On G-local G-schemes

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This is a joint work with Mitsuyasu Hashimoto.

1 Diagrams of schemes and modules over them

Let I be a small category, $\underline{Sch}$ denote the category of schemes. We think a contravariant functor $X_\bullet : I \to \underline{Sch}$. It can be thought as a diagram of schemes and morphisms. For each $i \in I$, denote the scheme $X_\bullet(i)$ by X_i. And for a morphism ϕ in I, denote the morphism $X_\bullet(\phi)$ by X_ϕ. We can define a category $\mathrm{Zar}(X_\bullet)$ as follows :

$$\begin{aligned}\mathrm{ob}\big(\mathrm{Zar}(X_\bullet)\big) &:= \{(i,U) \mid i \in \mathrm{ob}(I),\, U \in \mathrm{Zar}(X_i)\},\\ \mathrm{Hom}\big((i,U),(j,V)\big) &:= \{(\phi,h) \mid \phi : i \leftarrow j \text{ is a morphism in } I,\, h : U \to V \\ &\qquad \text{is a morphism such that it is the restriction of } X_\phi : X_i \to X_j\}\end{aligned}$$

In the definiton, for a scheme S, $\mathrm{Zar}(S)$ denote the category consisting of open subschemes of S and inclusion morphisms.

And we can define a Grothendieck topology on $\mathrm{Zar}(X_\bullet)$. A class of morphisms $\{(h_\lambda,\phi_\lambda) : (i_\lambda,U_\lambda) \to (i,U)\}$ is a covering of (i,U) if the following hold :

$$(1)\ i_\lambda = i \text{ and } \phi_\lambda = \mathrm{id} \text{ for any } \lambda, \qquad (2)\ U = \bigcup h_\lambda U_\lambda.$$

So we can think sheaves over $\mathrm{Zar}(X_\bullet)$.

Moreover, we define the sheaf of commutative rings $\mathcal{O}_{X_\bullet}$ on $\mathrm{Zar}(X_\bullet)$ by

$$\Gamma\big((i,U),\mathcal{O}_{X_\bullet}\big) := \Gamma(U,\mathcal{O}_{X_i}),$$

where $\mathcal{O}_{X_i}$ is the structure sheaf of X_i. So $\mathrm{Zar}(X_\bullet)$ is a ringed site, and we can think $\mathcal{O}_{X_\bullet}$-module sheaves. Denote the category of $\mathcal{O}_{X_\bullet}$-modules $\mathrm{Mod}(\mathrm{Zar}(X_\bullet))$ by $\mathrm{Mod}(X_\bullet)$, simply.

For $i \in I$, we can define a functor $[-]_i : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(X_i)$ by

$$\Gamma(U, \mathcal{M}_i) := \Gamma\big((i, U), \mathcal{M}\big).$$

This functor $[-]_i$ is called the restriction functor. The restriction functor $[-]_i$ has both a left adjoint and a right adjoint, so $[-]_i$ preserves limits and colimits, and it is exact (Hashimoto [3], (4.4)).

Let $\phi : i \to j$ be a morphism in I. For $(i, U) \in \mathrm{Zar}(X_\bullet)$ and an $\mathcal{O}_{X_\bullet}$-module $\mathcal{M}$, a morphism $\beta_\phi(\mathcal{M}) : \mathcal{M}_i \to (X_\phi)_* \mathcal{M}_j$ is defined by the following diagram of the sets of sections over U :

$$\begin{array}{ccccc}
\Gamma(U, \mathcal{M}_i) & \longrightarrow & \Gamma(X_\phi^{-1}U, \mathcal{M}_j) & = & \Gamma(U, (X_\phi)_* \mathcal{M}_i) \\
\| & & \| & & \\
\Gamma((i, U), \mathcal{M}) & \xrightarrow{f} & \Gamma((j, X_\phi^{-1}U), \mathcal{M}) & &
\end{array}$$

where f is the restrction with respect to the morphism $\Big(\phi, X_\phi|_{X_\phi^{-1}U}\Big)$.

And we can define a morphism $\alpha_\phi : X_\phi^*[-]_i \to [-]_j$ to be the composite

$$X_\phi^*[-]_i \xrightarrow{\beta_\phi} X_\phi^*(X_\phi)_*[-]_j \xrightarrow{\epsilon} [-]_j$$

where ϵ is the counit of the adjoint pair $(X_\phi^*, (X_\phi)_*)$.

Definition 1. Let $\mathcal{M}$ be an $\mathcal{O}_{X_\bullet}$-module.
(1) $\mathcal{M}$ is **equivariant** if α_ϕ is an isomorphism for each morphism ϕ in I.
(2) $\mathcal{M}$ is **locally coherent** (resp. **locally quasi-coherent**) if each $\mathcal{M}_i$ is a coherent (resp. quasi-coherent) $\mathcal{O}_{X_i}$-module for any $i \in I$.
(3) $\mathcal{M}$ is **coherent** (resp. **quasi-coherent**) if $\mathcal{M}$ is locally coherent (resp. locally quasi-coherent) and equivariant.

2 The diagram $B_G^M(X)$ and G-local G-scheme

Denote the set of natural numbers $\{0, 1, \cdots, n\}$ by $[n]$. Let Δ be the category defined as follows :

$$\begin{aligned}
\mathrm{ob}(\Delta) &= \{[0], [1], [2]\}, \\
\mathrm{Hom}([i], [j]) &= \text{the set of order-preserving injective maps } [i] \to [j].
\end{aligned}$$

Δ is represented by the following diagram (without identity maps) :

$$\Delta = \left([2] \;\; \begin{matrix} \xleftarrow{i_0} \\ \xleftarrow{i_1} \\ \xleftarrow{i_2} \end{matrix} \;\; [1] \;\; \begin{matrix} \xleftarrow{i_0} \\ \xleftarrow{i_1} \end{matrix} \;\; [0] \right)$$

where i_s is the order-preserving injection whose image does not contain s.

From now on, let S be a Noetherian scheme, G be an S-group scheme flat of finite type and X be a Noetherian G-scheme. G-scheme is an S-scheme with G-action. We define a diagram of schemes $B_G^M(X) \in \mathrm{Func}(\Delta^{\mathrm{op}}, \underline{Sch})$ by

$$B_G^M(X) := \left(G \times_S G \times_S X \;\; \begin{matrix} \xrightarrow{\mathrm{id}\times a} \\ \xrightarrow{\mu\times\mathrm{id}} \\ \xrightarrow{p_{23}} \end{matrix} \;\; G \times_S X \;\; \begin{matrix} \xrightarrow{a} \\ \xrightarrow{p_2} \end{matrix} \;\; X \right)$$

where $a : G \times X \to X$ is the action, $\mu : G \times G \to G$ is the product, and p_{23} and p_2 are projections.

We call a module over this diagram $B_G^M(X)$ **a** $(G, \mathcal{O}_X)$**-module**, and denote the category of $(G, \mathcal{O}_X)$-modules $\mathrm{Mod}(B_G^M(X))$ by $\mathrm{Mod}(G, X)$. And denote the fullsubcategory of locally quasi-coherent $(G, \mathcal{O}_X)$-modules, of quasi-coherent $(G, \mathcal{O}_X)$-modules and of coherent $(G, \mathcal{O}_X)$-modules by $\mathrm{Lqc}(G, X)$, $\mathrm{Qch}(G, X)$ and $\mathrm{Coh}(G, X)$, respectively.

Let Z be a closed subscheme of X. Denote the scheme theoritic image of the action $a : G \times Z \to X$ by Z^*. This subscheme Z^* has the following properties :

1. Z^* is the smallest G-stable (i.e. the action $a : G \times Z^* \to X$ factors through the inclusion $Z^* \hookrightarrow X$) closed subscheme which contains Z. So if Z is G-stable, then $Z^* = Z$.

2. Assume that G is an S-smooth group scheme with connected geometric fibers. If Z is irreducible (resp. reduced), then so is Z^*. So if Z is integral, then Z^* is integral, too.

Definition 2. A quasi-compact G-scheme X is G**-local** if X is has a unique minimal non-empty G-stable closed subscheme Y of X. In this case, we say that (X, Y) is G-local.

There are some examples of G-local G-schemes.

Example 3. (1) If G is trivial, a G-local G-scheme X is of the form $\operatorname{Spec} A$ where A is a local ring.
(2) Let $S = \operatorname{Spec}\mathbb{Z}$, $G = \mathbb{G}_m$ (multiplicative group) and A be a G-algebra. Let ω be the coaction $A \to A \otimes \mathbb{Z}[G]$ and $X(G)$ the charactor group of G. Now it holds $X(G) \simeq \mathbb{Z}$ as groups. For a character $\lambda \in X(G)$, set $A_\lambda = \{a \in A \mid \omega(a) = a \otimes \lambda\}$. Then $A = \bigoplus_{\lambda \in X(G)} A_\lambda$ hold. And for λ, $\mu \in X(G)$, $A_\mu A_\lambda = \{a_\lambda a_\mu \mid a_\lambda \in A_\lambda, a_\mu \in A_\mu\} \subset A_{\lambda+\mu}$. So the equation $A = \bigoplus A_\lambda$ means that $\mathbb{G}_m$-algebras are $\mathbb{Z}$-graded algebras and that an ideal I of $\mathbb{G}_m$-algebra A is $\mathbb{G}_m$-stable if and only if it is homogeneous.

So affine $\mathbb{G}_m$-scheme $X = \operatorname{Spec} A$ is $\mathbb{G}_m$-local if and only if A is an H-local $\mathbb{Z}$-graded ring in the sense of Goto and Watanabe [1].
(3) If $S = \operatorname{Spec} k$ with k an algebraically closed field, G is an linear algebraic group and B is a Borel subgroup of G, then $(G/B, G/B)$ is G-local and $(G/B, B/B)$ is B-local. But it is not affine unless $G = B$ So a G-local G-scheme is not neccesarily affine even if S and G are affine.
(4) Let k be a field, G a reductive group, C a k-algebra of finite type with G-action, $A := C^G$ and $P \in \operatorname{Spec} A$. Then $X = \operatorname{Spec} C_P$ is a G-local G-scheme.

Until the end of this article, let G be an S-smooth group scheme with connected geometric fibers. For example, a connected algebraic group over an algebraically closed field k has this property. And let (X, Y) be a Noetherian G-local G-scheme.

Under the assumption, the unique minimal non-empty G-stable closed subscheme Y of X is integral. In fact, each irreducible component of Y and the reduction Y_{red} of Y is G-stable, so Y is irreducible and reduced because of minimality of Y. So Y has the generic point. Let η be the generic point of Y, $\mathcal{I}$ the defining ideal of Y and $f : Y \to X$ the inclusion.

The localization at η is very important and useful.

Lemma 4. *The localization functor* $[-]_\eta : \operatorname{Qch}(G, X) \to \operatorname{Mod} \mathcal{O}_{X,\eta}$ *is faithful and exact.*

Proof. A localization functor is exact in general, so it is enough to prove that $[-]_\eta$ is faithful, i.e. $\mathcal{M}_\eta \neq 0$ for any quasi-coherent $(G, \mathcal{O}_X)$-module $\mathcal{M} \neq 0$. A quasi-coherent $(G, \mathcal{O}_X)$-module is represented as an inductive limit of coherent $(G, \mathcal{O}_X)$-modules, so we may assume that $\mathcal{M} \neq 0$ is coherent. Then $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}))$ is coherent, and $\underline{\operatorname{Ann}}\,\mathcal{M} := \ker(\mathcal{O}_X \to$

$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{M}))$ is a coherent G-ideal, so $\mathrm{Supp}\,\mathcal{M}$ is a non-empty G-stable closed subscheme. Since Y is minimal, $\eta \in Y \subset \mathrm{Supp}\,\mathcal{M}$. Then $\mathcal{M}_\eta \neq 0$. ■

By the lemma, we can prove a G-analogue of Nakayama's Lemma.

Theorem 5 (G-Nakayama's lemma). *For a coherent $(G, \mathcal{O}_X)$-module $\mathcal{M}$, if $f^*\mathcal{M} = 0$ then $M = 0$.*

Proof. $\kappa(\eta) \otimes_{\mathcal{O}_{X,\eta}} \mathcal{M}_\eta = (f^*\mathcal{M})_\eta = 0$, so $M_\eta = 0$ by the usual Nakayama's lemma for the local ring $\mathcal{O}_{X,\eta}$. And $[-]_\eta$ is faithful, so $M = 0$. ■

By localization at η, we also have criteria for coherentness and length-finiteness of quasi-coherent $(G, \mathcal{O}_X)$-modules.

Proposition 6. (1) *For $\mathcal{M} \in \mathrm{Qch}(G, X)$, the following are equivalent :*
- (a) *$\mathcal{M}$ is a Noetherian object of $\mathrm{Qch}(G, X)$.*
- (b) *$\mathcal{M}_{[0]}$ is a coherent $\mathcal{O}_X$-module.*
- (c) *$\mathcal{M}$ is a coherent $(G, \mathcal{O}_X)$-module.*
- (d) *$\mathcal{M}_\eta$ is a Noetherian $\mathcal{O}_{X,\eta}$-module.*

(2) *For $\mathcal{M} \in \mathrm{Qch}(G, X)$, the following are equivalent :*
- (a) *$\mathcal{M}$ is of finite length in $\mathrm{Qch}(G, X)$.*
- (b) *$\mathcal{M}$ is a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{I}^n\mathcal{M} = 0$ for some n.*
- (c) *$\mathcal{M}_\eta$ is $\mathcal{O}_{X,\eta}$-module of finite length.*

Proof. (1) (a)⇔(b). Hashimoto [3], Lemma 12.8. (b)⇒(c)⇒(d) are trivial. (d)⇒(a). Since $[-]_\eta$ is faithful and exact, then an ascending chain $\mathcal{N}_0 \subset \mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots$ of $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$ is stable if and only if an ascending chain $[\mathcal{N}_0]_\eta \subset [\mathcal{N}_1]_\eta \subset [\mathcal{N}_2]_\eta \cdots$ of $\mathcal{O}_{X,\eta}$-submodules of $\mathcal{M}_\eta$ is stable.
(2) (a)⇒(b). $\mathcal{M}$ is a coherent by (1). A descending chain $\mathcal{M} \supset \mathcal{I}^1\mathcal{M} \supset \mathcal{I}^2\mathcal{M} \supset \cdots$ is stable by (a). If $\mathcal{I}^n\mathcal{M} = \mathcal{I}^{n+1}\mathcal{M}$, then $\mathcal{I}_\eta^n\mathcal{M}_\eta = \mathcal{I}_\eta^{n+1}\mathcal{M}_\eta$. So $\mathcal{I}_\eta^n\mathcal{M}_\eta = 0$ by usual Nakayama's lemma, and then $\mathcal{I}^n\mathcal{M} = 0$ by faithfulness of $[-]_\eta$. (b)⇒(c) is trivial. (c)⇒(a) is similar to (1) (d)⇒(a) for a descending chain of $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$. ■

3 G-dualizing complex

For a Noetherian G-scheme Z, a complex $\mathbb{F} \in D(\mathrm{Mod}(G, Z))$ is G-**dualizing** if $\mathbb{F}$ has equivariant cohomology sheaves and if $\mathbb{F}_{[0]} \in D(\mathrm{Mod}\,Z)$

is a dualizing complex of Z. Since Δ is a finite ordered category, $\mathbb{F}$ is G-dualizing if and only if $\mathbb{F}$ has finite injective dimension, has coherent cohomology sheaves, and the natural map $\mathcal{O}_{B_G^M(Z)} \to R\,\underline{\mathrm{Hom}}^\bullet(\mathbb{F},\mathbb{F})$ is a quasi-isomorphism, see [3] Lemma 31.6.

For example, if Z is Gorenstein of finite Krull dimension, then $\mathcal{O}_Z$ itself is a G-dualizing complex of Z.

From now on, assume that X has a fixed G-dualizing complex $\mathbb{I}$.

4 The local cohomology

Let $g : X \setminus Y \hookrightarrow X$ be the open immersion. $u : \mathrm{Id} \to g_* g^*$ denote the unit of the adjoint pair (g_*, g^*). Then we think a functor $\underline{\Gamma}_Y = \ker u : \mathrm{Mod}(G,X) \to \mathrm{Mod}(G,X)$.

The functor $\underline{\Gamma}_Y$ is a left exact functor preserving $\mathrm{Lqc}(G,X)$ and $\mathrm{Qch}(G,X)$, see [4] Lemma 3.2. For $\mathcal{M} \in \mathrm{Lqc}(G,X)$, $\underline{\Gamma}_Y(\mathcal{M})$ is computed as follows :

$$\underline{\Gamma}_Y(\mathcal{M}) = \varinjlim_n \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}),$$

see [4] Lemma 3.21.

And the derived functor $R\,\underline{\Gamma}_Y : D(\mathrm{Mod}(G,X)) \to D(\mathrm{Mod}(G,X))$ preserves $D_{\mathrm{Qch}}(\mathrm{Mod}(G,X))$, see [4] Lemma 4.11. For $\mathbb{M} \in D(\mathrm{Mod}(G,X))$, $R^i\,\underline{\Gamma}_Y(\mathbb{M})$ is denoted by $\underline{\mathrm{H}}^i_Y(\mathbb{M})$.

Lemma 7. *For a G-dualizing complex $\mathbb{F}$ of X, the local cohomology sheaves $\underline{\mathrm{H}}^i_Y(\mathbb{F})$ vanish except for only one i.*

Proof. Over a Noetherian scheme S, $A \in \mathrm{Qch}\,S$ is an injective object of $\mathrm{Mod}\,S$ if and only if it is an injective object of $\mathrm{Qch}\,S$. So we can assume that each term of a dualizing complex $\mathbb{F}_S$ of S is quasi-coherent and injective. As this, we can assume that $\mathbb{F}$ is a K-injective complex whose terms are locally quasi-coherent.

Then the following diagram commutes :

$$\begin{array}{ccc} X \setminus Z & \xrightarrow{\;g\;} & X \\ {\scriptstyle f'}\uparrow & & {\scriptstyle f}\uparrow \\ \mathrm{Spec}\,\mathcal{O}_{X,\eta} \setminus \{\eta\} & \xrightarrow{\;g'\;} & \mathrm{Spec}\,\mathcal{O}_{X,\eta} \end{array} .$$

We calculate the functor $f^*\,\underline{\Gamma}_Y = f^* \ker(Id \xrightarrow{u} g_* g^*)$ by the commutative diagram :

$$\begin{aligned} f^*\,\underline{\Gamma}_Z = f^* \ker(\mathrm{Id} \xrightarrow{u} g_* g^*) &\simeq \ker(f^* \longrightarrow f^* g_* g^*) \\ \xrightarrow{\;\phi\;}\ \ker(f^* \longrightarrow g'_* g'^* f^*) &\simeq \ker(\mathrm{Id} \longrightarrow g'_* g'^*) f^* = \Gamma_{\mathcal{I}_\eta} f^*. \end{aligned}$$

Each term of $\mathbb{F}$ is locally quasi-coherent, so ϕ is isomorphic. So it holds $[\underline{\Gamma}_Z(\mathbb{F})]_\eta \simeq \Gamma_{\mathcal{I}_\eta}(\mathbb{F}_\eta)$. By definition, $\mathbb{F}_\eta$ is a dualizing complex of $\mathcal{O}_{X,\eta}$.

In general, for a local ring $(A, \mathfrak{m})$, local cohomology groups $\mathrm{H}^i_{\mathfrak{m}}(\mathbb{F})$ of a dualizing complex $\mathbb{F}$ of A with support $\{\mathfrak{m}\}$ vanish except for only one i, see Hartshorne [2] V.6. The functor $[-]_\eta$ is faithful and exact, so cohomology $\underline{\mathrm{H}}^i_Y(\mathbb{F})$ vanish except for only one i. ■

Let $\mathbb{F}$ be a G-dualizing complex of X. If it holds $\underline{\mathrm{H}}^0_Y(\mathbb{F}) \neq 0$, a G-dualizing complex $\mathbb{F}$ is called G-normalized. Assume that our G-dualizing complex $\mathbb{I}$ is G-normalized.

Definition 8. For a G-normalized G-dualizing complex $\mathbb{I}$, the non-vanishing local cohomology $\underline{\mathrm{H}}^0_Y(\mathbb{I})$ with support Y is denoted by $\mathcal{E}_X$, and we call it a G-**sheaf of Matlis**.

For a local ring $(A, \mathfrak{m})$, the non-vanishing local cohomology group $\mathrm{H}^i_{\mathfrak{m}}(\mathbb{F})$ of a dualizing complex $\mathbb{F}$ of A with support $\{\mathfrak{m}\}$ is the injective envelope $E_A(A/\mathfrak{m})$ of the residue field $A/\mathfrak{m}$. So we get an isomprphism $[\mathcal{E}_X]_\eta \simeq E_{\mathcal{O}_{X,\eta}}(\kappa(\eta))$ where $\kappa(\eta)$ is the residue field of the local ring $\mathcal{O}_{X,\eta}$.

A G-sheaf of Matlis $\mathcal{E}_X$ corresponds to the injective envelope $E_A(A/\mathfrak{m})$ of the residue field $A/\mathfrak{m}$ for a local ring $(A, \mathfrak{m})$. But it is not necesarily an injective $(G, \mathcal{O}_X)$-module.

Example 9. Let k be a field of characteristic 2, $V = k^2$ and $G = \mathbb{GL}(V)$. Let $X = \operatorname{Spec} A$ where $A = \operatorname{Sym} V^*$. Then $\mathcal{E}_X$ is a $(G, \mathcal{O}_X)$-module which is defined by $A^\dagger$ ($A^\dagger$ denote the graded dual module of A). It is not injective as a G-module, so $\mathcal{E}_X$ is not injective in $\mathrm{Qch}(G, X)$.

Moreover, G-sheaf of Matlis $\mathcal{E}_X = \underline{\mathrm{H}}^0_Y(\mathbb{I})$ depends on G-normalized G-dualizing complex $\mathbb{I}$, so it is not necesarily unique.

5 Main theorems

Theorem 10 (G-Matlis duality). *Let T be the functor $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(-, \mathcal{E}_X)$: $\mathrm{Mod}(G, X) \to \mathrm{Mod}(G, X)$, $\mathcal{F}$ denote the category of $(G, \mathcal{O}_X)$-modules of finite length. Then the followings hold :*
(1) T is an exact functor on $\mathrm{Coh}(G, X)$.
(2) If $\mathcal{M} \in \mathcal{F}$, then $T\mathcal{M} \in \mathcal{F}$ and the canonical map $\mathcal{M} \to TT\mathcal{M}$ is an isomorphism.

So the functor $T : \mathcal{F} \to \mathcal{F}$ is an anti-equivalence.

Proof. (1) If $\mathcal{N} \in \mathrm{Coh}(G,X)$ then $\mathcal{N}_\eta$ is a finitely generated $\mathcal{O}_{X,\eta}$-module, see Lemma 6. So it holds

$$\big[\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{N},\mathcal{E}_X)\big]_\eta \simeq \mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathcal{N}_\eta,[\mathcal{E}_X]_\eta). \tag{$\sharp$}$$

$[\mathcal{E}_X]_\eta$ is an injective $\mathcal{O}_{X,\eta}$-module, so the functor $\mathrm{Hom}_{\mathcal{O}_{X,\eta}}([-]_\eta,[\mathcal{E}_{X,\eta}])$ is exact. Then $T = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(-,\mathcal{E}_X)$ is exact because $[-]_\eta$ is faithful and exact.

(2) By Lemma 6, $\mathcal{M}_\eta$ is an $\mathcal{O}_{X,\eta}$-module of finite length for $\mathcal{M} \in \mathcal{F}$. Because of the isomorphism ($\sharp$) and usual Matlis duality for the local ring $\mathcal{O}_{X,\eta}$, $[T\mathcal{M}]_\eta$ is an $\mathcal{O}_{X,\eta}$-module of finite length. By Lemma 6 again, $T\mathcal{M}$ is of finite length.

$\mathcal{M}$ and $T\mathcal{M}$ are both coherent, then

$$[TT\mathcal{M}]_\eta \simeq \mathrm{Hom}_{\mathcal{O}_{X,\eta}}\big(\mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathcal{M}_\eta,[\mathcal{E}_X]_\eta),[\mathcal{E}_X]_\eta\big).$$

By usual Matlis duality, it is isomorphic to $\mathcal{M}_\eta$. So it holds $TT\mathcal{M} \simeq \mathcal{M}$ because of faithfulness of $[-]_\eta$. ■

Finally, we state a G-analogue of local duality theorem.

Theorem 11 (G-local duality). *Let $\mathbb{E}$ be a bounded below complex in* $\mathrm{Mod}(G,X)$ *with coherent cohomology. Then there is an isomorphism in* $\mathrm{Qch}(G,X)$ *:*

$$\underline{\mathrm{H}}^i_Y(\mathbb{E}) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}\big(\underline{\mathrm{Ext}}^{-i}_{\mathcal{O}_X}(\mathbb{E},\mathbb{I}),\mathcal{E}_X\big).$$

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