CLASSIFICATION OF 2-DIMENSIONAL GRADED NORMAL HYPERSURFACES WITH $a(R) = 1$.

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Introduction

Inspired by the talk of Kyoji Saito at the Toyama Conference, Aug. 2007, I tried the classification of 2-dimensional graded normal hypersurfaces with $a(R) = 1$ using Demazure's construction of normal graded rings. Since the classificaion is so simple and nearly automatic, I want to introduce it.

Although this classification is "known" in the literature (cf. [S], [P1]), it seems that the systematic method of classification is not known. So, I think this is worthwhile to be published in some form.

Also, I present here the classification of normal two-dimensional hypersurfaces with $a(R) = 2$ and normal graded complete intersections with $a(R) = 1$ and Proj $(R) \cong \mathbb{P}^1$.

1. Preliminaries

Let $R = k[u, v, w]/(f)$ be a 2-dimensional graded normal hypersurface, where k is an algebraically closed field of any characteristic. We put $X =$ Proj (R). Since $\dim R = 2$ and R is normal, X is a smooth curve. Then by the construction of Zariski and Demazure ([1], [5]), there is an ample $\mathbb{Q}$-Cartier divisor D (that is, ND is an ample divisor on X for some positive integer N), such that

$$R = R(X, D) = \oplus_{n\geq 0} H^0(X, O_X(nD)).T^n \subset k(X)[T],$$

where T is a variable over $k(X)$ and

$$H^0(X, O_X(nD)) = \{f \in k(X) \mid \mathrm{div}_X(f) + nD \geq 0\} \cup \{0\}.$$

Now, let us begin the classification. In the following, X is a smooth curve of genus g and D is a fractional divisor on X such that ND is an ample integral (Cartier) divisor for some $N > 0$.

We denote

$$D = D_0 + \sum_{i=1}^{r} \frac{p_i}{q_i} P_i \quad (\forall i,\ (p_i, q_i) = 1),$$

where D_0 is an integral divisor; a divisor with integer coefficients. In this case, we denote

$$D' = \sum_{i=1}^{r} \frac{q_i - 1}{q_i} P_i.$$

At the same time, by our assumption $R \cong k[u, v, w]/(f)$. If $\deg(u, v, w; f) = (a, b, c; h)$, then by [2],

$$a(R) = h - (a + b + c).$$

We always assume $\deg(u,v,w;f) = (a,b,c;h)$ and also that $a \le b \le c$.

Proposition 1.1. *(***Fundamental formulas***) Assume that $R = R(X,D)$ $\cong k[u,v,w]/(f)$ with $\deg(u,v,w;f) = (a,b,c;h)$ and $a(R) = h-(a+b+c) = 1$. Then we have the following equalities.*

(1) [W] Since R is Gorenstein with $a(R) = 1$, we have

$$D \sim K_X + D' = K_X + \sum_{i=1}^{r} \frac{q_i - 1}{q_i} P_i,$$

where, in general, $D_1 \sim D_2$ means that $D_1 - D_2 = \mathrm{div}_X(f)$ for some $f \in k(X)$.

(2) [Tomari's formula] If $P(R,t) = \sum_{n\ge 0} \dim R_n t^n$,

$$\lim_{t\to 1}(1-t)^2 P(R,t) = \deg D.$$

(3) Since $P(R,t) = \dfrac{1-t^h}{(1-t^a)(1-t^b)(1-t^c)}$, *we have*

$$\deg D = \frac{h}{abc} = \frac{1}{abc} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}.$$

Note that the latter expression is a decreasing function of a, b, c.

2. The classification of the hypersurfaces with $a(R) = 1$.

Henceforce, we put $D = K_X + \sum_{i=1}^{r} \dfrac{q_i - 1}{q_i} P_i$. **We always use the letter r in this meaning.** From 1.1 (3), the maximal value of $\deg D$ is taken when $a = b = c = 1$ and $\deg D = 4$ in that case.

Case A. The case $g > 0$.

Assume that $g \ge 1$. Since $\deg(D) \le 4$, and $\deg D \ge \deg K_X = 2g-2$, $g \le 3$ and if $g = 3$, $D = K_X$. We list the cases by giving the form of D and $(a,b,c;h)$. We can easily deduce the general form of the equation f from this data. Also, if f with the given weight has an isolated singularity, then $k[u,v,w]/(f) \cong R(X,D)$, where D is a divisor of given form.

(A-1) $\quad g = 3, D = K_X; \quad (1,1,1;4)$.

Next, consider the case $g = 2$. Note that $\dim R_1 = \dim H^0(K_X) = g = 2$, we have $a = b = 1$ and $\deg D = 1 + \dfrac{3}{c} \le \dfrac{5}{2}$ $\;(c \ge 2)$. Since, either $\deg D = 2, D = K_X$ or $\deg D \ge \dfrac{5}{2}$, we have 2 cases.

(A-2) $\quad g = 2, D = K_X; \quad (1,1,3;6)$.

(A-3) $\quad g = 2, D = K_X + \dfrac{1}{2}P; \quad (1,1,2;5)$.

Next, assume $g = 1$. In this case, $a = 1$, $2 \leq b \leq c$ and the maximal value of $\deg D$ is $\frac{3}{2}$. Since on the other hand, $\deg D \geq \frac{r}{2}$ and thus $r \leq 3$ and if $r = 3$, $D = \frac{1}{2}(P_1 + P_2 + P_3)$.

(A-4) $\quad g = 1, D = \frac{1}{2}(P_1 + P_2 + P_3); \quad (1, 2, 2; 6).$

Also, since $\dim R_2 = r$, if $r = 2$, then $a = 1, b = 2, c \geq 3$, $\deg(D) \leq \frac{7}{6}$.

(A-5) $\quad g = 1, D = \frac{1}{2}(P_1 + P_2); \quad (1, 2, 4; 8).$

(A-6) $\quad g = 1, D = \frac{1}{2}P_1 + \frac{2}{3}P_2; \quad (1, 2, 3; 7).$

If $g = 1$ and $D = \frac{q-1}{q}P$, we have $q - 1$ new generators in degrees $1, 3, \ldots, q$. Hence $q \leq 4$.

$a = 1, 3 \leq b, c$ and $\deg D \leq \frac{8}{9}$.

(A-7) $\quad g = 1, D = \frac{1}{2}P; \quad (1, 4, 6; 12).$

(A-8) $\quad g = 1, D = \frac{2}{3}P; \quad (1, 3, 5; 10).$

(A-9) $\quad g = 1, D = \frac{3}{4}P; \quad (1, 3, 4; 9).$

We have 9 types when $g \geq 1$.

Case B. The case $g = 0$ and $r \geq 4$.

In the following, we always assume $g = 0$. Since $\deg(K_X) = -2$ and $\deg D > 0$, we have $r \geq 3$. On the other hand, since $R_1 = H^0(K_X) = 0$, $a \geq 2$ and $\deg D \leq \frac{7}{8} < 1$. Since $\deg D \geq -2 + r/2$, we have $r \leq 5$.

In this subsection, we treat the cases where $r = 4, 5$.

Now, since $\deg[2D] = r - 4$, $\dim R_2 = 2, 1$, respectively, if $r = 5, 4$.

Thus if $r = 5$, then $a = b = 2$ and $c \geq 3$. Hence $\deg D \leq \frac{2}{3}$. Since $3.\frac{1}{2} + 2.\frac{2}{3} - 2 = \frac{5}{6} > \frac{2}{3}$, the only possible cases for $(q_1, \ldots, q_5)$ are $(2, 2, 2, 2, 2)$ and $(2, 2, 2, 2, 3)$.

(B-1) $\quad D = K_X + \frac{1}{2}(P_1 + P_2 + \ldots + P_5); \quad (2, 2, 5; 10).$

(B-2) $\quad D = K_X + \frac{1}{2}(P_1 + P_2 + P_3 + P_4) + \frac{2}{3}P_5; \quad (2, 2, 3; 8).$

Henceforce we assume $r = 4$ and express D by (q_1, q_2, q_3, q_4) and **we always assume** $q_1 \leq q_2 \leq q_3 \leq q_4$. In this case, $a = 2$ and $3 \leq b \leq c$. Hence $\deg D \leq \frac{1}{2}$. Since $4.\frac{2}{3} - 2 > \frac{1}{2}$, $q_1 = 2$ and $q_4 \geq 3$.

Let s be the number of $q_i > 2$ $(1 \leq s \leq 3)$. Then since $\deg[3D] = -6 + 8 - s$, $\dim R_3 = 0, 1, 2$ when $s = 1, 2, 3$, respectively.

If $s = 3$, $\dim R_2 + \dim R_3 = 3$ and we must have $(a, b, c; h) = (2, 3, 3; 9)$.

(B-3) $\quad D = K_X + \frac{1}{2}P_1 + \frac{2}{3}(P_2 + P_3 + P_4); \quad (2, 3, 3; 9).$

If $s = 2$, $a = 2, b = 3$ and $c \geq 4$ and $\deg D = \frac{1}{6} + \frac{1}{c} \leq \frac{5}{12}$. Also, since $-2 + (\frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{5}{12}$, we have 2 types.

(B-4) $\quad D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{2}{3}(P_3 + P_4); \quad (2, 3, 6; 12).$

(B-5) $\quad D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{2}{3}P_3 + \frac{3}{4}P_4; \quad (2, 3, 4; 10).$

Now we trat the case $(2, 2, 2, q)$, $q \geq 3$. In this case, $R_3 = 0$ and $\dim R_4 = 1$ or 2 according to $q = 3$ or $q \geq 4$. In the latter case, $\dim R_5 = 0$ or 1 according to $q = 4$ or $q \geq 5$. Hence, if $q \geq 5$, we have already 3 generators of R.

(B-6) $\quad D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{4}{5}P_4; \quad (2, 4, 5; 12).$

(B-7) $\quad D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{3}{4}P_4; \quad (2, 4, 7; 14).$

(B-8) $\quad D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{2}{3}P_4; \quad (2, 6, 9; 18).$

We have 8 types in this case.

Case C. The case $g = 0$ and $r = 3$.

We have to determine (q_1, q_2, q_3). In this case, $R_1 = R_2 = 0$ and $\dim R_3 = 1$ or 0 according to $q_1 = 2$ or $q_1 \geq 3$.

Case 1. $q_1 \geq 3$.

In this case, $a = 3$ and $4 \leq b \leq c$. Hence $\deg D \leq \frac{1}{4}$. Hence either $q_1 = 3$ or $q_1 = q_2 = q_3 = 4$.

(C-1) $\quad D = K_X + \frac{3}{4}(P_1 + P_2 + P_3); \quad (3, 4, 4; 12).$

Henceforce we assume $q_1 = 3$.

$R_4 \neq 0$ if and only if $q_2 \geq 4$. In this case, $a = 3, b = 4$ and $c \geq 5$. Hence $\deg D \leq \frac{13}{60} = \frac{2}{3} + \frac{3}{4} + \frac{4}{5} - 2$. Hence we have only 2 possibilities;

(C-2) $\quad D = K_X + \frac{2}{3}P_1 + \frac{3}{4}P_2 + \frac{4}{5}P_3; \quad (3, 4, 5; 13).$

(C-3) $D = K_X + \frac{2}{3}P_1 + \frac{3}{4}(P_2 + P_3)$; $(3, 4, 8; 16)$.

Next, assume $q_1 = q_2 = 3$. Hence $\deg D = \frac{q_3 - 1}{q_3} - \frac{2}{3}$. On the other hand, since $R_4 = 0$, $a = 3$, $b \geq 5$ and $c \geq 6$ and $\deg D \leq \frac{1}{6}$. This implies $q_3 \leq 6$.

(C-4) $D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{5}{6}P_3$; $(3, 5, 6; 15)$.

(C-5) $D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{4}{5}P_3$; $(3, 5, 9; 18)$.

(C-6) $D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{3}{4}P_3$; $(3, 8, 12; 24)$.

This completes the case $q_1 = 3$.

Case 2. $q_1 = 2$**.**

In this case, $a \geq 4$ and $R_4 \neq 0$ if and only if $q_2 \geq 4$.

First, we consider the case $q_1 = 2$ **and** $q_2 = 3$ $(q_3 \geq 7)$**.**

In this case, $\deg[4D] = -1 = \deg[5D] = \deg[7D], \deg[6D] = 0$. Hence $a = 6$ and $b \geq 8$. Hence $\deg D \leq \frac{1}{18} = \frac{8}{9} - \frac{5}{6}$. This shows that $7 \leq q_3 \leq 9$ and actually these cases gives the hypersurfaces.

(C-7) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{6}{7}P_3$; $(6, 14, 21; 42)$.

(C-8) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{7}{8}P_3$; $(6, 8, 15; 30)$.

(C-9) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{8}{9}P_3$; $(6, 8, 9; 24)$.

Next, we consider the case $q_1 = 2$ **and** $q_2 \geq 4$**.**

In this case, $\deg[4D] = 0$ and $a = 4, b \geq 5, c \geq 6$. Hence $\deg D \leq \frac{2}{15} = (\frac{1}{2} + \frac{4}{5} + \frac{5}{6}) - 2$. Hence $q_2 \leq 5$ and if $q_2 = 5$, the possibility is the following 2 cases.

(C-10) $D = K_X + \frac{1}{2}P_1 + \frac{4}{5}P_2 + \frac{5}{6}P_3$; $(4, 5, 6; 16)$.

(C-11) $D = K_X + \frac{1}{2}P_1 + \frac{4}{5}(P_2 + P_3)$; $(4, 5, 10; 20)$.

The remaining case is $q_1 = 2, q_2 = 4$ $(q_3 \geq 5)$**.**

Since $\dim R_4 = 1$ and $R_5 = 0$ and hence $a = 4, b \geq 6, c \geq 7$ and $\deg D = \frac{q_3 - 1}{q_3} - \frac{3}{4} \leq \frac{3}{28}$. Hence $5 \leq q_3 \leq 7$ and actually these cases give hypersurfaces.

This finishes the classification !

(C-12) $D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{4}{5}P_3$; $(4, 10, 15; 30)$.

(C-13) $D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{5}{6}P_3$; $(4, 6, 11; 22)$.

(C-14) $D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{6}{7}P_3$; $(4, 6, 7; 18)$.

3. The classification of hypersurfaces with $a(R) = 2$.

In this section, we classify normal graded hypersurfaces of dimension 2 with $a(R) = 2$.

We may assume that $R = R(X, D) \cong k[u, v, w]/(f)$ with

$$\deg(u, v, w; f) = (a, b, c; h); \quad h = a + b + c + 2.$$

We always assume $(a, b, c) = 1$. Since R is Gorenstein with $a(R) = 2$, $2D$ is linearly equivalent to $K_X + D'$. Hence we may assume that

$$D = E + \sum_{i=1}^{r} \frac{q_i - 1}{2q_i} P_i,$$

where $2E \sim K_X$ and every q_i is odd.

Since $\deg D = \dfrac{h}{abc} = \dfrac{a+b+c+2}{abc} \leq 5$, $2g - 2 \leq 2\deg D \leq 10$ and we have $g \leq 6$.

First, we divide the cases according to (1) $a \geq 2$, (2) $a = 1, b \geq 2$, or (3) $a = b = 1$.

Case 1. $a \geq 2$.

This is equivalent to say that $R_1 = H^0(X, \mathcal{O}_X(D)) = 0$. If this is the case, we have

$$\deg D \leq \frac{9}{2 \cdot 2 \cdot 3} = \frac{3}{4} < 1.$$

Since $\deg D \geq g - 1$, $g = 0$ or 1 in this case.

For a while, we assume that $g = 0$.

Now, we can write

$$D = -Q + \sum_{i=1}^{r} \frac{q_i - 1}{2q_i} P_i.$$

Hence $R_1 = R_2 = 0$ and $\dim R_3 = r - 2$ since $\dfrac{q_i - 1}{2q_i} \geq \dfrac{1}{3}$ for every q_i. Hence $a = 3$ and $\deg D \leq \dfrac{12}{3 \cdot 3 \cdot 4} = \dfrac{1}{3}$. This implies that $r \leq 4$ and if $r = 4$, then $D = -Q + \sum_{i=1}^{4} \frac{1}{3} P_i$.

(2-1) $\quad g = 0, D = -Q + \dfrac{1}{3}(P_1 + P_2 + P_3 + P_4), \quad (3, 3, 4; 12).$

Now, we assume $r = 3$ and $q_1 \leq q_2 \leq q_3$. Then since $\dim R_3 = 1$, we have $a = 3$ and $b \geq 5$. Also, $\dim R_4 = 0$ and $\dim R_5 = 2$ (resp. 1, resp. 0) if $q_3 \geq 5$ (resp. $q_1 = 3, q_2 \geq 5$, resp. $q_2 = 3$).

If $q_1 \geq 5$, $\deg D \leq \dfrac{15}{3 \cdot 5 \cdot 5} = \dfrac{1}{5}$. Hence we must have $D = -Q + \dfrac{2}{5}(P_1 + P_2 + P_3)$.

(2-2) $\quad g = 0, D = -Q + \dfrac{2}{5}(P_1 + P_2 + P_3), \quad (3, 5, 5; 15).$

Next, consider the case $q_1 = 3$ and $q_2 \geq 5$. In this case, $a = 3, b = 5$ and $c \geq 7$ and then $\deg D \leq \frac{17}{3 \cdot 5 \cdot 7} = \frac{1}{3} + \frac{2}{5} + \frac{3}{7} - 1$. Hence we are restricted to the following 2 cases.

(2-3) $\quad g = 0, D = -Q + \frac{1}{3}P_1 + \frac{2}{5}P_2 + \frac{3}{7}P_3, \quad (3, 5, 7; 17).$

Actually, if we put $D = -\frac{2}{3}(\infty) + \frac{2}{5}(0) + \frac{3}{7}(-1)$, then $R = k[F, G, H]$ with $F = \frac{1}{x(x+1)}T^3, G = \frac{1}{x^2(x+1)^2}T^5, H = \frac{1}{x^2(x+1)^3}T^7$ with the relation

$$F^4G = FH^2 + G^2H.$$

Hence $R \cong k[X, Y, Z]/(XZ^2 + Y^2Z - X^4Y)$.

(2-4) $\quad g = 0, D = -Q + \frac{1}{3}P_1 + \frac{2}{5}(P_2 + P_3), \quad (3, 5, 10; 20).$

If $q_2 = 3$, then $a = 3$ and $b \geq 7$. Hence $\deg D \leq \frac{21}{3 \cdot 7 \cdot 9} = 2\frac{1}{3} + \frac{4}{9} - 1$. Hence in this case, $q_3 = 5, 7$ or 9.

(2-5) $\quad g = 0, D = -Q + \frac{1}{3}(P_1 + P_2) + \frac{4}{9}P_3, \quad (3, 7, 9; 21).$

(2-6) $\quad g = 0, D = -Q + \frac{1}{3}(P_1 + P_2) + \frac{3}{7}P_3, \quad (3, 7, 15; 30).$

(2-7) $\quad g = 0, D = -Q + \frac{1}{3}(P_1 + P_2) + \frac{2}{5}P_3, \quad (3, 10, 15; 30).$

Now we have finished the case $a \geq 2$ and $g = 0$. Next, we treat the case $a \geq 2$ and $g = 1$. In this case, we put

$$D = E + \sum_{i=1}^{r} \frac{q_i - 1}{2q_i} P_i,$$

where $E \in \mathrm{Div}(X)$ with $E \neq 0$ and $2E \sim 0$. Since $[2D] = 0$ and $\deg[3D] = r > 0$, we have $a = 2$ and $b = 3$. Hence $\deg D \leq \frac{10}{2 \cdot 3 \cdot 3} < 1$ and actually, we have $r = 1$ or 2.

If $r = 2$, then $\deg[4D] = 2$ and we must have $(a, b, c) = (2, 3, 4)$ and $\deg D = \frac{11}{24}$. But since $\frac{q_1 - 1}{2q_1} + \frac{q_2 - 1}{2q_2} = \frac{11}{24}$ is impossible, this case does not occur. Hence we must have $r = 1$.

Since $a = 2, b = 3, c = 4$ is impossible as we have seen before, we must have $D = E + \frac{q-1}{2q}P$ with $E + P \geq 0$ and $\deg D \leq \frac{12}{2 \cdot 3 \cdot 5}$. We have 2 possibilities; $D = E + \frac{2}{5}P$ and $D = E + \frac{1}{3}P$. But the in latter case, we must have $a = 2, b = 3, c = 9$, which contradicts the fact $\deg D = \frac{1}{3}$.

Hence we are reduced to the case.

(2-8) $g = 1, D = E + \frac{2}{5}P$ with $2E \sim 0$ and $E \neq 0$, $(2, 3, 5; 12)$.

This finishes the case $a \geq 2$.

Case 2. $a = 1$ **and** $b \geq 2$**.**

In this case, $\deg D \leq \frac{7}{1 \cdot 2 \cdot 2} < 2$. Hence we have $g = 1$ or 2 in this case. Moreover, if $g = 1$, since $[2D] = 0$, we have $b \geq 3$ and $\deg D \leq \frac{9}{1 \cdot 3 \cdot 3} \leq 1$.

First, we assume $g = 1$ and $D = \sum_{i=1}^{r} \frac{q_i - 1}{2q_i} P_i$. Since $[2D] = 0$ in this case, $b \geq 3$ and we have $\deg D \leq \frac{9}{1 \cdot 3 \cdot 3} = 1$. Hence $r \leq 3$ in this case and if $r = 3$, $D = \frac{1}{3}(P_1 + P_2 + P_3)$.

(2-9) $g = 1, D = \frac{1}{3}(P_1 + P_2 + P_3)$, $(1, 3, 3; 9)$.

Next, we assume $r = 2$. Then $b = 3$ and $c \geq 5$. We have $\deg D \leq \frac{11}{1 \cdot 3 \cdot 5} = \frac{1}{3} + \frac{2}{5}$. Hence we have 2 possibilities;

(2-10) $g = 1, D = \frac{1}{3}P_1 + \frac{2}{5}P_2$, $(1, 3, 5; 11)$.

There is a linear relation between $T^{11}, GT^8, HT^6, G^2T^5, GHT^3, G^3T^2, H^2T, G^2H$, where $\deg T = 1, \deg G = 3$ and $\deg H = 5$.

(2-11) $g = 1, D = \frac{1}{3}(P_1 + P_2)$, $(1, 3, 6; 12)$.

Next, we assume $D = \frac{q-1}{2q}P$. In this case, $b \geq 5$ and $\deg D \leq \frac{15}{1 \cdot 5 \cdot 7} = \frac{3}{7}$. Hence we have 3 possibilities; $q = 3, 5, 7$.

(2-12) $g = 1, D = \frac{3}{7}P$, $(1, 5, 7; 15)$.

(2-13) $g = 1, D = \frac{2}{5}P$, $(1, 5, 8; 16)$.

(2-14) $g = 1, D = \frac{1}{3}P$, $(1, 6, 9; 18)$.

Next, we treat the case $g = 2$, $a = 1, b \geq 2$ and $D = E + \sum_{i=1}^{r} \frac{q_i - 1}{2q_i} P_i$, with $2E \sim K_X$. Since $[2D] \sim K_X$ in this case, we have $a = 1, b = 2$ and $c \geq 3$. Thus we have $\deg D \leq \frac{8}{1 \cdot 2 \cdot 3} = 1 + \frac{1}{3}$. Hence $r \leq 1$ and if $r = 1$, then $D = D = E + \frac{1}{3}P$.

(2-15) $g = 2, D = E$ with $2E \sim K_X$, $(1, 2, 5; 10)$.

(2-16) $g = 2, D = \frac{1}{3}P, \quad (1, 2, 3; 8).$

This finishes the case $a = 1, b \geq 2$.

Case 3. $a = b = 1$.

In this case, $\deg D = \frac{c+4}{c}$. Since $g \geq 3$ in this case, $\deg D \geq 2$ and we have $c \leq 4$.

(2-17) $g = 6, D = E$ with $2E \sim K_X, \quad (1, 1, 1; 5).$

(2-18) $g = 5, D = E$ with $2E \sim K_X, \quad (1, 1, 2; 6).$

(2-19) $g = 3, D = E + \frac{1}{3}P$ with $2E \sim K_X, \quad (1, 1, 3; 7).$

(2-20) $g = 3, D = E$ with $2E \sim K_X, \quad (1, 1, 4; 8).$

4. Complete intersections with $a(R) = 1$.

In my talk at the conference, I talked about classification of normal graded complete intersections of dimension 2 with $a(R) = 1$. Until now, I can not find a satisfactory way to classify them. Here, I will show the results when the genus of the curve is 0.

Proposition 4.1. *Let $R = \oplus_{n \geq 0} R_n$ be a normal graded complete intersection 0f dimension 2 with $R_0 = k$, a field, $a(R) = 1$ and $R_1 = 0$. Then the embedded dimension of R is at most 4.*

This follows from the fact $\mathfrak{m}H^1(X, \mathcal{O}_X) = 0$, where $\mathfrak{m}$ is the graded maximal ideal of R and $X \to \operatorname{Spec}(R)$ is a resolution of singularities of R. By the Briançon-Skoda type argument, we can assert that $\mathfrak{m}^3 \subset J$, where J is a minimal reduction of $\mathfrak{m}$. Then by the argument as in [NW], §2, we can deduce that the embedded dimension of R is at most 4. Conversely, if R is Gorenstein with the embedded dimension 4, then R is a complete intersection by the famous result of J.-P. Serre.

Until now, I can not find a satisfactory method of classification for this case. Actually, what I do is only to restrict the embedding dimension. So, I list only the results in this case. We list the divisor D on $X = \mathbb{P}^1$ with $R(X, D) \cong k[u, v, w, z]/(f, g)$. We also put $\deg(u, v, w, z; f, g) = (a, b, c, d; g, h)$ with $g + h = a + b + c + d + 1$ with $a \leq b \leq c \leq d$ and $g \leq h$ in the following table.

(3-1) $D = K_X + \frac{1}{2}(P_1 + \ldots + P_6), \quad (2, 2, 2, 3; 4, 6).$

(3-2) $D = K_X + \frac{1}{2}(P_1 + \ldots + P_4) + \frac{3}{4}P_5, \quad (2, 2, 3, 4; 6, 6).$

(3-3) $D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{2}{3}(P_4 + P_5), \quad (2, 2, 3, 3; 5, 6).$

(3-4) $D = K_X + \frac{2}{3}(P_1 + \ldots + P_4), \quad (2, 3, 3, 3; 6, 6).$

(3-4) $D = K_X + \frac{2}{3}(P_1 + \ldots + P_4), \quad (2,3,3,3;6,6).$

(3-5) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}(P_2 + P_3) + \frac{3}{4}P_4, \quad (2,3,3,4;6,7).$

(3-5) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}(P_2 + P_3) + \frac{3}{4}P_4, \quad (2,3,3,4;6,7).$

(3-6) $D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{3}{4}(P_3 + P_4), \quad (2,3,4,4;6,8).$

(3-7) $D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{2}{3}P_3 + \frac{4}{5}P_4, \quad (2,3,4,5;7,8).$

(3-8) $D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{5}{6}P_4, \quad (2,4,5,6;8,10).$

(3-9) $D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{5}{6}P_4, \quad (2,4,5,6;8,10).$

(3-10) $D = K_X + \frac{3}{4}(P_1 + P_2) + \frac{4}{5}P_3, \quad (3,4,4,5;8,9).$

(3-11) $D = K_X + \frac{2}{3}P_1 + \frac{4}{5}(P_2 + P_3), \quad (3,4,5,5;8,10).$

(3-12) $D = K_X + \frac{2}{3}P_1 + \frac{3}{4}P_2 + \frac{5}{6}P_3, \quad (3,4,5,6;9,10).$

(3-13) $D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{6}{7}P_3, \quad (3,5,6,7;10,12).$

(3-14) $D = K_X + \frac{1}{2}P_1 + \frac{5}{6}(P_2 + P_3), \quad (4,5,6,6;10,12).$

(3-15) $D = K_X + \frac{1}{2}P_1 + \frac{4}{5}P_2 + \frac{6}{7}P_3, \quad (4,5,6,7;11,12).$

(3-16) $D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{7}{8}P_3, \quad (4,6,7,8;12,14).$

(3-17) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{9}{10}P_3, \quad (6,8,9,10;16,18).$

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