Auslander-Reiten conjecture on Gorenstein rings [1]

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1. Introduction

The generalized Nakayama conjecture which was given by M. Auslander and I. Reiten is as follows [3] : Let Λ be an artin algebra. Any indecomposable injective Λ-module appears as a direct summand in the minimal injective resolution of Λ.

They showed that above conjecture holds for all artin algebras if and only if the following conjecture holds for all artin algebras.
Let Λ be an Artin algebra and M be a finitely generated Λ-module. If $\mathrm{Ext}^i_\Lambda(M, M \oplus \Lambda) = 0$ $(\forall i > 0)$, then M is projective.
M. Auslander, S. Ding, and Ø. Solberg widened the context to algebras over commutative local rings [2].

(ARC) Let R be a commutative Noetherian local ring and M be a finitely generated R-module. If $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ $(\forall i > 0)$, then M is free.

They showed in [2] that if R is a complete intersection, then R satisfies (ARC). We shall show the following main theorem.

Theorem 1. *Let R be a Gorenstein ring. If R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$ with $\mathrm{ht}\, p \le 1$, then R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$.*

2. Main Results

Through in this paper, we denote by R the d-dimensional commutative Gorenstein local ring with the unique maximal ideal $\mathfrak{m}$. We also denote by $\mathrm{mod}\, R$ the category of finitely generated R-modules and by $\mathrm{CM}\, R$ the full subcategory of $\mathrm{mod}\, R$ consisting of all maximal Cohen-Macaulay modules.

We give a following condition to consider the Auslander-Reiten conjecture.

(ARC) For $M \in \mathrm{mod}\, R$, suppose $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ $(i > 0)$, then M is free.

The main theorem of this paper is following;

Theorem 1. *If R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$ with $\mathrm{ht}\, p \le 1$, then R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$.*

It is difficult to check the freeness of modules in general. We give a following theorem to check the freeness of vector bundles.

Theorem 2. *We assume $\dim R = d \ge 2$. Let $M \in \mathrm{CM}\, R$ be a vector bundle. Suppose $\mathrm{Ext}^{d-1}_R(M, M) = 0$, then M is free.*

[1] The detailed version of this paper will be submitted for publication elsewhere.

We say M is a *vector bundle* if M_p is a free R_p-module for all prime ideal p which is not maximal ideal $\mathfrak{m}$. We want to omit the assumption M is a vector bundle in Theorem 2. But there is a counterexample if M is not a vector bundle.

Example 3. Let k be a field. We set $R = k[x, y, z]/(xy)$ be a 2-dimensional hypersurface and $M = R/(x)$. In this case, we can check that $\mathrm{Ext}^i_R(M, M) = 0$ if and only if i is odd. In particular, we see that $\mathrm{Ext}^{2-1}_R(M, M) = 0$ even if M is not free.

We prepare a lemma to show Theorem 2.

Lemma 4. [9, Lemma 3.10.] *Let R be a d-dimensional Cohen-Macaulay local ring and ω be a canonical module. We denote by $(-)^\vee$ the canonical dual $\mathrm{Hom}_R(-, \omega)$. For vector bundles M and $N \in \mathrm{CM}\, R$, we have a following isomorphism;*

$$\mathrm{Ext}^d_R(\underline{\mathrm{Hom}}(N, M), \omega) \cong \mathrm{Ext}^{d+1}_R(M, (\mathrm{tr}\, N)^\vee)$$

Here, $\underline{\mathrm{Hom}}(N, M)$ is the set of stable homomorphisms.

Proof of Theorem 2. Let $M \in \mathrm{CM}\, R$ be a vector bundle and we assume $\mathrm{Ext}^{d-1}_R(M, M) = 0$.

We take a minimal free resolution of M;

$$F_\bullet : \cdots \to F_1 \to F_0 \to M \to 0.$$

Apply $(-)^* := \mathrm{Hom}_R(-, R)$, we get exact sequence;

$$0 \to M^* \to F_0^* \to F_1^* \to \mathrm{tr}\, M \to 0.$$

Since R is Gorenstein and M is maximal Cohen-Macaulay, we have $\Omega^2 M \cong (\mathrm{tr}\, M)^* (\cong (\mathrm{tr}\, M)^\vee)$. Therefore, we have

$$\begin{aligned} \mathrm{Ext}^{d+1}_R(M, (\mathrm{tr}\, N)^\vee) &\cong \mathrm{Ext}^{d+1}_R(M, (\mathrm{tr}\, N)^*) \\ &\cong \mathrm{Ext}^{d+1}_R(M, \Omega^2 M) \\ &\cong \mathrm{Ext}^{d-1}_R(M, M) = 0. \end{aligned}$$

Since M is vector bundle,

$$\underline{\mathrm{Hom}}_R(M, M)_p \cong \underline{\mathrm{Hom}}_{R_p}(M_p, M_p) = 0 \ (\forall p \neq \mathfrak{m}).$$

Thus we have $\underline{\mathrm{Hom}}_R(M, M)$ has finite length and we have

$$\begin{aligned} \underline{\mathrm{Hom}}_R(M, M) &\cong \mathrm{Ext}^d_R(\mathrm{Ext}^d_R(\underline{\mathrm{Hom}}_R(M, M), R), R) \\ &\cong \mathrm{Ext}^d_R(\mathrm{Ext}^{d+1}_R(M, (\mathrm{tr}\, M)^\vee), R) = 0 \end{aligned}$$

Thus we get M is free. □

Proof of Theorem 1. We put $\mathfrak{P} := \{ p \in \mathrm{Spec} R \mid R_p \text{ does not satisfy (ARC)} \}$ and assume $\mathfrak{P} \neq \phi$. Let q be a minimal element in $\mathfrak{P}$ and replace R with R_q. By the minimalty, R is a $d(\geq 2)$-dimrnsional Gorenstein local ring which does not satisfy (ARC) but R_p satisfy (ARC) for all prime $p \neq \mathfrak{m}$. There exists $M \in \mathrm{mod}\, R$ s.t. $\mathrm{Ext}^i_R(M, M \oplus R) = 0 \ (\forall i > 0)$ but M is not free. Since $\mathrm{Ext}^i_R(M, R) = 0 \ (i > 0)$, M is maximal Cohen-Macaulay. For any $p \neq \mathfrak{m}$, $\mathrm{Ext}^i_{R_p}(M_p, M_p \oplus R_p) = 0 \ (\forall i > 0)$ and R_p satisfies (ARC), we have M_p is a free

R_p-module. Thus we get M is vector bundle. Furthermore, $\mathrm{Ext}_R^{d-1}(M, M) = 0$ implies M is free. ($\because$Theorem 2.) Therefore we get contradiction and we have $\mathfrak{P} = \phi$. $\square$

Finally, we remark that normal domain satisfies Serre's (R_1)-condition and regular local ring satisfies (ARC), we get the following as a corollay of Theorem 1.

Corollary 5. *Gorenstein normal domain satisfies (ARC).*

References

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