ANALYTIC SPREAD OF SQUAREFREE MONOMIAL IDEALS

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Introduction

This is a joint work with Naoki Terai (Saga Univ.) and Ken-ichi Yoshida (Nagoya Univ.).

Let S be a polynomial ring with each variable has degree 1 over an infinite field k, and I a squarefree monomial ideal of S. The *arithmetical rank* of I is defined by

$$\operatorname{ara} I := \min \left\{ r : \text{there exist } a_1, \dots, a_r \in I \text{ such that } \sqrt{(a_1, \dots, a_r)} = \sqrt{I} \right\}.$$

It is known by Lyubeznik [2] that $\operatorname{pd}_S S/I \leq \operatorname{ara} I$, where $\operatorname{pd}_S S/I$ denotes the *projective dimension* of S/I. Let J be a minimal reduction of I. The number of a minimal set of generators of J, which is independent on the choice of J, is called the *analytic spread* of I. We denote it by $l(I)$. Since $\sqrt{J} = \sqrt{I}$ holds, we have

$$\operatorname{pd}_S S/I \leq \operatorname{ara} I \leq l(I).$$

Schmitt–Vogel lemma [4, Lemma, pp. 249] is an important and useful tool in the study of the arithmetical rank. Using this lemma, Schmitt–Vogel proved $\operatorname{ara} I = \operatorname{pd}_S S/I$ for

$$(*) \qquad I = (x_{11}, \dots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \dots, x_{qi_q}),$$

where x_{ij} are variables in S pairwise distinct. Note that this ideal I is the Alexander dual of a complete intersection ideal.

In this report, we refine Schmitt–Vogel lemma for reductions and prove $l(I) = \operatorname{pd}_S S/I$ for the ideal $(*)$ as its application.

1. Main Theorem

In this section, we consider a commutative ring R with unitary. Let I, J be ideals in R with $J \subset I$. We say J is a *reduction* of I if there exists $s \in \mathbb{N}$ such that $I^{s+1} = JI^s$. It is easy to see that if J is a reduction of I, then $\sqrt{J} = \sqrt{I}$. The main theorem of this report is the following:

Theorem 1. *Let R be a commutative ring with unitary. Let $P_0, P_1, \dots, P_r \subset R$ be finite subsets, and we set*

$$P = \bigcup_{\ell=0}^{r} P_\ell,$$
$$g_\ell = \sum_{a \in P_\ell} a, \quad \ell = 0, 1, \dots, r.$$

Assume that

(C1) $\sharp P_0 = 1$.

(C2) *For all $\ell > 0$ and $a, a'' \in P_\ell$ ($a \neq a''$), there exist some ℓ' ($0 \leq \ell' < \ell$), $a' \in P_{\ell'}$, and $b \in (P)$ such that $aa'' = a'b$.*

Then we have $(g_0, g_1, \dots, g_r)$ is a reduction of (P).

The difference between our theorem and Schmitt–Vogel lemma is the assumption of the existence of $b \in (P)$ in (C2). The second condition of Schmitt–Vogel lemma is

(C2)' For all $\ell > 0$ and $a, a'' \in P_\ell$ ($a \neq a''$), there exist some ℓ' ($0 \leq \ell' < \ell$) and $a' \in P_{\ell'}$ such that $aa'' \in (a')$;

and the conclusion is $\sqrt{(g_0, g_1, \dots, g_r)} = \sqrt{(P)}$.

Remark 2. Schmitt–Vogel lemma allows us to add some exponent $e(a)$ for each $a \in P_\ell$ in the sum g_ℓ, i.e., we may put

$$g_\ell = \sum_{a \in P_\ell} a^{e(a)}.$$

Thus we can take g_ℓ as homogeneous if R is graded. But our theorem is not allowed to add such $e(a)$.

2. Proof of Main Theorem

In this section, we prove Theorem 1.

As first, we fix notation. Put $I = (P)$, $J = (g_0, g_1, \dots, g_r)$, and

$$I_\ell = \left(\bigcup_{j=0}^{\ell} P_j \right), \qquad \ell = 0, 1, \dots, r.$$

It is enough to show $I_\ell^{2^\ell} \subset JI^{2^\ell - 1}$ for $\ell = 0, 1, \dots, r$. We show this by induction on ℓ. In fact, we show

$$I_\ell^{2^\ell} \subset I_{\ell-1}^{2^{\ell-1}} I^{2^\ell - 2^{\ell-1}} + JI^{2^\ell - 1}, \qquad \ell = 0, 1, \dots, r.$$

If $\ell = 0$, then $I_0 = (P_0) = (g_0) \subset J$ because $\sharp P_0 = 1$. Let us consider the case of $\ell > 0$. Take $a_1, \dots, a_{2^\ell} \in \bigcup_{j=0}^{\ell} P_j$. We may assume $a_1, \dots, a_m \in P_\ell$ and $a_{m+1}, \dots, a_{2^\ell} \in \bigcup_{j=0}^{\ell-1} P_j$.

First, we assume that we can renumbering $a_1, \dots, a_m$ such that

$$\{a_1, a''_1\}, \dots, \{a_{\lfloor m/2 \rfloor}, a''_{\lfloor m/2 \rfloor}\},$$

where $a_\lambda \neq a''_\lambda$, $a''_\lambda = a_{\lfloor m/2 \rfloor + \lambda}$ ($\lambda = 1, \ldots, \lfloor m/2 \rfloor$), and $\lfloor \alpha \rfloor$ denotes the maximal integer which does not exceed α. Then we can use the condition (C2), that is, there are $a'_\lambda \in \bigcup_{j=0}^{\ell-1} P_j$ and $b_\lambda \in I$ such that $a_\lambda a''_\lambda = a'_\lambda b_\lambda$. Thus

$$\begin{aligned} a_1 \cdots a_{2^\ell} &= \left(\prod_{\lambda=1}^{\lfloor m/2 \rfloor} a'_\lambda b_\lambda \right) a_{2\lfloor m/2 \rfloor + 1} \cdots a_{2^\ell} \\ &= \left(\prod_{\lambda=1}^{\lfloor m/2 \rfloor} a'_\lambda \right) a_{m+1} \cdots a_{2^\ell} \left(\prod_{\lambda=1}^{\lfloor m/2 \rfloor} b_\lambda \right) a_{2\lfloor m/2 \rfloor + 1} \cdots a_m. \end{aligned}$$

Note that $m \leq 2^\ell$ and $\lfloor m/2 \rfloor \geq (m-1)/2$. Then it is easy to see that $\lfloor m/2 \rfloor + 2^\ell - m \geq 2^{\ell-1} - 1/2$. Since $\lfloor m/2 \rfloor + 2^\ell - m \in \mathbb{Z}$, we have $\lfloor m/2 \rfloor + 2^\ell - m \geq 2^{\ell-1}$. Therefore

$$a_1 \cdots a_{2^\ell} \in I_{\ell-1}^{2^{\ell-1}} I^{2^\ell - 2^{\ell-1}}.$$

Next, we consider the case that we cannot make $\lfloor m/2 \rfloor$ pairs of distinct elements. This case occurs if and only if there exist $a \in P_\ell$ (uniquely) such that

$$a = a_1 = \cdots = a_{\lfloor (m-1)/2 \rfloor + 2},$$

by renumbering $a_1, \ldots, a_m$. Then

$$\begin{aligned} a_1 a_2 \cdots a_{2^\ell} &= a a_2 \cdots a_{2^\ell} \\ &= \left(g_\ell - \sum_{a'' \in P_\ell,\ a'' \neq a} a'' \right) a_2 \cdots a_{2^\ell} \\ &= g_\ell a_2 \cdots a_{2^\ell} - \sum_{a'' \in P_\ell,\ a'' \neq a} a'' a_2 \cdots a_{2^\ell}. \end{aligned}$$

The first term belongs to $JI^{2^\ell - 1}$. Thus we consider $a'' a_2 \cdots a_{2^\ell}$ in the second term only. Since $\max\{\sharp\{i : a_i = a\} : a \in P_\ell\}$ is strictly reduced, the problem can be reduced to the first case.

Q.E.D.

3. An application

In this section, we apply Theorem 1 to some ideals and calculate the analytic spread of them.

Consider the ideal

$$(*) \qquad I = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q}),$$

where $x_{11}, \ldots, x_{qi_q}$ are all distinct variables. Then one can easily see that

$$\operatorname{pd}_S S/I = \sum_{s=1}^{q} i_s - q + 1.$$

Schmitt–Vogel [4] proved $\operatorname{ara} I = \operatorname{pd}_S S/I$ (see also Schenzel–Vogel [3]). They proved it by applying

$$P_\ell = \{x_{1\ell_1} \cdots x_{q\ell_q} \ : \ \ell_1 + \cdots + \ell_q = \ell + q\}, \qquad \ell = 0, 1, \ldots, r$$

to Schmitt–Vogel lemma, where $r = \sum_{s=1}^{q} i_s - q$. Since these $P_0, P_1, \dots, P_r$ also satisfy the assumption of Theorem 1, we have the following corollary:

Corollary 3. *Let* $I = (x_{11}, \dots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \dots, x_{qi_q})$. *Then we have*

$$l(I) = \mathrm{pd}_S\, S/I.$$

In particular, $(g_0, g_1, \dots, g_r)$ *is a minimal reduction of* I.

Although $l(I) = \mathrm{pd}_S\, S/I$ is also proven by computing the dimension of fiber cone, we construct a minimal reduction of I explicitly.

By giving an example, we remark that the relation between our theorem and the reduction number.

Let $I = (x_{11}, x_{12}) \cap (x_{21}, x_{22}) \cap (x_{31}, x_{32})$. This is a special case of the ideal $(*)$ and $\mathrm{pd}_S\, S/I = 2+2+2-3+1 = 4$. The minimal reduction of I which derived from Corollary 3 is generated by the following 4 elements:

$$\begin{aligned} g_0 &= x_{11}x_{21}x_{31}, \\ g_1 &= x_{12}x_{21}x_{31} + x_{11}x_{22}x_{31} + x_{11}x_{21}x_{32}, \\ g_2 &= x_{12}x_{22}x_{31} + x_{12}x_{21}x_{32} + x_{11}x_{22}x_{32}, \\ g_3 &= x_{12}x_{22}x_{32}. \end{aligned}$$

Put $J = (g_0, g_1, g_2, g_3)$. Then what is the reduction number $r_J(I)$ of J ? From the our proof of Theorem 1, we can only see $r_J(I) \le 2^3 - 1 = 7$. But $I^3 = JI^2$ holds. In fact, $r_J(I) = 2$. Thus the upper bound of $r_J(I)$ derived from Theorem 1 is very big in general.

References

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