A modification of Ikeda's theorem

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1 Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of A with $\operatorname{grade} I \geq 2$. Assume that A is a homomorphic image of a Gorenstein local ring and that the field $A/\mathfrak{m}$ is infinite. Let t be an indeterminate over A. We define $\mathrm{R}(I) := A[It] \subseteq A[t]$, $\mathrm{R}'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}]$, and $\mathrm{G}(I) := \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I)$ and call them respectively the Rees algebra, the extended Rees algebra, and the associated graded ring of I. Let $\mathrm{K}_{\mathrm{R}(I)}$, $\mathrm{K}_{\mathrm{R}'(I)}$, and $\mathrm{K}_{\mathrm{G}(I)}$ denote the graded canonical modules of $\mathrm{R}(I)$, $\mathrm{R}'(I)$, and $\mathrm{G}(I)$, respectively. Let $\mathrm{a}(\mathrm{G}(I))$ stand for the a-invariant of $\mathrm{G}(I)$. We always assume A is a quasi-Gorenstein ring, which means that the canonical module of A is a free A-module of rank 1. The purpose of this paper is to prove the following result, which is a modification of theorem given by Ikeda [I].

Theorem 1.1. *Assume that $\mathrm{R}(I)$ is a Cohen-Macaulay ring and $\mathrm{a}(\mathrm{G}(I)) = -2$. Then the following two conditions are equivalent.*

(1) $\mathrm{R}(I)$ *is a Gorenstein ring.*

(2) $\mathrm{K}_{\mathrm{R}'(I)} \cong \mathrm{R}'(I)(-1)$ *as graded* $\mathrm{R}'(I)$*-modules.*

Let us give some consequences of the theorem above. We define $\widetilde{I} := \cup_{n\geq 0} I^{n+1} : I^n$, which is called the Ratliff-Rush closure of I. We set $\mathcal{F} = \{\widetilde{I^i}\}_{i\in\mathbb{Z}}$ and $\mathrm{R}'(\mathcal{F}) := \sum_{i\in\mathbb{Z}} \widetilde{I^i} t^i \subseteq A[t, t^{-1}]$. Let k be a positive integer. With this notation, the first corollary can be stated as follows.

Corollary 1.2. *Assume that $\mathrm{R}(I^k)$ is a Cohen-Macaulay ring and $\mathrm{a}(\mathrm{G}(I^k)) = -2$. Then the following two conditions are equivalent.*

(1) $\mathrm{R}(I^k)$ *is a Gorenstein ring.*

(2) $\mathrm{K}_{\mathrm{R}'(I)} \cong \mathrm{R}'(\mathcal{F})(-k)$ *as graded* $\mathrm{R}'(I)$*-modules.*

To state the second corollary of the theorem, we set up some notation. We put $d = \dim A$. Let $\mathfrak{a}(A) := \prod_{i=0}^{d-1} (0) :_A \mathrm{H}^i_{\mathfrak{m}}(A)$ and let $\mathrm{NCM}(A) := \{\mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}}$ is not a Cohen-Macaulay ring$\}$. Then $\mathrm{NCM}(A) = \mathrm{V}(\mathfrak{a}(A))$. Put $s = \dim \mathrm{NCM}(A)$

and there is a system of parameters $x_1, x_2, \dots, x_d$ for A such that $x_{s+1}, x_{s+2} \dots, x_d \in \mathfrak{a}(A)$. For each $i \leq s$, we put $J_i := (x_{i+1}, x_{i+2}, \dots, x_d)$. Then we have

Corollary 1.3. *Assume that $d \geq 2$. Let $s = 0$ and $I = J_0$. Then the following two conditions are equivalent.*

(1) $\mathrm{R}(I^k)$ *is a Gorenstein ring.*

(2) A *is a Cohen-Macaulay ring and* $k = d - 1$.

The implication (2) ⇒ (1) is already known (see [O], 4.3). The converse implication (1) ⇒ (2) is a result in this paper. The third one is the following

Corollary 1.4. *Assume that $d \geq 3$. Let $s \leq 1$ and $I = \bigcup_{\ell \geq 0} J_1 : x_1^\ell$. Then the following two conditions are equivalent.*

(1) $\mathrm{R}(I^k)$ *is a Gorenstein ring.*

(2) A *has finite local cohomology modules and* $k = d - 2$.

The implication (2) ⇒ (1) is already shown in the last symposium. The converse implication (1) ⇒ (2) is a result in this paper.

2 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. We put $R = \mathrm{R}(I)$, $R' = \mathrm{R}'(I)$, and $G = \mathrm{G}(I)$. To begin with we note

Lemma 2.1. *Let a be an integer and let $\kappa = \{\kappa_i\}_{i \geq -a-1}$ be an I-filtration of A such that $\kappa_{-a-1} = A$ and $\kappa_{-a-1} \supsetneq \kappa_{-a}$. Set $\mathrm{gr}_A(\kappa) = \bigoplus_{i \geq -a} \kappa_{i-1}/\kappa_i$ that is a graded G-module. If there is an embedding $G(a) \hookrightarrow \mathrm{gr}_A(\kappa)$ of graded G-modules, then $\kappa_i = I^{i+a+1}$ for all integers $i \geq -a-1$.*

Proof. See the proof of Theorem 3.2 in the paper [GI]. □

Let the ideal I be generated by elements $a_1, a_2, \dots, a_n \in A$. We may assume a_1 is an regular element of A. Let $X_1, X_2, \dots X_n, Y$ are indeterminates over A. We consider the A-algebra homomorphisms

$$\varphi : A[X_1, X_2, \dots X_n] \to R$$

such that $\varphi(X_i) = a_i t$ for all $1 \leq i \leq n$ and

$$\varphi' : A[X_1, X_2, \dots X_n, Y] \to R'$$

such that $\varphi'(X_i) = a_i t$ for all $1 \leq i \leq n$ and $\varphi'(Y) = t^{-1}$. Put $P = A[X_1, X_2, \dots X_n]$ and $F_i = X_i Y - a_i$. Then we get the following equality.

Claim 2.2. $\ker\varphi' = (F_1, F_2, \dots F_n)P[Y] + \ker\varphi P[Y]$.

Proof. Take any element $F \in \ker\varphi'$. Dividing F by $F_1, F_2, \dots F_n$, we can write $F = \sum_{i=1}^n Q_i F_i + H + H'$, where $Q_i \in P[Y]$, $H \in P$, and $H' \in A[Y]$. Then $\varphi'(F) = H(a_1 t, a_2 t, \dots, a_n t) + H'(t^{-1})$, which is an element of $A[t, t^{-1}]$. Since $\varphi'(F) = 0$, we get $H' \in A$, and hence $H + H' \in \ker\varphi$. $\square$

Set $f_i = a_i t Y - a_i$, which is an element of $R[Y]$. We note f_1 is a regular element on $R[Y]$ because so is a_1. Look at the graded R-algebra homomorphism

$$\psi : R[Y] \to R'$$

induced by the injection $R \to R'$ of graded rings such that $\psi(Y) = t^{-1}$. Then the claim above implies the following equality.

Lemma 2.3. $\ker\psi = (f_1, f_2, \dots f_n)$.

Therefore we get the exact sequence $0 \to \dfrac{\ker\psi}{f_1 R[Y]} \to \dfrac{R[Y]}{f_1 R[Y]} \to R' \to 0$ of graded $R[Y]$-modules. Let us now prove the theorem.

Proof of Theorem 1.1. Assume that R is a Gorenstein ring. Then $\mathrm{K}_{R[Y]} \cong R[Y](m)$ for some $m \in \mathbb{Z}$. Taking the $\mathrm{K}_{R[Y]}$-dual of the graded exact sequence

$$0 \to R[Y](1) \xrightarrow{Y} R[Y] \to R \to 0$$

of graded $R[Y]$-modules, we get the graded exact sequence

$$0 \to R[Y](m) \xrightarrow{Y} R[Y](m-1) \to R(-1) \to 0$$

of graded $R[Y]$-modules because $\mathrm{K}_R \cong R(-1)$ as graded R-modules. Therefore $m = 0$. Put $S = \frac{R[Y]}{f_1 R[Y]}$ and we obtain that S is a Gorenstein graded ring with $\mathrm{K}_S \cong S$ as graded S-modules (recall that $\deg f_1 = 0$). Since Y is a regular element on S, we have $\mathrm{K}_{S/YS} \cong [S/YS](-1)$ as graded S-modules. Put $L = \frac{\ker\psi}{f_1 R[Y]}$. Then $R' \cong S/L$ and $G \cong S/YS + L$ as graded rings, so that $\mathrm{K}_{R'} \cong \mathrm{Hom}_S(S/L, S)$ and $\mathrm{K}_G \cong \mathrm{Hom}_{S/Y}(S/YS + L, [S/YS](-1))$ as graded S-modules. Hence we get $\mathrm{K}_{R'} \cong (0) :_S L$ and $\mathrm{K}_G \cong [YS :_S L/YS](-1)$ as graded S-modules. We can check

Claim 2.4. $YS :_S L = [(0) :_S L] + YS$.

The above claim implies that the natural map $\pi : (0) :_S L \to [YS :_S L]/YS$ is surjective. And we have $\ker\pi = Y[(0) :_S L]$. Therefore $\mathrm{K}_{R'}/t^{-1}\mathrm{K}_{R'} \cong \mathrm{K}_G(1)$ as graded G-modules. Thanks to [I], we get $\mathrm{K}_{R'}/t^{-1}\mathrm{K}_{R'} \cong G(-1)$ as graded G-modules. Let $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ stand for the canonical I-filtration of A (see [GI], 1.1 and notice that the canonical filtration exists if the base ring A satisfies Serre's condition (S_2)), namely the I-filtration ω fulfills $I^i \subseteq \omega_{i-a-1}$ for all $i \in \mathbb{Z}$ and

$K_{R'(I)} \cong \sum_{i\in\mathbb{Z}} \omega_i t^i \subseteq A[t,t^{-1}]$ as graded $R'(I)$-modules. Therefore we get $\omega_{i+1} = I^i$ by Lemma 2.1, and hence $K_{R'(I)} \cong R'(I)(-1)$ as graded $R'(I)$-modules.

Conversely, assume that $K_{R'(I)} \cong R'(I)(-1)$ as graded $R'(I)$-modules. Hence $K_{R'}/t^{-1}K_{R'} \cong G(-1)$ as graded G-modules. The embedding $K_{R'}/t^{-1}K_{R'}(-1) \hookrightarrow K_G$ follows from the exact sequence $0 \to R'(1) \xrightarrow{t^{-1}} R' \to G \to 0$ of graded R'-modules, so that we can find an embedding $G(-2) \hookrightarrow K_G$ of graded R'-modules. By [TVZ], there is an I-filtration $\kappa = \{\kappa_i\}_{i\geq 0}$ of A such that $\kappa_1 = A$, $\kappa_1 \supsetneq \kappa_2$, $K_R \cong \bigoplus_{i\geq 1}\kappa_i$ as a graded R-module, and $K_G = \bigoplus_{i\geq 2}\kappa_{i-1}/\kappa_i$ as a graded G-module because R is a Cohen-Macaulay ring. Therefore we get $\kappa_i = I^{i-1}$ by Lemma 2.1, and hence $K_{R(I)} \cong R(I)(-1)$ as graded $R(I)$-modules. Then R is a Gorenstein ring. $\square$

We remark that Corollary 1.3 does not hold true without the assumption that the ring A is quasi-Gorenstein. Let us close this paper with the following typical example in [HR], (2.2).

Example 2.5. *Let $k[[s,t]]$ be a formal power series ring over a field k. Let $A = k[[s^2,t,s^3,st]]$ and $I = (s^2,t)$. Then $R(I^2)$ is a Gorenstein ring but A is not a Cohen-Macaulay ring.*

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