Representations of the solvable group acting on a homogeneous Siegel domain

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1. Introduction. Let $(\mathfrak{g}, j, \omega)$ be a normal j -algebra and G the solvable Lie group corresponding to $\mathfrak{g}$. As is shown by Piatetskii-Shapiro [12], G is realized as an affine transformation group of a homogeneous Siegel domain D , so that one can define representations of G on the space of holomorphic functions on D in a natural way (see (4.1)). Our concern is to get unitary representations of G embedded in these representations. A similar situation for the case of symmetric Siegel domains has been investigated by Vergne and Rossi [15] (see also Wallach [17] and Clerc [3]) about the analytic continuation of the holomorphic discrete series representation of a semisimple Lie group. In the present paper, we determine all of such unitary representations of G by relating them to the positivity of certain invariant distributions (Riesz distributions) on the dual cone and classify them by using the theory of orbit method.

2. Preliminaries. We first recall the definition of normal j -algebras. Let $\mathfrak{g}$ be a real split solvable Lie algebra, j a linear operator on $\mathfrak{g}$ such that $j^2 = -\text{id}_{\mathfrak{g}}$, and ω a linear form on $\mathfrak{g}$. The triple $(\mathfrak{g}, j, \omega)$ is called a *normal j -algebra* if the following two conditions are satisfied:

- (i) $[Y_1, Y_2] + j[jY_1, Y_2] + j[Y_1, jY_2] - [jY_1, jY_2] = 0$ for all $Y_1, Y_2 \in \mathfrak{g}$,
 - (ii) a bilinear form $(Y_1|Y_2)_{\omega} := \omega([Y_1, jY_2])$ defines a j -invariant inner product on $\mathfrak{g}$.
- Let $\mathfrak{a}$ be the orthogonal complement of the subspace $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ with respect to the inner product $(\cdot|\cdot)_{\omega}$. Then $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$. Put $r := \dim \mathfrak{a}$, and for a linear form $\alpha \in \mathfrak{a}^*$, set $\mathfrak{g}_{\alpha} := \{Y \in \mathfrak{g}; [C, Y] = \alpha(C)Y \text{ for all } C \in \mathfrak{a}\}$. The following proposition is fundamental for our study.

Proposition 1 ([12, Chapter 2, Sections 3 and 5]). (i) *There is a linear basis $\{A_1, \dots, A_r\}$ of $\mathfrak{a}$ such that if one puts $E_l := -jA_l$, then $[A_k, E_l] = \delta_{kl}E_l$ ($1 \leq k, l \leq r$).*

(ii) Let $\alpha_1, \dots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $A_1, \dots, A_r$. Then one has a decomposition $\mathfrak{g} = \mathfrak{g}(1) \oplus \mathfrak{g}(1/2) \oplus \mathfrak{g}(0)$ with

$$(2.1) \quad \begin{aligned} \mathfrak{g}(1) &:= \sum_{k=1}^r \oplus \mathbb{R}E_k \oplus \sum_{1 \leq k < m \leq r} \oplus \mathfrak{g}_{(\alpha_m + \alpha_k)/2}, & \mathfrak{g}(1/2) &:= \sum_{k=1}^r \oplus \mathfrak{g}_{\alpha_k/2}, \\ \mathfrak{g}(0) &:= \mathfrak{a} \oplus \sum_{1 \leq k < m \leq r} \oplus \mathfrak{g}_{(\alpha_m - \alpha_k)/2}. \end{aligned}$$

(iii) One has $[\mathfrak{g}(\mu), \mathfrak{g}(\nu)] \subset \mathfrak{g}(\mu + \nu)$ ($\mu, \nu = 0, 1/2, 1$), where $\mathfrak{g}(\mu) := \{0\}$ if $\mu > 1$.

The assertion (iii) tells us that the subspace $\mathfrak{g}(0)$ is a subalgebra of $\mathfrak{g}$ and that the corresponding Lie group $H := \exp \mathfrak{g}(0)$ acts on $\mathfrak{g}(1/2)$ and $\mathfrak{g}(1)$ respectively by the adjoint actions. Put $E := E_1 + \dots + E_r \in \mathfrak{g}(1)$ and let Ω be the H -orbit through E . Then Ω is a regular open convex cone in $\mathfrak{g}(1)$ and H acts on Ω simply transitively. The linear operator j preserves the subspace $\mathfrak{g}(1/2)$, so that j defines a complex structure on $\mathfrak{g}(1/2)$. We define a $\mathfrak{g}(1)_{\mathbb{C}}$ -valued Hermitian map Q on $\mathfrak{g}(1/2)$ by $Q(u, u') := ([ju, u'] + i[u, u'])/4$ ($u, u' \in \mathfrak{g}(1/2)$). Then Q is Ω -positive, that is, $Q(u, u) \in \overline{\Omega} \setminus \{0\}$ for all $u \in \mathfrak{g}(1/2) \setminus \{0\}$. Now we define the Siegel domain D corresponding to $(\mathfrak{g}, j, \omega)$ by $D := \{(z, u) \in \mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1/2); \operatorname{Im} z - Q(u, u) \in \Omega\}$.

By Proposition 1 (iii), the space $\mathfrak{n}(Q) := \mathfrak{g}(1) \oplus \mathfrak{g}(1/2)$ is a nilpotent subalgebra of $\mathfrak{g}$. Put $N(Q) := \exp \mathfrak{n}(Q)$. Then the solvable Lie group G is a semidirect product $N(Q) \rtimes H$. We realize G as an affine transformation group on $\mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1/2)$ by setting $g_0 \cdot (z, u) := (t_0 \cdot z + x_0 + 2iQ(t_0 \cdot u, u_0) + iQ(u_0, u_0), t_0 \cdot u + u_0)$ for $g_0 = \exp(x_0 + u_0)t_0 \in G$ ($x_0 \in \mathfrak{g}(1)$, $u_0 \in \mathfrak{g}(1/2)$, $t_0 \in H$) and $(z, u) \in \mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1/2)$. Then G acts on D simply transitively.

For $s = (s_1, \dots, s_r) \in \mathbb{C}^r$, let χ_s be the one-dimensional representation of G such that $\chi_s(\exp(\sum_{k=1}^r c_k A_k)) = e^{s_1 c_1 + \dots + s_r c_r}$ ($c_1, \dots, c_r \in \mathbb{R}$). We define a function Δ_s on Ω by $\Delta_s(t \cdot E) := \chi_s(t)$ ($t \in H$) and extend it to the domain $\Omega + i\mathfrak{g}(1) \subset \mathfrak{g}(1)_{\mathbb{C}}$ by analytic continuation.

3. Riesz distributions on the dual cone. Let Ω^* be the dual cone of Ω , that is, $\Omega^* := \{\xi \in \mathfrak{g}(1)^*; \langle x, \xi \rangle > 0 \text{ for all } x \in \overline{\Omega} \setminus \{0\}\}$. The group H acts on Ω^* simply transitively by the contragredient action. For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \{0, 1\}^r$, let E_{ε}^* be a linear form on $\mathfrak{g}(1)$ given by $\langle \sum_{k=1}^r x_{kk} E_k + \sum_{m > k} X_{mk}, E_{\varepsilon}^* \rangle := \sum_{k=1}^r \varepsilon_k x_{kk}$ for $x_{kk} \in \mathbb{R}$ and $X_{mk} \in \mathfrak{g}_{(\alpha_m + \alpha_k)/2}$. Let $\mathcal{O}_{\varepsilon}^*$ be the H -orbit in $\mathfrak{g}(1)^*$ through E_{ε}^* . Then the H -orbit decomposition of the closure of the cone Ω^* is given as

$$(3.1) \quad \overline{\Omega^*} = \bigsqcup_{\varepsilon \in \{0, 1\}^r} \mathcal{O}_{\varepsilon}^*.$$

Now we consider tempered distributions $\mathcal{R}_s^*$ ($s \in \mathbb{C}^r$) on $\mathfrak{g}(1)^*$ characterized by the Laplace transform:

$$(3.2) \quad \langle \mathcal{R}_s^*, e^{-\langle y, \cdot \rangle} \rangle = \Delta_{-s^*}(y) \quad (y \in \Omega),$$

where $s^* := (s_r, \dots, s_1) \in \mathbb{C}^r$. These $\mathcal{R}_s^*$ are relatively invariant under H and supported by $\overline{\Omega^*}$. We call $\mathcal{R}_s^*$ the *Riesz distributions* on Ω^* (see [7] for the precise definition of the Riesz distributions).

The results of [8] tell us that the structure of the positive Riesz distributions is closely related to the orbit structure (3.1) of $\overline{\Omega^*}$. Put $q_k(\varepsilon) := \sum_{m>k} \varepsilon_m \dim \mathfrak{g}_{(\alpha_m - \alpha_k)/2}$ ($k = 1, \dots, r$),

$$\mathfrak{X}(\varepsilon) := \{s \in \mathbb{C}^r; \operatorname{Re} s_k > q_k(\varepsilon)/2 \text{ (if } \varepsilon_k = 1), \operatorname{Re} s_k = q_k(\varepsilon)/2 \text{ (if } \varepsilon_k = 0)\},$$

and $\mathfrak{X} := \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathfrak{X}(\varepsilon)$. For $s \in \mathbb{C}^r$, we set $\operatorname{Re} s^* := (\operatorname{Re} s_r, \dots, \operatorname{Re} s_1)$.

Proposition 2. (i) *The Riesz distribution $\mathcal{R}_{\operatorname{Re} s^*}^*$ is positive if and only if $s \in \mathfrak{X}$.*
(ii) *When $s \in \mathfrak{X}(\varepsilon)$, the distribution $\mathcal{R}_{\operatorname{Re} s^*}^*$ is an H -relatively invariant positive measure on $\mathcal{O}_\varepsilon^*$.*

4. Representations of G on spaces of holomorphic functions on D .

Let $\mathcal{H}(D)$ be the space of holomorphic functions on D . For $s \in \mathbb{C}^r$, we define a representation π_s on $\mathcal{H}(D)$ by

$$(4.1) \quad \pi_s(g)F(p) := \chi_{-s/2}(g)F(g^{-1} \cdot p) \quad (g \in G, p \in D, F \in \mathcal{H}(D)).$$

We consider the subspace $\mathcal{H}_s(D)$ of $\mathcal{H}(D)$ satisfying the following two conditions:

- (i) $\mathcal{H}_s(D)$ has a Hilbert space structure with reproducing kernel,
- (ii) $(\pi_s, \mathcal{H}_s(D))$ is a unitary representation of G .

The condition (i) is equivalent to the statement that the Hilbert space $\mathcal{H}_s(D)$ is continuously embedded into a topological vector space $\mathcal{H}(D)$ with the compact open topology (see [3]). We note that non-zero $\mathcal{H}_s(D)$ is unique, though it might not exist for some $s \in \mathbb{C}^r$.

Proposition 3. *If non-zero $\mathcal{H}_s(D)$ exists, the reproducing kernel K^s of $\mathcal{H}_s(D)$ is given (up to a positive constant) by*

$$K^s((z, u), (z', u')) = \Delta_{-\operatorname{Re} s}((z - \bar{z}')/i - 2Q(u, u')) \quad ((z, u), (z', u') \in D).$$

By Proposition 3 and the argument given in [15, Chapter 3], we see that the non-vanishing of the Hilbert space $\mathcal{H}_s(D)$ is equivalent to the fact that the function $\Delta_{-\operatorname{Re} s}$ on Ω is of positive type. On the other hand, a Bochner type theorem [4, Satz 5.1] states that a homogeneous function f on Ω is of positive type if and only

if f is the Laplace transform of a positive measure on $\overline{\Omega^*}$. Therefore (3.2) and Proposition 2 (i) lead us to the following theorem.

Theorem 4. *Non-zero $\mathcal{H}_s(D)$ exists if and only if $s \in \mathfrak{X}$.*

In view of Theorem 4 and the result of [15], the set $\mathfrak{X}$ can be regarded as a non-symmetric analogue of the Wallach sets for symmetric Siegel domains.

Next we describe each $\mathcal{H}_s(D)$ ($s \in \mathfrak{X}$) through the Fourier-Laplace transformation. For $\xi \in \overline{\Omega^*}$, put $Q_\xi := 2\xi \circ Q$ and $N_\xi := \{v \in \mathfrak{g}(1/2) ; Q_\xi(v, v) = 0\}$. Then Q_ξ induces a positive definite Hermitian form on $\mathfrak{g}(1/2)/N_\xi$. Let $\mathcal{F}_\xi$ be the space of holomorphic functions ϕ on $\mathfrak{g}(1/2)$ such that

(i) $\phi(u + v) = \phi(u)$ for all $u \in \mathfrak{g}(1/2)$ and $v \in N_\xi$,

(ii) $\|\phi\|_\xi^2 := \int_{\mathfrak{g}(1/2)/N_\xi} |\phi(u)|^2 e^{-Q_\xi(u, u)} dm_\xi(u) < \infty$,

where dm_ξ is the Lebesgue measure on $\mathfrak{g}(1/2)/N_\xi$ normalized in such a way that $\int_{\mathfrak{g}(1/2)/N_\xi} e^{-Q_\xi(u, u)} dm_\xi(u) = 1$. In other words, $\mathcal{F}_\xi$ is the Fock space on $\mathfrak{g}(1/2)/N_\xi$ with reproducing kernel $e^{Q_\xi(\cdot, \cdot)}$ (see [1]). Keeping Proposition 2 (ii) in mind, we define $\mathcal{L}_s$ ($s \in \mathfrak{X}(\varepsilon)$) to be the space of measurable functions f on $\mathcal{O}_\varepsilon^* \times \mathfrak{g}(1/2)$ such that

(i) $f(\xi, \cdot) \in \mathcal{F}_\xi$ for almost all $\xi \in \mathcal{O}_\varepsilon^*$ with respect to the measure $d\mathcal{R}_{\text{Re } s^*}^*$,

(ii) $\|f\|^2 := \int_{\mathcal{O}_\varepsilon^*} \|f(\xi, \cdot)\|_\xi^2 d\mathcal{R}_{\text{Re } s^*}^*(\xi) < \infty$.

Then $\mathcal{L}_s$ forms a Hilbert space, realizing the direct integral $\int_{\mathcal{O}_\varepsilon^*}^\oplus \mathcal{F}_\xi d\mathcal{R}_{\text{Re } s^*}^*(\xi)$.

Theorem 5. *For $s \in \mathfrak{X}(\varepsilon)$, one has the unitary isomorphism $\Phi_s : \mathcal{L}_s \ni f \mapsto F \in \mathcal{H}_s(D)$ given by*

$$F(z, u) := \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z, \xi \rangle} f(\xi, u) d\mathcal{R}_{\text{Re } s^*}^*(\xi) \quad ((z, u) \in D).$$

We define the so-called Fock representation τ_ξ of $N(Q)$ on $\mathcal{F}_\xi$ by

$$\tau_\xi(\exp(x_0 + u_0))\phi(u) := e^{-i\langle x_0, \xi \rangle + Q_\xi(u, u_0) - Q_\xi(u_0, u_0)/2} \phi(u - u_0) \quad (u, u_0 \in \mathfrak{g}(1/2), x_0 \in \mathfrak{g}(1))$$

and a unitary representation $\tilde{\pi}_s$ ($s \in \mathfrak{X}(\varepsilon)$) of G on $\mathcal{L}_s$ by

$$\begin{aligned} \tilde{\pi}_s(t)f(\xi, u) &:= \chi_{\bar{s}/2}(t)f(t^{-1} \cdot \xi, t^{-1} \cdot u) \quad (t \in H), \\ \tilde{\pi}_s(n)f(\xi, \cdot) &:= \tau_\xi(n)f(\xi, \cdot) \quad (n \in N(Q)), \end{aligned}$$

where $\bar{s} := (\bar{s}_1, \dots, \bar{s}_r) \in \mathbb{C}^r$.

Proposition 6. *The operator $\Phi_s : \mathcal{L}_s \rightarrow \mathcal{H}_s(D)$ is an intertwining operator between $\tilde{\pi}_s$ and π_s . In particular, Φ_s induces the direct integral decomposition of $\pi_s|_{N(Q)}$:*

$$\pi_s|_{N(Q)} \simeq \int_{\mathcal{O}_\varepsilon^*}^\oplus \tau_\xi|_{N(Q)} d\mathcal{R}_{\text{Re } s^*}^*(\xi).$$

5. The orbit method and the classification of π_s . Since $\mathcal{H}_s(D)$ has a reproducing kernel, we obtain the following proposition from Kunze's result [11]:

Proposition 7. *The unitary representation $(\pi_s, \mathcal{H}_s(D))$ ($s \in \mathfrak{X}$) of G is irreducible.*

On the other hand, since G is exponential solvable, the equivalence classes of IUR's (irreducible unitary representations) of G are classified by the orbit method. Namely, there is a bijection, called *the Kirillov-Bernat correspondence*, between elements of the unitary dual $\hat{G}$ and coadjoint orbits in $\mathfrak{g}^*$ (see [2] for this). In this section, we shall describe the coadjoint orbits in $\mathfrak{g}^*$ corresponding to our $(\pi_s, \mathcal{H}_s(D))$, so that the classification of these IUR's is obtained.

We first identify each $\zeta = (\zeta_1, \dots, \zeta_r) \in \mathbb{R}^r$ with the linear form on $\mathfrak{g}$ given by $\langle \sum_{k=1}^r c_k A_k + Y, \zeta \rangle := \sum_{k=1}^r \zeta_k c_k$ ($c_1, \dots, c_r \in \mathbb{R}$, $Y \in [\mathfrak{g}, \mathfrak{g}]$), and for $\varepsilon \in \{0, 1\}^r$, we set $Z(\varepsilon) := \{ \zeta \in \mathfrak{g}^*; \zeta_k = 0 \text{ for all } k \text{ such that } \varepsilon_k = 1 \}$. Let $\mathfrak{h}_\xi$ ($\xi \in \mathfrak{g}(1)^*$) denote the Lie algebra of the stabilizer H_ξ at ξ in H .

Proposition 8. *Let $\zeta \in Z(\varepsilon)$ with $\varepsilon \in \{0, 1\}^r$. Then the coadjoint orbit in $\mathfrak{g}^*$ through $-E_\varepsilon^* + \zeta$ is described as*

$$(5.1) \quad \text{Ad}^*(G)(-E_\varepsilon^* + \zeta) = \bigsqcup_{\xi \in \mathcal{O}_\varepsilon^*} \left(-\xi + \zeta + (\mathfrak{g}(1) \oplus N_\xi \oplus \mathfrak{h}_\xi)^\perp \right),$$

where $(\mathfrak{g}(1) \oplus N_\xi \oplus \mathfrak{h}_\xi)^\perp := \{ \eta \in \mathfrak{g}^*; \langle Y, \eta \rangle = 0 \text{ for all } Y \in \mathfrak{g}(1) \oplus N_\xi \oplus \mathfrak{h}_\xi \}$. Distinct (ε, ζ) 's give distinct orbits.

We note that the coadjoint orbit through $\zeta \in Z(0, \dots, 0) = \mathbb{R}^r$ is a one-point set $\{\zeta\}$. On the other hand, the coadjoint orbit through $-E_{(1, \dots, 1)}^*$ is the open set $-\Omega^* + \mathfrak{g}(1/2)^* + \mathfrak{g}(0)^*$ in $\mathfrak{g}^*$.

Let $\Theta(\varepsilon, \zeta)$ be the set given by

$$\Theta(\varepsilon, \zeta) := \{ s \in \mathbb{C}^r; \text{Re } s_k > q_k(\varepsilon)/2 \text{ (if } \varepsilon_k = 1), \quad s_k = q_k(\varepsilon)/2 - 2i\zeta_k \text{ (if } \varepsilon_k = 0) \}.$$

Then we have $\mathfrak{X}(\varepsilon) = \bigsqcup_{\zeta \in Z(\varepsilon)} \Theta(\varepsilon, \zeta)$, so that Theorem 4 implies that $\mathcal{H}_s(D) \neq \{0\}$ exists if and only if s belongs to some $\Theta(\varepsilon, \zeta)$.

Theorem 9. (i) *If $s \in \Theta(\varepsilon, \zeta)$, then the IUR $(\pi_s, \mathcal{H}_s(D))$ of G corresponds to the coadjoint orbit through $-E_\varepsilon^* + \zeta \in \mathfrak{g}^*$ by the Kirillov-Bernat correspondence.*

(ii) *Two IUR's $(\pi_s, \mathcal{H}_s(D))$ and $(\pi_{s'}, \mathcal{H}_{s'}(D))$ are equivalent if and only if s and s' belong to the same $\Theta(\varepsilon, \zeta)$.*

When $s = (-2i\zeta_1, \dots, -2i\zeta_r) \in \Theta(0, \zeta)$, it can be checked directly that $\mathcal{H}_s(D)$ equals the space of constant functions on D and the representation π_s is equivalent to the unitary character $\chi_{i\zeta}$. We see from Theorem 9 that this IUR π_s corresponds

to the one-point set $\{\zeta\} = \text{Ad}^*(G)\zeta$. On the other hand, Theorem 9 also states that the unitary representations of G realized naturally on the various weighted Bergman spaces on D investigated by Gindikin [7] (see also Rossi and Vergne [14]) correspond to the same open orbit $\text{Ad}^*(G)(-E_{(1,\dots,1)}^*)$ mentioned above, so that all of such representations are equivalent.

The details [10] will appear elsewhere.

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