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Koszul algebras and Gröbner bases of quadrics

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Abstract: We present results that appear in the papers [C, CTV, CRV] joint with M.E.Rossi, N.V.Trung and G.Valla and also some new results contained in [C1]. These results concern Koszul and G-quadratic properties of algebras associated with points, curves, cubics and spaces of quadrics of low codimension.

1. Introduction

Let R be a standard graded K-algebra, that is, an algebra of the form $R = K[x_1, \dots, x_n]/I$ where $K[x_1, \dots, x_n]$ is a polynomial ring over the field K and I is a homogeneous ideal with respect to the grading $\deg(x_i) = 1$. Let M be a finitely generated graded R-module. Consider the (essentially unique) minimal graded R-free resolution of M

$$\cdots \to R^{\beta_i} \to R^{\beta_{i-1}} \to \cdots \to R^{\beta_1} \to R^{\beta_0} \to M \to 0$$

The rank β_i of the i-th module in the minimal free resolution of M is called the i-th *Betti number* of M. One can also keep track of the graded structure of the resolution. It follows that the free modules in the resolution are indeed direct sums of "shifted" copies of R:

$$R^{\beta_i} = \oplus_j R(-j)^{\beta_{ij}}$$

where $R(-a)$ denotes the free module with the generator in degree a and that the matrices representing the maps between free modules have homogeneous entries. The number β_{ij} is called the (i, j)-*th graded Betti number* of M. The resolution is finite if $\beta_i = 0$ for $i \gg 0$.

For which algebras R does every module M has a finite minimal free resolution? The answer is given by (the graded version of) the Auslander-Buchsbaum-Serre theorem:

Theorem 1.1. *(Auslander-Buchsbaum-Serre) Let R be a standard graded K-algebra. The following are equivalent:*

1. *Every finitely generated graded R-module M has a finite minimal free resolution as an R-module.*
2. *The field K, regarded as an R-module via the identification $K = R/\oplus_{i>0} R_i$, has a finite minimal free resolution as an R-module.*
3. *R is regular, i.e. R is (isomorphic to) a polynomial ring.*

If R is not regular then the resolution of K is infinite. The *Poincaré series* $P_R(z)$ of R is the formal power series whose coefficients are the Betti numbers of K, i.e.

$$P_R(z) = \sum_{i \geq 0} \beta_i^R(K) z^i$$

where $\beta_i^R(K)$ is the i-th Betti number of K as an R-module.

Serre asked in [S] whether the Poincaré series $P_R(z)$ is a rational series, that is whether there exist polynomials $a(z), b(z) \in \mathbf{Q}[z]$ such that $P_R(z) = a(z)/b(z)$. The positive answer to Serre's question became well-known under the name of Serre's conjecture. This conjecture has been proved for several classes of algebras. For instance it holds for complete intersections (Tate, Assmus) and for algebras defined by monomials (Backelin). But in 1981 Anick [A] discovered algebras with irrational Poincaré series, such as:

$$\mathbf{Q}[x_1, x_2, \ldots, x_5]/(x_1^2, x_2^2, x_4^2, x_5^2, x_1x_2, x_4x_5, x_1x_3 + x_3x_4 + x_2x_5) + m^3$$

More recently Roos and Sturmfels have shown that irrational Poincaré series arise also in the realm of toric rings, see[RS].

The Poincaré series of R takes into account the rank of the free modules in the minimal free resolution of K. One can also consider the degrees of the generators of the free modules. This leads to the introduction of estimates for the growth of the degrees of the syzygies (like for instance Backelin's rate) and to the definition of Koszul algebras:

Definition 1.2. (Priddy) A standard graded K-algebra R is Koszul if for all i the generators of the i-th free module in the minimal free resolution of K have degree i. Equivalently, R is Koszul if the entries of matrices representing the maps in the minimal free resolution of K are homogeneous of degree 1.

Example 1.3. Let $R = K[x]/(x^n)$ with $n > 1$. Then the resolution of K as an R-module is

$$\ldots \xrightarrow{x^{n-1}} R \xrightarrow{x} R \xrightarrow{x^{n-1}} R \xrightarrow{x} R \to K \to 0$$

Hence R is Koszul iff $n = 2$.

The algebra R is said to be:

(1) quadratic if its defining ideal I is generated by quadrics (i.e. homogeneous elements of degree 2).
(2) G-quadratic if I has a Gröbner basis of quadrics with respect to some system of coordinates and some term order.
(3) LG-quadratic (the L stands for lifting) if there exist a G-quadratic algebra S and a S-regular sequence $y_1, \ldots, y_s$ of elements of degree 1 in S such that $S/(y_1, \ldots, y_s) \simeq R$.

One has:

$$(2) \Rightarrow (3) \Rightarrow \text{ Koszul } \Rightarrow (1)$$

Implications (2) $\Rightarrow$ (3) and Koszul $\Rightarrow$ (1) cannot be reversed in general. We do not know examples of Koszul algebras which are not LG-quadratic.

By a theorem of Tate (see [F]) every quadratic complete intersection is Koszul, but not all of them are G-quadratic. Non-G-quadratic complete intersection of quadrics are given in [ERT]. The easiest example of non-G-quadratic and quadratic complete intersection is given by 3 general quadrics in 3 variables. But every complete intersection of quadrics is LG-quadratic as the following argument of G.Caviglia shows.

Example 1.4. If $R = K[x_1, \ldots, x_n]/(q_1, \ldots, q_m)$ is a complete intersection of quadrics then $R = S/(y_1, \ldots, y_m)$ where

$$S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]/(y_1^2 + q_1, \ldots, y_m^2 + q_m).$$

That $y_1^2+q_1,\dots,y_m^2+q_m$ is a Gröbner basis is an easy consequence of Buchberger criterion. That $y_1,\dots,y_m$ form a S-regular sequence follows by an Hilbert function computation.

The Koszul property can be characterized in terms of the Poincaré series. Denote by $H_R(z)$ the Hilbert series of R. Then one has:

$$R \text{ is Koszul } \Leftrightarrow P_R(z)H_R(-z)=1$$

In particular, Koszul algebras have rational Poincaré series.

2. Filtrations, points and curves

Given an algebra R it can be very difficult to detect whether R is Koszul or not. One can compute the first few matrices in the resolution and check whether they are linear. If they are not, then R is not Koszul. If instead they are linear, one can then compute few more matrices. But the growth of the size of the matrices (i.e. the growth of the Betti numbers) is in general very fast. And it is known that the first non-linear syzygy can appear in arbitrarily high homological degree even for algebras with a given Hilbert function.

A quite efficient method to prove that an algebra is Koszul is that given by filtration arguments of various kinds. These notions have been used by various authors. Inspired by the work of Eisenbud, Reeves and Totaro [ERT], Bruns, Herzog and Vetter [BHV] and of Herzog, Hibi and Restuccia [HHR], we have defined:

Definition 2.1. Let R be a standard graded algebra and let F be a family of ideals of R. Then F is said to be a Koszul filtration of R if the following conditions hold:

(1) Every ideal in F is generated by linear forms,
(2) The ideal (0) and the maximal homogeneous ideal $\oplus_{i>0}R_i$ are in F,
(3) For every non-zero I in F there exists J in F such that $J\subset I$, I/J is cyclic and $J:I$ is also in F.

Definition 2.2. Let R be a standard graded algebra. A Gröbner flag of R is a Koszul filtration F of R which consists of a single complete flag. In other words, a Gröbner flag is a set of ideals $F=\{(0),(V_1),(V_2),\dots,(V_n)=(R_1)\}$ where V_i is a i-dimensional subspace of R_1, $V_i\subset V_{i+1}$ and $(V_i):(V_{i+1})=(V_j)$ for some j depending on i.

As the names suggest, we have:

Theorem 2.3. (1) *Let F be a Koszul filtration of R. Then $\operatorname{Tor}_i^R(R/I,K)_j=0$ for all $i\neq j$ and for all $I\in F$. In particular, R is Koszul.*
(2) *If R has a Gröbner flag then R is G-quadratic.*

Example 2.4. Let $R=K[x_1,\dots,x_n]/I$ with I a quadratic monomial ideal. Then the set F of the ideals of R generated by subsets of $\{x_1,\dots,x_n\}$ is a Koszul filtration of R. To check it, one has only to observe that $I:(x_i)$ is generated by variables mod I.

The property of having a Koszul filtration is stronger than just being Koszul as the following example shows:

Example 2.5. Let R be a complete intersection of 5 generic quadrics in 5. As said already above, R is Koszul. But it does not have a Koszul filtration since its defining ideal does not contain quadrics of rank < 3.

As well, to have a Gröbner flag is more than G-quadratic. For instance the algebra $R = K[x, y, z]/(x^2, y^2, xz, yz)$ is obviously G-quadratic but one can easily see that R does not have a Gröbner flag.

However many classes of algebras which are known to be Koszul have indeed a Koszul filtration or even a Gröbner flag. For instance:

Theorem 2.6. *(Kempf) Let X be a set of s (distinct) points of the projective space $\mathbf{P}^n$ and let $R(X)$ denote the coordinate ring of X. If $s \leq 2n$ and the points are in general linear position then the ring $R(X)$ is Koszul, see* [K].

We have shown that:

Theorem 2.7. *With the assumption of Kempf's theorem, the ring $R(X)$ has a Gröbner flag.*

One may ask whether Kempf's theorem holds also for a larger number of points. This is not the case.

Example 2.8. There exists a set of 9 points in $\mathbf{P}^4$ which are in general linear position and whose coordinate ring is quadratic but non-Koszul. It is obtained via Gröbner-lifting form the ideal number (55) in Roos' list [R].

On the other hand for "generic points" (indeterminates coordinates) we have the following:

Theorem 2.9. *Let X be a set of "generic points" in $\mathbf{P}^n$. Then $R(X)$ is Koszul if and only if $|X| \leq 1 + n + (n^2/4)$.*

Let C be a smooth algebraic curve of genus g over an algebraically closed field of characteristic zero. If C is not hyperelliptic, then the canonical sheaf on C gives a canonical embedding $C \to \mathbf{P}^{g-1}$ and the coordinate ring R_C of C in this embedding is the canonical ring of C. It is known that R_C is quadratic unless C is a trigonal curve of genus $g \geq 5$ or a plane quintic. Another important application of the filtration arguments is the following theorem.

Theorem 2.10. *Let R_C be the coordinate ring of a curve C in its canonical embedding. Assume that R_C is quadratic. Then R_C is Koszul.*

This is due to Vishik and Finkelberg [VF]; other proofs are given by Polishchuk [P], and by Pareschi and Purnaprajna [PP]. We are able to show that:

Theorem 2.11. *Let R_C be as in the Theorem 2.10. Then R_C has a Gröbner flag.*

For integers n, d, s the "pinched Veronese" $\mathrm{PV}(n, d, s)$ is the K-algebra generated by the monomials of degree d in n variables and with at most s non-zero exponents, that is,

$$\mathrm{PV}(n, d, s) = K[x_1^{a_1} \cdots x_n^{a_n} : \quad \sum a_j = d \quad \text{and} \quad \#\{j : a_j > 0\} \leq s].$$

It is an open question whether $\mathrm{PV}(n, d, s)$ is Koszul when quadratic (they are not all quadratic: $\mathrm{PV}(4, 5, 2)$ is not). G.Caviglia shown in [Ca] that the first not trivial pinched Veronese $\mathrm{PV}(3, 3, 2)$ is Koszul by using a combination of filtrations and ad hoc arguments.

3. Artinian Gorenstein algebras of cubics

The algebras R_C are 2-dimensional Gorenstein domains with h-vector $1+nz+nz^2+z^3$ and Theorem 2.10 asserts that they are Koszul as soon as they are quadratic. One might ask:

Question 3.1. Let R be a quadratic Gorenstein algebra with h-vector $1+nz+nz^2+z^3$. Is R Koszul?

Without loss of generality, one can assume that the algebra is Artinian. Artinian Gorenstein algebras are described via Macaulay inverse system. Let us recall how. Let $S=K[x_1,\dots,x_n]$ be a polynomial ring over a field K of characteristic 0. Let f be a non-zero polynomial of S which is homogeneous of degree, say, s. Let I_f be the ideal of S of the polynomials $g(x_1,\dots,x_n)$ such that

$$g(\partial/\partial x_1,\dots,\partial/\partial x_n)f=0.$$

Set $R_f=S/I_f$. It is known that R_f is a Gorenstein Artinian algebra with socle in degree s and that every such an algebra arises in this way. In particular, in the case $s=3$ the Hilbert series of R_f is equal to $1+nz+nz^2+z^3$ (provided f is not a cone). So Question 3.1 is about algebras R_f with f a cubic form. We are able to show the following:

Theorem 3.2.
(1) *Let f be a cubic in S. Assume there exist linear forms y,z such that $\partial f/\partial yz=0$ and $\partial f/\partial y$ and $\partial f/\partial z$ are quadrics of rank $n-1$. Then R_f has a Koszul filtration.*
(2) *If f is smooth then R_f is not G-quadratic.*
(3) *For the generic cubic f, the ring R_f is Koszul and not G-quadratic.*

Furthermore:

Theorem 3.3.
(1) *Let f be a cubic in S. Assume there exists linear form y such that $\partial f/\partial y^2=0$ and $\partial f/\partial y$ is quadric of rank $n-1$. Then R_f has a Gröbner flag.*
(2) *For the generic singular cubic f, the ring R_f is G-quadratic.*

We are not able to answer Question 3.1 in general. But in [CRV] we have shown that Question 3.1 has an affermative answer $n=3,4$. In both cases the characterization of the f such that R_f quadratic (or Koszul) is very elegant:

Theorem 3.4. *For $n=3$ or 4, the following are equivalent:*
(1) *R_f is quadratic.*
(2) *R_f is Koszul.*
(3) *The ideal of 2-minors of the Hessian matrix $(\partial f/\partial x_i x_j)$ of f has codimension n.*

Furthermore for $n=3$ these conditions are equivalent also to:
(4) *f is not in the closure of the $\mathrm{GL}_3(K)$-orbit of the Fermat cubic $x_1^3+x_2^3+x_3^3$.*

G.Caviglia shown in his unpublished master thesis that property (1),(2) and (3) of 3.4 are equivalent also in the case of $n=5$.

Another interesting question is whether the assumption of 3.2(1) indeed characterize Koszul property for R_f. In this case the answer is no, as the following example shows.

Example 3.5. Let f be the Veronese cubic, that is the determinant of a 3×3 symmetric matrix filled with 6 distinct variables $x_1, \dots, x_6$ and let H be its Hessian matrix. The cubic f has a remarkable property: $\det H$ is f^2 up to scalar and the ideal of 5-minors of H is $(x_1, \dots, x_6)^2 f$. These facts imply that f does not satisfy the assumption of 3.2(1), nevertheless R_f is Koszul (even G-quadratic).

Also, one could also ask whether R_f is LG-quadratic provided it is quadratic. We have reasons to believe that the answer to this question might be positive.

4. Space of quadrics of low codimension

Another point of view we have taken is the following. Let V be a vector space of quadrics of dimension d in n variables. Set $c = \binom{n+1}{2} - d$ the codimension of V in the space of quadrics. Let R_V be the quadratic algebra defined by the ideal generated by V. A theorem of Backelin [B] asserts that if $c \leq 2$ then R_V is Koszul. We have proved in [C] that:

Theorem 4.1. (1) *If $c < n$ then the ring R_V is G-quadratic for a generic V.*
(2) *If $c \leq 2$ then R_V is G-quadratic with, essentially, one exception given by $V = \langle x^2, xy, y^2 - xz, yz \rangle$ in $K[x, y, z]$.*

The "exceptional" algebra $K[x, y, z]/(x^2, xy, y^2 + xz, yz)$ has Hilbert series

$$1 + 3z + 2z^2 + z^3 + z^4 + z^5 + \dots$$

and is LG-quadratic since we may deform it to

$$K[x, y, z, t]/(x^2 + xt, xy + yt, yz + xt, y^2 + xz)$$

which is G-quadratic in the given coordinate system and with respect to revlex $t > x > y > z$.

It follows from 4.1 that every quadratic Artinian algebra R with $\dim R_2 \leq 2$ is G-quadratic.

A recent conjecture of Polishchuk [P2] on Koszul configurations of points, suggests that Artinian quadratic algebras R with $\dim R_2 = 3$ should be Koszul. This is what we have proved in [C1]:

Theorem 4.2. *Let R be an Artinian quadratic algebras with* $\dim R_2 = 3$. *Then:*

1. *R is Koszul.*
2. *R is G-quadratic unless it is (up to trivial extension) a complete intersection of 3 general quadrics in 3 variables.*

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