda \left( I^{\mathrm{NA}} - J^{\mathrm{NA}} \right)$, while $I^{\mathrm{NA}} - J^{\mathrm{NA}} \geq \frac{1}{n} J^{\mathrm{NA}}$ by Proposition 7.8. We thus get $M_B^{\mathrm{NA}} \geq \frac{\lambda}{n} J^{\mathrm{NA}}$, which proves (iii). $\square$

*Proof of Corollary 8.4.* If $K_X + B$ is numerically trivial, then Lemma 7.19 gives $M_B^{\mathrm{NA}} = H_B^{\mathrm{NA}}$. The result is thus a direct consequence of Theorem 8.1 and Theorem 8.2. $\square$

*Proof of Corollary 8.6.* Since $K_X + B \equiv -\lambda L$, Lemma 7.19 becomes

$$M_B^{\mathrm{NA}} = H_B^{\mathrm{NA}} - \lambda \left( I^{\mathrm{NA}} - J^{\mathrm{NA}} \right) \tag{8.5}$$

We thus get $H_B^{\mathrm{NA}} \geq \lambda \left( I^{\mathrm{NA}} - J^{\mathrm{NA}} \right)$, and hence $H_B^{\mathrm{NA}} \geq \frac{\lambda}{n} J^{\mathrm{NA}}$ by Proposition 7.8. By Theorem 8.2, this implies that $(X, B)$ is klt. $\square$

The following result gives a slightly more precise version of the computations of [OSa12, Theorem 1.4] and [Der14a, Theorem 3.24].

**Proposition 8.16.** *Let $B$ be an effective boundary on $X$ such that $(X, B)$ is klt and $(K_X + B) \equiv -\lambda L$ with $\lambda > 0$. Assume also that $\varepsilon := \mathrm{lct}((X, B), L) - \frac{n}{n+1}\lambda > 0$. Then we have*

$$M_B^{\mathrm{NA}} \geq \varepsilon I^{\mathrm{NA}} \geq \frac{(n+1)}{n} \varepsilon J^{\mathrm{NA}}.$$

*In particular, the polarized pair $((X, B), L)$ is J-uniformly K-stable.*

*Proof.* By Proposition 8.15 we have $H^{\mathrm{NA}} \geq \left( \frac{n}{n+1}\lambda + \varepsilon \right) I^{\mathrm{NA}}$, and hence

$$M_B^{\mathrm{NA}} \geq \varepsilon I^{\mathrm{NA}} + \lambda \left( J^{\mathrm{NA}} - \frac{1}{n+1} I^{\mathrm{NA}} \right).$$

The result follows since we have

$$\frac{1}{n+1} I^{\mathrm{NA}} \leq J^{\mathrm{NA}} \leq \frac{n}{n+1} I^{\mathrm{NA}}$$

by Proposition 7.8. $\square$

## Appendix A. Asymptotic Riemann-Roch on a normal variety

The following result is of course well-known, but we provide a proof for lack of suitable reference. In particular, the sketch provided in [Oda13a, Lemma 3.5] assumes that the line bundle in question is ample, which is not enough for the application to the intersection theoretic formula for the Donaldson-Futaki invariant.

**Theorem A.1.** *If $Z$ is a proper normal variety over an algebraically closed field $k$ of dimension $d$, and $L$ is a line bundle on $Z$, then*

$$\chi(Z, mL) = (L^d) \frac{m^d}{d!} - (K_Z \cdot L^{d-1}) \frac{m^{d-1}}{2(d-1)!} + O\left( m^{d-2} \right).$$

*A proof in characteristic* 0. When $Z$ is smooth, the result follows from the Riemann-Roch formula, which reads

$$\chi(Z, mL) = \int \left(1 + c_1(mL) + \cdots + \frac{c_1(mL)^d}{d!}\right)\left(1 + \frac{c_1(Z)}{2} + \dots\right).$$

Assume now that $Z$ is normal, pick a resolution of singularities $\mu : Z' \to Z$ and set $L' := \mu^* L$. The Leray spectral sequence and the projection formula imply that

$$\chi(Z', mL') = \sum_j (-1)^j \chi\left(Z, \mathcal{O}(mL) \otimes R^j \mu_* \mathcal{O}\right).$$

Since $Z$ is normal, $\mu$ is an isomorphism over an open subset with complement of codimension at least 2. As a result, for each $j \geq 1$ the support of the coherent sheaf $R^j \mu_* \mathcal{O}$ has codimension at least 2, and hence $\chi\left(Z, \mathcal{O}(mL) \otimes R^j \mu_* \mathcal{O}\right) = O\left(m^{d-2}\right)$ (cf. [Kle66, §1]). We thus get

$$\begin{aligned}\chi(Z, mL) &= \chi(Z', mL') + O(m^{d-2}) \\ &= (L'^d)\frac{m^d}{d!} - (K_{Z'} \cdot L'^{d-1})\frac{m^{d-1}}{2(d-1)!} + O\left(m^{d-2}\right)\end{aligned}$$

and the projection formula yields the desired result, since $\mu_* K_{Z'} = K_Z$ as cycle classes. □

*The general case.* By Chow's lemma, there exists a birational morphism $Z' \to Z$ with $Z'$ projective and normal. By the same argument as above, it is enough to prove the result for $Z'$, and we may thus assume that $Z$ is projective to begin with.

We argue by induction on $d$. The case $d = 0$ being clear. Now assume $d \geq 1$ and let $H$ be a very ample line bundle on $Z$ such that $L + H$ is also very ample. By the Bertini type theorem for normality of [Fle77, Satz 5.2], general elements $B \in |H|$ and $A \in |L + H|$ are also normal, with $L = A - B$. The short exact sequence

$$0 \to \mathcal{O}_Z((m+1)L - A) \to \mathcal{O}_Z(mL) \to \mathcal{O}_B(mL) \to 0$$

shows that

$$\chi(Z, (m+1)L - A) = \chi(Z, mL) - \chi(B, mL).$$

We similarly find

$$\chi(Z, (m+1)L) = \chi(Z, (m+1)L - A) + \chi(A, (m+1)L),$$

and hence

$$\chi(Z, (m+1)L) - \chi(Z, mL) = \chi(A, (m+1)L) - \chi(B, mL).$$

Since $A$ and $B$ are normal Cartier divisors on $X$, the adjunction formulae $K_A = (K_Z + A)|_A$ and $K_B = (K_Z + B)|_B$ hold, as they are equalities between Weil divisor classes on a normal variety that hold outside a closed subset of codimension at least 2. By the induction hypothesis, we thus get

$$\begin{aligned}&\chi(Z, (m+1)L) - \chi(Z, mL) \\ &= (L^{d-1} \cdot A)\left(\frac{m^{d-1}}{(d-1)!} + \frac{m^{d-2}}{(d-2)!}\right) - \left((K_Z + A) \cdot A \cdot L^{d-2}\right)\frac{m^{d-2}}{2(d-2)!} \\ &-(L^{d-1} \cdot B)\frac{m^{d-1}}{(d-1)!} + \left((K_Z + B) \cdot B \cdot L^{d-2}\right)\frac{m^{d-2}}{2(d-2)!} + O(m^{d-3}). \\ &= (L^d)\frac{m^{d-1}}{(d-1)!} + \left[(L^d) - \tfrac{1}{2}(K_Z \cdot L^{d-1})\right]\frac{m^{d-2}}{(d-2)!} + O(m^{d-3}),\end{aligned}$$

$$= P(m+1) - P(m) + O(m^{d-3})$$

with

$$P(m) := (L^d)\frac{m^d}{d!} - (K_Z \cdot L^{d-1})\frac{m^{d-1}}{2(d-1)!}.$$

The result follows. □

## Appendix B. The equivariant Riemann-Roch theorem for schemes

We summarize the general equivariant Riemann-Roch theorem for schemes, which extends to the equivariant setting the results of [Ful, Chap. 18], and is due to Edidin-Graham [EG98, EG00].

Let $G$ be a linear algebraic group, and $X$ be a scheme with a $G$-action. The Grothendieck group $K^0_G(X)$ of virtual $G$-linearized vector bundles forms a commutative ring with respect to tensor products, and is functorial under pull-back. On the other hand, the Grothendieck group $K^G_0(X)$ of virtual $G$-linearized coherent sheaves on $X$ is $K^0_G(X)$-module with respect to tensor products, and every proper $G$-equivariant morphism $f : X \to Y$ induces a push-forward homomorphism $f_! : K^G_0(X) \to K^G_0(Y)$ defined by

$$f_![\mathcal{F}] := \sum_{q\in\mathbb{N}} (-1)^q [R^q f_* \mathcal{F}].$$

Note that $K^0_G(\operatorname{Spec} k) = K^G_0(\operatorname{Spec} k)$ identifies with the representation ring $R(G)$, so that all the above groups are in particular $R(G)$-modules.

Equivariant Chow homology and cohomology groups are constructed in [EG98], building on an idea of Totaro. The $G$-equivariant Chow cohomology ring

$$\mathrm{CH}^\bullet_G(X) = \bigoplus_{d\in\mathbb{N}} \mathrm{CH}^d_G(X)$$

can have $\mathrm{CH}^d_G(X) \neq 0$ for infinitely many $d \in \mathbb{N}$, and we set

$$\widehat{\mathrm{CH}}^\bullet_G(X) = \prod_{d\in\mathbb{N}} \mathrm{CH}^d_G(X).$$

The $G$-equivariant first Chern class defines a morphism $c^G_1 : \operatorname{Pic}^G(X) \to \mathrm{CH}^1_G(X)$, which is an isomorphism when $X$ is smooth [EG98, Corollary 1]. In particular, we have natural isomorphisms

$$\operatorname{Hom}(G, \mathbb{G}_m) \simeq \operatorname{Pic}^G(\operatorname{Spec} k) \simeq \mathrm{CH}^1_G(\operatorname{Spec} k).$$

The $G$-equivariant Chern character is a ring homomorphism

$$\mathrm{ch}^G : K^0_G(X) \to \widehat{\mathrm{CH}}^\bullet_G(X)_\mathbb{Q},$$

functorial with respect to pull-back and such that

$$\mathrm{ch}^G(L) = e^{c^G_1(L)} = \left(\frac{c^G_1(L)^d}{d!}\right)_{d\in\mathbb{N}}$$

for a $G$-linearized line bundle $L$.

On the other hand, the $G$-equivariant Chow homology group

$$\mathrm{CH}^G_\bullet(X) = \bigoplus_{p\in\mathbb{Z}} \mathrm{CH}^G_p(X)$$

is a $\mathrm{CH}^\bullet_G(X)$-module, with $\mathrm{CH}^d_G(X)\cdot\mathrm{CH}^G_p(X)\subset\mathrm{CH}_{p-d}(X)$. While $\mathrm{CH}^G_p(X)=0$ for $p>\dim X$, it is in general non-zero for infinitely many (negative) $p$ in general, and we set again

$$\widehat{\mathrm{CH}}{}^G_\bullet(X)=\prod_{p\in\mathbb{Z}}\mathrm{CH}^G_p(X),$$

a $\widehat{\mathrm{CH}}{}^\bullet_G(X)$-module.

When $X$ is smooth and pure dimensional, the action of $\mathrm{CH}^d_G(X)$ on $[X]\in\mathrm{CH}^G_{\dim X}(X)$ defines a 'Poincaré duality' isomorphism

$$\mathrm{CH}^d_G(X)\simeq\mathrm{CH}^G_{\dim X-d}(X).$$

Via the Chern character, both $K^G_0(X)$ and $\mathrm{CH}^G_\bullet(X)_\mathbb{Q}$ become $K^0_G(X)$-modules, and the general Riemann-Roch theorem of [EG00, Theorem 3.1] constructs a $K^0_G(X)$-module homomorphism

$$\tau^G:K^G_0(X)\to\widehat{\mathrm{CH}}{}^G_\bullet(X)_\mathbb{Q}:=\prod_{p\in\mathbb{N}}\mathrm{CH}^G_p(X)_\mathbb{Q},$$

functorial with respect to push-forward under proper morphisms, and normalized by $\tau(1)=1$ on $K^0_G(\operatorname{Spec}k)$, so that $\tau^G=\mathrm{ch}^G$ on $R(G)$.

When $X$ is proper, the *equivariant Euler characteristic* of a $G$-linearized coherent sheaf $\mathcal{F}$ on $X$ is defined as

$$\chi^G(X,\mathcal{F}):=\mathrm{ch}^G(\pi_![\mathcal{F}])\in\widehat{\mathrm{CH}}{}^\bullet_G(\operatorname{Spec}k).$$

with $\pi:X\to\operatorname{Spec}k$ the structure morphism. The equivariant Riemann-Roch formula then reads

$$\chi^G(X,E\otimes\mathcal{F})=\pi_*\left(\mathrm{ch}^G(E)\cdot\tau^G(\mathcal{F})\right)$$

for every $G$-linearized coherent sheaf $\mathcal{F}$ and vector bundle $E$ on $X$.

When $G=T$ is an algebraic torus, the equivariant Euler characteristic admits a more explicit description. Let $M:=\mathrm{Hom}(T,\mathbb{G}_m)$ be character lattice, with the first Chern class isomorphism $M\simeq\mathrm{Pic}^T(\operatorname{Spec}k)\simeq\mathrm{CH}^1_T(\operatorname{Spec}k)$. By [EG97, Lemma 2], it induces a graded ring isomorphism

$$\widehat{\mathrm{S}}^\bullet M:=\prod_{d\in\mathbb{N}}\mathrm{S}^d M\simeq\widehat{\mathrm{CH}}{}^\bullet_T(\operatorname{Spec}k)$$

On the other hand, the representation ring $R(T)$ identifies with the ring of Laurent polynomials $\mathbb{Z}[M]$, a $T$-module $V$ being sent to $\sum_{\lambda\in M}(\dim V_\lambda)\lambda$. Under these identifications, the Chern character

$$\mathrm{ch}^T\colon R(T)\to\widehat{\mathrm{CH}}{}^\bullet_T(\operatorname{Spec}k)_\mathbb{Q}$$

corresponds to the ring homomorphism $\mathbb{Z}[M]\to\widehat{\mathrm{S}}^\bullet M_\mathbb{Q}$ mapping $\lambda\in M$ to $e^\lambda=\left(\frac{\lambda^d}{d!}\right)_{d\in\mathbb{N}}$. Viewed as an element of $\widehat{\mathrm{S}}^\bullet M_\mathbb{Q}$, the equivariant Euler characteristic of a $T$-linearized coherent sheaf $\mathcal{F}$ on a proper scheme $X$ can thus be described as

$$\chi^T(X,\mathcal{F})=\sum_{\lambda\in M}\left(\sum_{q=0}^{\dim X}(-1)^q\dim H^q(X,\mathcal{F})_\lambda\right)e^\lambda. \tag{B.1}$$

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