Symbolic Rees rings of space monomial curves in characteristic p and existence of negative curves in characteristic 0

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This is a joint work with Naoyuki Matsuoka from Meiji University.

We refer the reader to [7] for detail and proofs.

Our aim is to study finite generation of symbolic Rees rings of the defining ideal of the space monomial curves (t^a, t^b, t^c) for pairwise coprime integers a, b, c such that $(a, b, c) \neq (1, 1, 1)$. If such a ring is not finitely generated over a base field, then it is a counterexample to the Hilbert's fourteenth problem. Finite generation of such rings is deeply related to existence of negative curves on certain normal projective surfaces. We prove that, in the case of $(a + b + c)^2 > abc$, a negative curve exists. Using a computer, we shall show that a negative curve exists if all of a, b, c are at most 300. As a corollary, the symbolic Rees rings of space monomial curves are shown to be finitely generated if a base field is of positive characteristic and all of a, b, c are less than or equal to 300.

1 Symbolic Rees rings of monomial curves and Hilbert's fourteenth problem

Throughout of this note, we assume that rings are commutative with unit.

For a prime ideal P of a ring A, $P^{(r)}$ denotes the r-th symbolic power of P, i.e.,

$$P^{(r)} = P^r A_P \cap A.$$

By definition, it is easily seen that $P^{(r)} P^{(r')} \subset P^{(r+r')}$ for any $r,\ r' \geq 0$, therefore,

$$\bigoplus_{r \geq 0} P^{(r)} T^r$$

is a subring of the polynomial ring $A[T]$. This subring is called the *symbolic Rees ring* of P, and denoted by $R_s(P)$.

Let k be a field and m be a positive integer. Let $a_1, \ldots, a_m$ be positive integers. Consider the k-algebra homomorphism

$$\phi_k : k[x_1, \ldots, x_m] \longrightarrow k[t]$$

given by $\phi_k(x_i) = t^{a_i}$ for $i = 1, \ldots, m$, where $x_1, \ldots, x_m, t$ are indeterminates over k. Let $\mathfrak{p}_k(a_1, \ldots, a_m)$ be the kernel of ϕ_k. We sometimes denote $\mathfrak{p}_k(a_1, \ldots, a_m)$ simply by $\mathfrak{p}$ or $\mathfrak{p}_k$ if no confusion is possible.

Theorem 1.1 *Let k be a field and m be a positive integer. Let $a_1, \ldots, a_m$ be positive integers. Consider the prime ideal $\mathfrak{p}_k(a_1, \ldots, a_m)$ of the polynomial ring $k[x_1, \ldots, x_m]$.*

Let $\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t, T$ be indeterminates over k. Consider the following injective k-homomorphism

$$\xi : k[x_1, \ldots, x_m, T] \longrightarrow k(\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t)$$

given by $\xi(T) = \alpha_2/\alpha_1$ and $\xi(x_i) = \alpha_1\beta_i + t^{a_i}$ for $i = 1, \ldots, m$.

Then,

$$k(\alpha_1\beta_1+t^{a_1}, \alpha_1\beta_2+t^{a_2}, \ldots, \alpha_1\beta_m+t^{a_m}, \alpha_2/\alpha_1) \cap k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] = \xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m)))$$

holds true.

Remark 1.2 Let k be a field. Let R be a polynomial ring over k with finitely many variables. For a field L satisfying $k \subset L \subset Q(R)$, Hilbert asked in 1900 whether the ring $L \cap R$ is finitely generated as a k-algebra or not. It is called the *Hilbert's fourteenth problem.*

The first counterexample to this problem was discovered by Nagata [10] in 1958. An easier counterexample was found by Paul C. Roberts [11] in 1990. Further counterexamples were given by Kuroda, Mukai, etc.

On the other hand, Goto, Nishida and Watanabe [2] proved that $R_s(\mathfrak{p}_k(7n-3, (5n-2)n, 8n-3))$ is not finitely generated over k if the characteristic of k is zero, $n \geq 4$ and $n \not\equiv 0$ (3). By Theorem 1.1, we know that they are new counterexamples to the Hilbert's fourteenth problem.

Remark 1.3 With notation as in Theorem 1.1, we set

$$\begin{aligned} D_1 &= \alpha_1\frac{\partial}{\partial\alpha_1} + \alpha_2\frac{\partial}{\partial\alpha_2} - \beta_1\frac{\partial}{\partial\beta_1} - \cdots - \beta_m\frac{\partial}{\partial\beta_m} \\ D_2 &= a_1t^{a_1-1}\frac{\partial}{\partial\beta_1} + \cdots + a_mt^{a_m-1}\frac{\partial}{\partial\beta_m} - \alpha_1\frac{\partial}{\partial t}. \end{aligned}$$

Assume that the characteristic of k is zero.

Then, one can prove that $\xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m)))$ is equal to the kernel of the derivations D_1 and D_2, i.e.,

$$\xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m))) = \{f \in k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] \mid D_1(f) = D_2(f) = 0\}.$$

2 Symbolic Rees rings of space monomial curves

In the rest of this paper, we restrict ourselves to the case $m = 3$. For the simplicity of notation, we write x, y, z, a, b, c for x_1, x_2, x_3, a_1, a_2, a_3, respectively. We regard the polynomial ring $k[x, y, z]$ as a $\mathbb{Z}$-graded ring by $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$.

$\mathfrak{p}_k(a, b, c)$ is the kernel of the k-algebra homomorphism

$$\phi_k : k[x, y, z] \longrightarrow k[t]$$

given by $\phi_k(x) = t^a$, $\phi_k(y) = t^b$, $\phi_k(z) = t^c$.

By a result of Herzog [3], we know that $\mathfrak{p}_k(a, b, c)$ is generated by at most three elements.

We are interested in the symbolic powers of $\mathfrak{p}_k(a, b, c)$. If $\mathfrak{p}_k(a, b, c)$ is generated by two elements, then the symbolic powers always coincide with ordinary powers because $\mathfrak{p}_k(a, b, c)$ is a complete intersection. However, it is known that, if $\mathfrak{p}_k(a, b, c)$ is minimally generated by three elements, the second symbolic power is strictly bigger than the second ordinary power.

We are interested in finite generation of the symbolic Rees ring $R_s(\mathfrak{p}_k(a, b, c))$. It is known that this problem is reduced to the case where a, b and c are pairwise coprime, i.e.,

$$(a, b) = (b, c) = (c, a) = 1.$$

In the rest of this paper, we always assume that a, b and c are pairwise coprime.

Let $\mathbb{P}_k(a, b, c)$ be the weighted projective space $\mathrm{Proj}(k[x, y, z])$. Then

$$\mathbb{P}_k(a, b, c) \setminus \{V_+(x, y), \quad V_+(y, z), \quad V_+(z, x)\}$$

is a regular scheme. In particular, $\mathbb{P}_k(a, b, c)$ is smooth at the point $V_+(\mathfrak{p}_k(a, b, c))$. Let $\pi : X_k(a, b, c) \to \mathbb{P}_k(a, b, c)$ be the blow-up at $V_+(\mathfrak{p}_k(a, b, c))$. Let E be the exceptional divisor, i.e.,

$$E = \pi^{-1}(V_+(\mathfrak{p}_k(a, b, c))).$$

We sometimes denote $\mathfrak{p}_k(a, b, c)$ (resp. $\mathbb{P}_k(a, b, c)$, $X_k(a, b, c)$) simply by $\mathfrak{p}$ or $\mathfrak{p}_k$ (resp. $\mathbb{P}$ or $\mathbb{P}_k$, X or X_k) if no confusion is possible.

It is easy to see that

$$\mathrm{Cl}(\mathbb{P}) = \mathbb{Z}H \simeq \mathbb{Z},$$

where H is the Weil divisor corresponding to the reflexive sheaf $\mathcal{O}_{\mathbb{P}}(1)$. Set $H = \sum_i m_i D_i$, where D_i's are subvarieties of $\mathbb{P}$ of codimension one. We may choose D_i's such that $D_i \not\ni V_+(\mathfrak{p})$ for any i. Then, set $A = \sum_i m_i \pi^{-1}(D_i)$.

One can prove that

$$\mathrm{Cl}(X) = \mathbb{Z}A + \mathbb{Z}E \simeq \mathbb{Z}^2.$$

Since all Weil divisor on X are $\mathbb{Q}$-Cartier, we have the intersection pairing

$$\mathrm{Cl}(X) \times \mathrm{Cl}(X) \longrightarrow \mathbb{Q},$$

that satisfies

$$A^2 = \frac{1}{abc}, \quad E^2 = -1, \quad A.E = 0.$$

Here, we have the following natural identification:

$$H^0(X, \mathcal{O}_X(nA - rE)) = \begin{cases} \left[\mathfrak{p}^{(r)}\right]_n & (r \geq 0) \\ S_n & (r < 0) \end{cases}$$

Therefore, the *total coordinate ring* (or *Cox ring*)

$$TC(X) = \bigoplus_{n, r \in \mathbb{Z}} H^0(X, \mathcal{O}_X(nA - rE))$$

is isomorphic to the extended symbolic Rees ring

$$R_s(\mathfrak{p})[T^{-1}] = \cdots \oplus ST^{-2} \oplus ST^{-1} \oplus S \oplus \mathfrak{p}T \oplus \mathfrak{p}^{(2)}T^2 \oplus \cdots.$$

It is well-known that $R_s(\mathfrak{p})[T^{-1}]$ is Noetherian if and only if so is $R_s(\mathfrak{p})$.

Remark 2.1 By Huneke's criterion [5] and a result of Cutkosky [1], the following four conditions are equivalent:

(1) $R_s(\mathfrak{p})$ is a Noetherian ring, or equivalently, finitely generated over k.

(2) $TC(X)$ is a Noetherian ring, or equivalently, finitely generated over k.

(3) There exist positive integers r, s, $f \in \mathfrak{p}^{(r)}$, $g \in \mathfrak{p}^{(s)}$, and $h \in (x, y, z) \setminus \mathfrak{p}$ such that

$$\ell_{S_{(x,y,z)}}(S_{(x,y,z)}/(f, g, h)) = rs \cdot \ell_{S_{(x,y,z)}}(S_{(x,y,z)}/(\mathfrak{p}, h)),$$

where $\ell_{S_{(x,y,z)}}$ is the length as an $S_{(x,y,z)}$-module.

(4) There exist curves C and D on X such that

$$C \neq D, \quad C \neq E, \quad D \neq E, \quad C.D = 0.$$

Here, a curve means a closed irreducible reduced subvariety of dimension one.

The condition (4) as above is equivalent to that just one of the following two conditions is satisfied:

(4-1) There exist curves C and D on X such that

$$C \neq E, \quad D \neq E, \quad C^2 < 0, \quad D^2 > 0, \quad C.D = 0.$$

(4-2) There exist curves C and D on X such that

$$C \neq E, \quad D \neq E, \quad C \neq D, \quad C^2 = D^2 = 0.$$

Definition 2.2 A curve C on X is called a *negative curve* if

$$C \neq E \text{ and } C^2 < 0.$$

It is proved that two distinct negative curves never exist.

In the case where the characteristic of k is positive, Cutokosky [1] proved that $R_s(\mathfrak{p})$ is finitely generated over k if there exists a negative curve on X.

We remark that there exists a negative curve on X if and only if there exists positive integers n and r such that

$$\frac{n}{r} < \sqrt{abc} \text{ and } [\mathfrak{p}^{(r)}]_n \neq 0.$$

We are interested in existence of a negative curve. Let a, b and c be pairwise coprime positive integers. By the following lemma, if there exists a negative curve on $X_{k_0}(a,b,c)$ for a field k_0 of characteristic 0, then there exists a negative curve on $X_k(a,b,c)$ for any field k.

Lemma 2.3 *Let a, b and c be pairwise coprime positive integers.*

1. *Let K/k be a field extension. Then, for any integers n and r,*

$$[\mathfrak{p}_k(a,b,c)^{(r)}]_n \otimes_k K = [\mathfrak{p}_K(a,b,c)^{(r)}]_n.$$

2. *For any integers n, r and any prime number p,*

$$\dim_{\mathbb{F}_p}[\mathfrak{p}_{\mathbb{F}_p}(a,b,c)^{(r)}]_n \geq \dim_{\mathbb{Q}}[\mathfrak{p}_{\mathbb{Q}}(a,b,c)^{(r)}]_n$$

holds, where $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{F}_p$ is the prime field of characteristic p. Here, $\dim_{\mathbb{F}_p}$ (resp. $\dim_{\mathbb{Q}}$) denotes the dimension as an $\mathbb{F}_p$-vector space (resp. $\mathbb{Q}$-vector space).

Remark 2.4 Let a, b, c be pairwise coprime positive integers. Assume that there exists a negative curve on $X_{k_0}(a,b,c)$ for a field k_0 of characteristic zero.

By Lemma 2.3, we know that there exists a negative curve on $X_k(a,b,c)$ for any field k. Therefore, if k is a field of characteristic positive, then the symbolic Rees ring $R_s(\mathfrak{p}_k)$ is finitely generated over k by a result of Cutkosky [1]. However, if k is a field of characteristic zero, then $R_s(\mathfrak{p}_k)$ is not necessary Noetherian. In fact, assume that k is of characteristic zero and $(a,b,c) = (7n-3, (5n-2)n, 8n-3)$ with $n \not\equiv 0$ (3) and $n \geq 4$ as in Goto-Nishida-Watanabe [2]. Then there exists a negative curve, but $R_s(\mathfrak{p}_k)$ is not Noetherian.

Definition 2.5 Let a, b, c be pairwise coprime positive integers. Let k be a field.

We define the following three conditions:

(C1) There exists a negative curve on $X_k(a,b,c)$, i.e., $[\mathfrak{p}_k(a,b,c)^{(r)}]_n \neq 0$ for some positive integers n, r satisfying $n/r < \sqrt{abc}$.

(C2) There exist positive integers n, r satisfying $n/r < \sqrt{abc}$ and $\dim_k S_n > r(r+1)/2$.

(C3) There exist positive integers q, r satisfying $abcq/r < \sqrt{abc}$ and $\dim_k S_{abcq} > r(r+1)/2$.

Here, $\dim_k$ denotes the dimension as a k-vector space.

By the following lemma, we know the implications

$$(C3) \Longrightarrow (C2) \Longrightarrow (C1)$$

since $\dim_k[\mathfrak{p}^{(r)}]_n = \dim_k S_n - \dim_k[S/\mathfrak{p}^{(r)}]_n$.

Lemma 2.6 *Let a, b, c be pairwise coprime positive integers. Let r and n be non-negative integers. Then,*

$$\dim_k[S/\mathfrak{p}^{(r)}]_n \leq r(r+1)/2$$

holds true for any field k.

Remark 2.7 It is easy to see that $[\mathfrak{p}_k(a,b,c)]_n \neq 0$ if and only if $\dim_k S_n \geq 2$. Therefore, if we restrict ourselves to $r = 1$, then (C1) and (C2) are equivalent.

On can prove that, if (C1) is satisfied with $r \leq 2$ for a field k of characteristic zero, then (C2) is satisfied.

Assume that k is a field of characteristic zero. Let a, b and c be pairwise coprime integers such that $1 \leq a, b, c \leq 300$. As we shall see in Theorem 5.1, a negative curve exists unless $(a,b,c) = (1,1,1)$. In these cases, calculations by a computer show that (C2) is satisfied if (C1) holds with $r \leq 5$.

We shall discuss the difference between (C1) and (C2) in Section 5.1.

Remark 2.8 Let a, b and c be pairwise coprime positive integers. Assume that $\mathfrak{p}_k(a,b,c)$ is a complete intersection, i.e., generated by two elements.

Permuting a, b and c, we may assume that

$$\mathfrak{p}_k(a,b,c) = (x^b - y^a,\ z - x^\alpha y^\beta)$$

for some $\alpha, \beta \geq 0$ satisfying $\alpha a + \beta b = c$. If $ab < c$, then

$$\deg(x^b - y^a) = ab < \sqrt{abc}.$$

If $ab > c$, then

$$\deg(z - x^\alpha y^\beta) = c < \sqrt{abc}.$$

If $ab = c$, then (a,b,c) must be equal to $(1,1,1)$. Ultimately, there exists a negative curve if $(a,b,c) \neq (1,1,1)$.

3 The case where $(a+b+c)^2 > abc$

In the rest of this paper, we set $\xi = abc$ and $\eta = a+b+c$ for pairwise coprime positive integers a, b and c.

For $v = 0, 1, \ldots, \xi - 1$, we set

$$S^{(\xi,v)} = \oplus_{q \geq 0} S_{\xi q + v}.$$

This is a module over $S^{(\xi)} = \oplus_{q \geq 0} S_{\xi q}$.

Lemma 3.1

$$\dim_k [S^{(\xi,v)}]_q = \dim_k S_{\xi q + v} = \frac{1}{2}\left\{\xi q^2 + (\eta + 2v)q + 2\dim_k S_v\right\}$$

holds for any $q \geq 0$.

Lemma 3.2 Assume that a, b and c are pairwise coprime positive integers such that $(a,b,c) \neq (1,1,1)$. Then, $\eta - \sqrt{\xi} \neq 0, 1, 2$.

Theorem 3.3 *Let* a, b *and* c *be pairwise coprime integers such that* $(a,b,c) \neq (1,1,1)$. *Then, we have the following:*

1. *Assume that* $\sqrt{abc} \notin \mathbb{Z}$. *Then, (C3) holds if and only if* $(a+b+c)^2 > abc$.
2. *Assume that* $\sqrt{abc} \in \mathbb{Z}$. *Then, (C3) holds if and only if* $(a+b+c)^2 > 9abc$.
3. *If* $(a+b+c)^2 > abc$, *then, (C2) holds. In particular, a negative curve exists in this case.*

Remark 3.4 If $(a+b+c)^2 > abc$, then $R_s(\mathfrak{p})$ is Noetherian by a result of Cutkosky [1].

If $(a+b+c)^2 > abc$ and $\sqrt{abc} \notin \mathbb{Q}$, then the existence of a negative curve follows from Nakai's criterion for ampleness, Kleimann's theorem and the cone theorem (e.g. Theorem 1.2.23 and Theorem 1.4.23 in [8], Theorem 4-2-1 in [6]).

The condition $(a+b+c)^2 > abc$ is equivalent to $(-K_X)^2 > 0$. If $-K_X$ is ample, then the finite generation of the total coordinate ring follows from Proposition 2.9 and Corollary 2.16 in Hu-Keel [4].

If $(a,b,c) = (5,6,7)$, then the negative curve C is the proper transform of the curve defined by $y^2 - zx$. Therefore, C is linearly equivalent to $12A - E$. Since $(a+b+c)^2 > abc$, $(-K_X)^2 > 0$. Since

$$-K_X.C = (18A - E).(12A - E) = 0.028 \cdots > 0,$$

$-K_X$ is ample by Nakai's criterion.

If $(a,b,c) = (7,8,9)$, then the negative curve C is the proper transform of the curve defined by $y^2 - zx$. Therefore, C is linearly equivalent to $16A - E$. Since $(a+b+c)^2 > abc$, $(-K_X)^2 > 0$. Since

$$-K_X.C = (24A - E).(16A - E) = -0.23\cdots < 0,$$

$-K_X$ is not ample by Nakai's criterion.

4 Degree of a negative curve

Proposition 4.1 *Let a, b and c be pairwise coprime integers, and k be a field of characteristic zero. Suppose that a negative curve exists, i.e., there exist positive integers n and r satisfying $[\mathfrak{p}_k(a,b,c)^{(r)}]_n \neq 0$ and $n/r < \sqrt{abc}$.*

Set n_0 and r_0 to be

$$\begin{aligned} n_0 &= \min\{n \in \mathbb{N} \mid \exists r > 0 \text{ such that } n/r < \sqrt{\xi} \text{ and } [\mathfrak{p}^{(r)}]_n \neq 0\} \\ r_0 &= \lfloor \frac{n}{\sqrt{\xi}} \rfloor + 1, \end{aligned}$$

where $\lfloor \frac{n}{\sqrt{\xi}} \rfloor$ is the maximum integer which is less than or equal to $\frac{n}{\sqrt{\xi}}$.

Then, the negative curve C is linearly equivalent to $n_0 A - r_0 E$.

Remark 4.2 Let a, b and c be pairwise coprime integers, and k be a field of characteristic zero. Assume that the negative curve C exists, and C is linearly equivalent to $n_0 A - r_0 E$.

Then, by Proposition 4.1, we obtain

$$\begin{aligned} n_0 &= \min\{n \in \mathbb{N} \mid [\mathfrak{p}^{(\lfloor \frac{n}{\sqrt{\xi}} \rfloor + 1)}]_n \neq 0\} \\ r_0 &= \lfloor \frac{n_0}{\sqrt{\xi}} \rfloor + 1. \end{aligned}$$

Proposition 4.3 *Let a, b and c be pairwise coprime positive integers such that $\sqrt{\xi} > \eta$. Assume that (C2) is satisfied, i.e., there exist positive integers n_1 and r_1 such that $n_1/r_1 < \sqrt{\xi}$ and $\dim_k S_{n_1} > r_1(r_1+1)/2$. Suppose $n_1 = \xi q_1 + v_1$, where q_1 and v_1 are integers such that $0 \leq v_1 < \xi$.*

Then, $q_1 < \frac{2 \dim_k S_{v_1}}{\sqrt{\xi} - \eta}$ holds.

In particular,

$$n_1 = \xi q_1 + v_1 < \frac{2\xi \max\{\dim_k S_t \mid 0 \leq t < \xi\}}{\sqrt{\xi} - \eta} + \xi.$$

5 Calculation by computer

5.1 Examples that do not satisfy (C2)

Suppose that (C2) is satisfied, i.e., there exist positive integers n_1 and r_1 such that $n_1/r_1 < \sqrt{\xi}$ and $\dim_k S_{n_1} > r_1(r_1+1)/2$. Put $n_1 = \xi q_1 + v_1$, where q_1 and v_1 are integers such that $0 \le v_1 < \xi$. If $\sqrt{\xi} > \eta$, then $q_1 < \frac{2\dim_k S_{v_1}}{\sqrt{\xi}-\eta}$ holds by Proposition 4.3.

By a following programming on MATHEMATICA ([7]), we can check whether (C2) is satisfied or not in the case where $\sqrt{\xi} > \eta$.

Calculations by a computer show that (C2) is not satisfied in some cases, for example, $(a,b,c) = (9,10,13), (13,14,17)$.

The examples due to Goto-Nishida-Watanabe [2] have negative curves with $r = 1$. Therefore, by Remark 2.7, they satisfy the condition (C2).

In the case where $(a,b,c) = (9,10,13), (13,14,17)$, the authors do not know whether $R_s(\mathfrak{p}_k)$ is Noetherian or not in the case where the characteristic of k is zero, however the negative curve do exists as in Theorem 5.1 below.

If we input $(a,b,c) = (5,26,43)$, then the output is $(n,r) = (1196,16)$. Therefore, (C2) is satisfied with $(n,r) = (1196,16)$. However, the negative curve on $X_{\mathbb{C}}(5,26,43)$ is linearly equivalent to $515A - 7E$ by a calculation in the next subsection.

5.2 Existence of a negative curve

Theorem 5.1 *Let a, b and c be pairwise coprime positive integers such that $(a,b,c) \neq (1,1,1)$. Assume that the characteristic of k is zero.*

If all of a, b and c are at most 300, then there exists a negative curve on X.

Let a, b and c be pairwise coprime positive integers such that $(a,b,c) \neq (1,1,1)$ and $1 \le a,b,c \le 300$. Then, by Theorem 5.1, there exists a negative curve in the case where k is of characteristic zero. Then, by Remark 2.4, $R_s(\mathfrak{p}_k(a,b,c))$ is Noetherian in the case where k is of positive characteristic. Thus, we obtain the following corollary immediately.

Corollary 5.2 *Let a, b and c be pairwise coprime positive integers such that all of a, b and c are at most 300. Assume that the characteristic of k is positive.*

Then the symbolic Rees ring $R_s(\mathfrak{p}_k(a,b,c))$ is Noetherian.

Remark 5.3 Assume that the characteristic of k is zero. Let a, b and c be pairwise coprime positive integers such that $a+b+c < \sqrt{abc}$, $(a,b,c) \neq (1,1,1)$ and $1 \le a \le b \le c \le 300$.

More than 90% in these cases satisfy (C2).

Using this program, it is possible to know n_0 and r_0 such that the negative curve is linearly equivalent to $n_0A - r_0E$ (cf. Remark 4.2).

Calculations show the following.
The maximal value of r_0 is nine.
In the case where $r_0 \leq 5$, (C2) is satisfied, i.e.,

$$\dim_k S_{n_0} > r_0(r_0+1)/2.$$

Suppose $(a,b,c) = (9,10,13)$. In the case where the characteristic of k is zero, the negative curve is linearly equivalent to $305A - 9E$. We know that the negative curve is also linearly equivalent to $305A - 9E$ if the characteristic of k is sufficiently large. On the other hand, the negative curve is linearly equivalent to $100A - 3E$ if the characteristic of k is three as in Morimoto-Goto [9]. Therefore, the linear equivalent class that contains the negative curve depends on the characteristic of a base field. Assume that the characteristic is a sufficiently large prime number. Let C be the negative curve and D be a curve that satisfies (4-1) in Remark 2.1. Suppose that D is linearly equivalent to $n_1A - r_1E$. Since $C.D = 0$, we know

$$\frac{n_1}{r_1} = \frac{9^2 \cdot 10 \cdot 13}{305}.$$

Therefore, r must be a multiple of 305.

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