Stanley–Reinser rings which are complete intersections locally [1]

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1. Introduction

By a simplicial complex Δ on the vertex set $V = [n] = \{1, 2, \ldots, n\}$, we mean that Δ is a family of subsets of V which satisfies the following conditions:

(i) $\{i\} \in \Delta$ for every $i \in V$ (ii) $F \in \Delta$, $G \subseteq F$ imply $G \in \Delta$.

An element of Δ is called a *face* of Δ. The *dimension* of Δ, denoted by $\dim \Delta$, is the maximum of the dimension $\dim F = \sharp(F) - 1$, where F runs through all faces of Δ and $\sharp(F)$ denotes the cardinality of a set F. A simplicial complex Δ is called *pure* if all facets (maximal faces with respect to inclusion) of Δ have the same dimension.

For a face F of Δ,

$$\mathrm{link}_\Delta(F) = \{G \in \Delta \;:\; F \cup G \in \Delta,\, F \cap G = \emptyset\}$$

is called the *link* of F. For a subset W of V,

$$\Delta_W = \{F \in \Delta \;:\; F \subseteq W\}$$

is called the *restriction* to W of Δ.

Throughout this talk, let K be a field, and let $S = K[X_1, \ldots, X_n]$ be a polynomial ring over K, unless otherwise specified. The ring S can be viewed as a standard graded K-algebra (i.e., $S = \bigoplus_{n\in\mathbb{N}}$ is an $\mathbb{N}$-graded ring with $S_0 = K$, $S = K[S_1]$) with the unique homogeneous maximal ideal $\mathfrak{m} = (X_1, \ldots, X_n)$. For a simplicial complex Δ, the *Stanley–Reisner ideal* I_Δ and the *Stanley–Reisner ring* $K[\Delta]$ are defined by

$$\begin{aligned} I_\Delta &= (X_{i_1} \cdots X_{i_p} \;:\; 1 \le i_1 < \cdots < i_p \le n,\, \{i_1, \ldots, i_p\} \notin \Delta)S, \\ K[\Delta] &= S/I_\Delta. \end{aligned}$$

Note that any squarefree monomial ideal $I \subseteq S$ with $\operatorname{indeg} I \geq 2$ can be written as $I = I_\Delta$ for some simplicial complex Δ, and that $K[\Delta]$ is a graded reduced K-algebra with $\dim K[\Delta] = \dim \Delta + 1$. See [BH, St] about simplicial complexes and Stanley–Reisner rings.

Let $R = S/I$ be an arbitrary standard graded K-algebra. The ring R is said to be *Buchsbaum* (resp. to have (FLC)) if $\mathrm{Ext}^i_S(S/\mathfrak{m}, R) \to H^i_\mathfrak{m}(R))$ is

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surjective (resp. $H^i_{\mathfrak{m}}(R)$ has finite length) for every $i < \dim R$. In particular, any Buchsbaum ring has (FLC).

The Stanley–Reisner ring $K[\Delta]$ has (FLC) if and only if Δ is pure and $K[\mathrm{link}_\Delta\{i\}]$ is Cohen–Macaulay for every $i \in V$. When this is the case, $K[\Delta]$ is Buchsbaum; see e.g., [St].

Let Δ be a simplicial complex, and let $G(I_\Delta) = \{m_1, \dots, m_\mu\}$ denote the minimal set of monomial generators of I_Δ. Then one can easily check the following fact.

Fact 1.1. *Let $I_\Delta = (m_1, \dots, m_\mu)$ be as above. Then I_Δ is a complete intersection (i.e., I_Δ is generated by a regular sequence) if and only if* $\gcd(m_i, m_j) = 1$ *for every i, j with $i \neq j$.*

In general, if $I \subseteq S$ is generated by a regular sequence, then S/I^ℓ is Cohen–Macaulay for every integer $\ell \geq 1$. When I is generically a complete intersection (i.e., I_P is a complete intersection for all minimal prime ideal P over I), the converse is also true; see [CN]. Hence, for example, I_Δ is a complete intersection if and only if S/I_Δ^ℓ is Cohen–Macaulay for every $\ell \geq 1$.

In [GT], Goto and Takayama introduced the notion of generalized complete intersection complexes and characterized those complexes: a simplicial complex Δ is said to be a *generalized complete intersection* complex if Δ is pure and $K[\mathrm{link}_\Delta\{i\}]$ is a complete intersection for any vertex $i \in V$. The following theorem gives a motivation of our study.

Theorem 1.2 (Goto–Takayama (see also [GT]))**.** *Let Δ be a simplicial complex on $V = [n]$. Then the following conditions are equivalent*:

(1) *$K[\Delta]$ is a generalized complete intersection in the sense of* [GT]*.*
(2) *S/I_Δ^ℓ has (FLC) for every $\ell \geq 1$.*

Clearly, a complete intersection is a generalized complete intersection. In [GT], they gave examples which are not complete intersections but generalized complete intersection complexes. However, their complexes Δ are disconnected or $\dim \Delta = 1$. So it is natural to ask the following question:

Question 1.3. Assume that a simplicial complex Δ is connected and $\dim \Delta \geq 2$. If Δ is a generalized complete intersection complex, then is it a complete intersection?

The main aim of this talk is to give a complete answer to this question. Before stating our result, let us define the following notion:

Definition 1.4. A simplicial complex $K[\Delta]$ (or Δ) is called *a locally complete intersection* (resp. Gorenstein, Cohen–Macaulay) if $K[\Delta]_P$ is a complete intersection (resp. Gorenstein, Cohen–Macaulay) for every $P \in \mathrm{Proj}\, K[\Delta]$.

Note that $K[\Delta]$ is a locally complete intersection if and only if $K[\Delta]_{X_i}$ is a complete intersection for every $1 \leq i \leq n$. Moreover, since $k[\Delta]_{X_i} \cong K[\mathrm{link}_\Delta\{i\}][X_i, X_i^{-1}]$ we have:

Lemma 1.5. *Let Δ be a simplicial complex on $V = [n]$. Then the following conditions are equivalent*:

(1) $K[\Delta]$ *is a locally complete intersection.*
(2) $K[\Delta]_{X_i}$ *is a complete intersection for every* $i \in V$.
(3) $K[\mathrm{link}_\Delta\{i\}]$ *is a complete intersection for every* $i \in V$.

In particular, Δ is a generalized complete intersection if and only if Δ is pure and a locally complete intersection.

Corollary 1.6. *Let Δ be a simplicial complex on V. If $K[\Delta]$ is a complete intersection (resp. Gorenstein, Cohen–Macaulay), then so is $K[\mathrm{link}_\Delta(F)]$ for any face F of Δ.*

Proof. It immediately follows from the fact $\mathrm{link}_{\mathrm{link}_\Delta\{i\}}(F \setminus \{i\}) = \mathrm{link}_\Delta(F)$ for $i \in F$. □

Example 1.7. Let Δ be a simplicial complex corresponding to 5-gon. That is, $K[\Delta] = K[X_1, X_2, X_3, X_4, X_5]/(X_1X_3, X_1X_4, X_2X_4, X_2X_5, X_3X_5)$. Then $K[\Delta]$ is a locally complete intersection but *not* a complete intersection.

Indeed, $K[\mathrm{link}_\Delta\{1\}] \cong K[X_2, X_5]/(X_2X_5)$ is a complete intersection. Similarly, $K[\mathrm{link}_\Delta\{i\}]$ is also a complete intersection for other $i \in [5]$.

The following theorem is a main result in this talk; see also Section 2.

Theorem 1.8. *Let Δ be a simplicial complex on $V = [n]$ with $\dim \Delta \geq 2$. Assume that Δ is a locally complete intersection. Then it is a disjoint union of finitely many simplicial complexes whose Stanley–Reisner rings are complete intersections.*

In the case $\dim \Delta = 1$, we can also characterize locally complete intersection complexes. See Section 3.

2. Proof of the main theorem

In this section, we will prove the main theorem. First of all, we remark the following lemma.

Lemma 2.1. *Assume that $V = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$. Let Δ_i be a locally complete intersection complex on V_i for $i = 1, 2$. Then a disjoint union $\Delta_1 \cup \Delta_2$ is also a locally complete intersection complex on V.*

Proof. Put $V_1 = [m]$ and $V_2 = [n]$. If we write

$$K[\Delta_1] = K[X_1, \ldots, X_m]/I_{\Delta_1} \quad \text{and} \quad K[\Delta_2] = K[Y_1, \ldots, Y_n]/I_{\Delta_2},$$

then

$$K[\Delta] \cong K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/(I_{\Delta_1},\ I_{\Delta_2},\ \{X_iY_j\}_{1 \leq i \leq m,\ 1 \leq j \leq n}).$$

Hence

$$K[\Delta]_{X_i} \cong K[\Delta]_{X_i} \quad \text{and} \quad K[\Delta]_{Y_j} \cong K[\Delta_2]_{Y_j}.$$

are complete intersection rings. Thus Δ is also a locally complete intersection. □

Remark 2.2. In the above lemma, we suppose that both Δ_1 and Δ_2 are generalized complete intersections. Then $\Delta_1 \cup \Delta_2$ is a generalized complete intersection if and only if $\dim \Delta_1 = \dim \Delta_2$.

Example 2.3. Let Δ be the disjoint union of the standard $(m-1)$-simplex and the standard $(n-1)$-simplex. Then Δ is a locally complete intersection complex by Lemma 2.1. Moreover, $K[\Delta]$ is isomorphic to

$$K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/(X_iY_j \,:\, 1 \le i \le m,\, 1 \le j \le n)$$

and it is a generalized complete intersection if and only if $m = n$.

By virtue of Lemma 2.1, it suffices to show the following theorem.

Theorem 2.4. *Let Δ be a simplicial complex on $V = [n]$. Assume that Δ is connected and $\dim \Delta \ge 2$. Then the following conditions are equivalent*:

(1) $K[\Delta]$ *is a complete intersection.*
(2) $K[\Delta]$ *is a locally complete intersection.*
(2)' $K[\Delta]$ *is a generalized complete intersection.*

From now on, assume that Δ is a locally complete intersection, connected complex which is not a complete intersection. Suppose that $\dim \Delta \ge 1$. Note that Δ is pure since Δ is connected and locally complete intersection and hence Δ satisfies Serre condition (S_2). Let $G(I_\Delta) = \{m_1, \ldots, m_\mu\}$ denote the minimal set of monomial generators of I_Δ. Then $\mu \ge 2$ and $\deg m_i \ge 2$ for every $i = 1, 2, \ldots, \mu$, and that there exists i, j $(1 \le i < j \le n)$ such that $\gcd(m_i, m_j) \ne 1$.

In order to prove Theorem 2.4, it is enough to show that $\dim \Delta = 1$. In what follows, X_i, $Y_j, \ldots$ denote corresponding variables to vertices x_i, $y_j, \ldots$.

Lemma 2.5. *We may assume that* $\deg m_i = \deg m_j = 2$.

Proof. Take m_j, m_k $(j \ne k)$ such that $\gcd(m_j, m_k) \ne 1$. If $\deg m_j = \deg m_k = 2$, then there is nothing to prove.

Now suppose that $\deg m_k \ge 3$. By [GT, Lemmas 3.4, 3.5], we may assume that $\deg m_j = 2$ and $\gcd(m_j, m_k) = X_p$. Write $m_k = X_pX_{i_1}\cdots X_{i_r}$ and $m_j = X_pX_q$. Then [GT, Lemma 3.6] implies that $X_{i_1}X_q \in G(I_\Delta)$. Set $m_i = X_{i_1}X_q \in I_\Delta$. Then $\deg m_i = \deg m_j = 2$ and $\gcd(m_i, m_j) = X_q \ne 1$, as required. □

The following lemma is simple but important.

Lemma 2.6. *Let x_1, x_2, y be distinct vertices such that X_1Y, $X_2Y \in I_\Delta$. For any $z \in V \setminus \{x_1, x_2, y\}$, at lease one of monomials X_1Z, X_2Z and YZ belongs to I_Δ.*

Proof. It immediately follows from the fact that $K[\mathrm{link}_\Delta\{z\}]$ is a complete intersection. □

Lemma 2.7. *There exist some integers k, $\ell \ge 2$ such that*

(1) $V = \{x_1, \ldots, x_k, y_1, \ldots, y_\ell\}$.
(2) $X_1Y_1, \ldots, X_kY_1 \in I_\Delta$.

(3) $\sharp\{i \,:\, 1 \le i \le k,\ X_iY_j \notin I_\Delta\} \le 1$ *holds for each* $j = 2, \ldots, \ell$.

Proof. By Lemma 2.5, there exists vertices $x_1, x_2, y_1 \in V$ such that X_1Y_1, $X_2Y_1 \in I_\Delta$. Thus one can write $V = \{x_1, \ldots, x_k, y_1, \ldots, y_\ell\}$ such that

$$\begin{array}{ccccc} X_1Y_1, & X_2Y_1, & \ldots, & X_kY_1 & \in & I_\Delta, \\ Y_1Y_2, & Y_1Y_3, & \ldots, & Y_1Y_\ell & \notin & I_\Delta. \end{array}$$

If $\ell = 1$, then $\Delta = \Delta_{\{y_1\}} \cup \Delta_{\{x_1,\ldots,x_k\}}$ is a disjoint union since $\{y_1, x_i\} \notin \Delta$ for all i. This contradicts the connectedness of Δ. Hence $\ell \ge 2$. Thus it is enough to show (3) in this notation.

Now suppose that there exists an integer j with $2 \le j \le \ell$ such that

$$\sharp\{i \,:\, 1 \le i \le k,\ X_iY_j \notin I_\Delta\} \ge 2.$$

When $k = 2$, we have X_1Y_j, $X_2Y_j \notin I_\Delta$. On the other hand, as X_1Y_1, $X_2Y_1 \in I_\Delta$ and $Y_j \ne X_1, X_2, Y_1$, we obtain that at least one of X_1Y_j, X_2Y_j, Y_1Y_j belongs to I_Δ. It is impossible. So we may assume that $k \ge 3$ and $X_{k-1}Y_j$, $X_kY_j \notin I_\Delta$. Then $\{x_{k-1}\}$, $\{x_k\}$ and $\{y_1\}$ belong to $\mathrm{link}_\Delta\{y_j\}$, and $X_{k-1}Y_1$, X_kY_1 form part of a minimal system of generators of $I_{\mathrm{link}_\Delta\{y_j\}}$. This contradicts the assumption that $K[\mathrm{link}_\Delta\{y_j\}]$ is a complete intersection. □

In what follows, we fix the notation as in Lemma 2.7. First, we suppose that there exists i_0 with $1 \le i_0 \le k$ such that

$$\sharp\{j \,:\, 1 \le j \le \ell,\ X_{i_0}Y_j \notin I_\Delta\} = 1.$$

In this case, we may assume that $X_1Y_2 \notin I_\Delta$ and $X_1Y_j \in I_\Delta$ for all $3 \le j \le \ell$ without loss of generality. Note that $X_2Y_2, \ldots, X_kY_2 \in I_\Delta$ by Lemma 2.7. We claim that $\{x_1, y_2\}$ is a facet of Δ. As $X_iY_2 \in I_\Delta$ for each $i = 2, \ldots, k$, we have that $\{x_1, y_2, x_i\} \notin \Delta$. Similarly, $\{x_1, y_2, y_j\} \notin \Delta$ since $X_1Y_j \in I_\Delta$ for $j = 1$ or $3 \le j \le \ell$. Hence $\{x_1, y_2\}$ is a facet of Δ, and $\dim \Delta = 1$ because Δ is pure.

By the observation as above, we may assume that for every i with $1 \le i \le k$,

$$\sharp\{j \,:\, 1 \le j \le \ell,\ X_iY_j \notin I_\Delta\} \ge 2$$

or $X_iY_j \in I_\Delta$ holds for all $j = 1, \ldots, \ell$.

Now suppose that there exists j_1, j_2 with $1 \le j_1 < j_2 \le \ell$ such that $X_iY_{j_1}$, $X_iY_{j_2} \notin I_\Delta$. Then $X_rY_{j_1}$, $X_rY_{j_2} \in I_\Delta$ for all $r \ne i$ by Lemma 2.7. It follows that $X_rX_i \in I_\Delta$ from Lemma 2.6. Then we can relabel x_i (say $y_{\ell+1}$). Repeating this procedure, we can get one of the following cases:

Case 1: $V = \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ such that $X_iY_j \in I_\Delta$ for all i, j with $1 \le i \le r$, $1 \le j \le s$.

Case 2: $V = \{x_1, x_2, y_1, \ldots, y_m, z_1, \ldots, z_p, w_1, \ldots, w_q\}$ such that

$$\begin{cases} X_1Y_j \in I_\Delta, & X_2Y_j \in I_\Delta \quad (j = 1, \ldots, m) \\ X_1Z_j \notin I_\Delta, & X_2Z_j \in I_\Delta \quad (j = 1, \ldots, p) \\ X_1W_j \in I_\Delta, & X_2W_j \notin I_\Delta \quad (j = 1, \ldots, q) \end{cases}$$

holds for some $m \ge 1$, $p, q \ge 2$.

If Case 1 occurs, then $\Delta = \Delta_{\{x_1,\dots,x_r\}} \cup \Delta_{\{y_1,\dots,y_s\}}$ is a disjoint union. This contradicts the assumption. Thus Case 2 must occur. If $\{x_1, x_2\} \in \Delta$, then it is a facet and so $\dim \Delta = 1$. Hence we may assume that $\{x_1, x_2\} \notin \Delta$. However, since Δ is connected, there exists a path between x_1 and x_2.

Cases (2-a): the case where $\{z_1, w_k\} \in \Delta$ for some k with $1 \le k \le q$.

We may assume that $\{z_1, w_1\} \in \Delta$. Now suppose that $\dim \Delta \ge 2$. Then since $\{z_1, w_1\}$ is *not* a facet, there exists $u \in V \setminus \{x_1, x_2\}$ such that $\{z_1, w_1, u\} \in \Delta$. If $u = z_j$ $(2 \le j \le p)$ (resp. $u = y_i$ $(1 \le i \le m)$), then $G(I_{\mathrm{link}_\Delta \{w_1\}})$ contains X_2Z_1 and X_2Z_j (resp. X_2Y_i); see figure below. It is impossible since $\mathrm{link}_\Delta\{w_1\}$ is a complete intersection. When $u = w_k$, we can obtain a contradiction by a similar argument as above. Therefore $\dim \Delta = 1$.

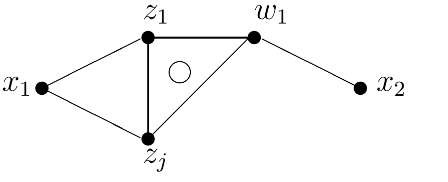

Figure: the case $\{z_1, z_j, w_1\} \in \Delta$ in Case (2-a)

Cases (2-b): the case where $\{z_j, w_k\} \notin \Delta$ for all j, k.

Then we may assume that (i) $\{z_1, y_1\} \in \Delta$ and (ii) $\{y_1, y_2\} \in \Delta$ or $\{y_1, w_1\} \in \Delta$. Now suppose that $\dim \Delta \ge 2$. Then since $\{z_1, y_1\}$ is *not* a facet, we have

$$\{z_1, y_1, y_i\} \in \Delta, \ \{z_1, y_1, w_k\} \in \Delta \text{ or } \{z_1, y_1, z_j\} \in \Delta.$$

When $\{z_1, y_1, y_i\} \in \Delta$, we obtain that $\{X_1Y_1, X_1Y_i\} \in G(I_{\mathrm{link}_\Delta\{z_1\}})$. This is a contradiction. When $\{z_1, y_1, w_k\} \in \Delta$, we can obtain a contradiction by a similar argument as in Case (2-a). Thus it is enough to consider the case $\{z_1, y_1, z_j\} \in \Delta$.

First we suppose that $\{y_1, y_2\} \in \Delta$.

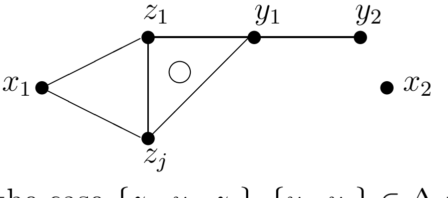

Figure: the case $\{z_1, y_1, z_j\}, \{y_1, y_2\} \in \Delta$ in Case (2-b)

Then $\mathrm{link}_\Delta\{y_1\}$ contains an egde $\{z_1, z_j\}$ and $\{y_2\}$. Since $\mathrm{link}_\Delta\{y_1\}$ is also connected, we can find vertices z_α, y_β such that $\{z_\alpha, y_\beta\} \in \mathrm{link}_\Delta\{y_1\}$. In particular, $\{z_\alpha, y_\beta, y_1\} \in \Delta$. This yields a contradiction because X_1Y_1, X_1Y_β is contained in $G(I_{\mathrm{link}_\Delta\{z_\alpha\}})$.

Next suppose that $\{y_1, w_1\} \in \Delta$.

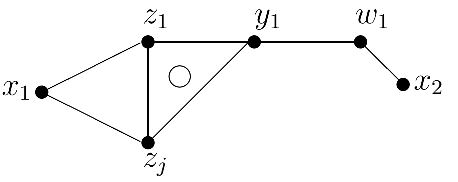

Figure: the case $\{z_1, y_1, z_j\}, \{y_1, w_1\} \in \Delta$ in Case (2-b)

Then $\mathrm{link}_\Delta\{y_1\}$ contains an egde $\{z_1, z_j\}$ and $\{w_1\}$. Since $\mathrm{link}_\Delta\{y_1\}$ is also connected, we can also find vertices z_α, y_β such that $\{z_\alpha, y_\beta\} \in \mathrm{link}_\Delta\{y_1\}$ (notice that $\{z_j, w_k\} \notin \Delta$). Hence we have $\dim \Delta = 1$. We complete the proof of Theorem 2.4.

Let Δ be a simplicial complex with $\dim \Delta \geq 2$. The Stanley–Reisner ring $K[\Delta]$ satisfies the Serre condition (S_2), that is, $\mathrm{depth}\, K[\Delta]_P \geq \min\{2, \mathrm{height}\, P\}$, if and only if Δ is pure and $\mathrm{link}_\Delta(F)$ is connected for every face F with $\dim \mathrm{link}_\Delta(F) \geq 1$.

Corollary 2.8. *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Assume that $K[\Delta]$ satisfies (S_2). Then the following conditions are equivalent*:

(1) *$K[\Delta]$ is a complete intersection.*
(2) *For any face F with $\dim \mathrm{link}_\Delta F = 1$, $\mathrm{link}_\Delta F$ is a complete intersection.*
(3) *There exists $W \subseteq V$ such that $\dim \Delta_{V \setminus W} \leq \dim \Delta - 3$ which satisfies the following condition*:

"*$\mathrm{link}_\Delta\{x\}$ is a complete intersection for all $x \in W$.*"

Proof. Note that Δ is pure. Put $d = \dim \Delta + 1$.

$(1) \Longrightarrow (3)$: It is enough to put $W = V$.

$(3) \Longrightarrow (2)$: Let W be a subset of V satisfying the condition (3). Let F be a face with $\dim \mathrm{link}_\Delta(F) = 1$. Since Δ is pure, $\sharp(F) = d - 1 - \dim \mathrm{link}_\Delta(F) = d - 2$. As $\dim \Delta_{V \setminus W} \leq d - 4$, F is not contained in $V \setminus W$. Thus there exists $i \in F$ such that $i \in W$. Then since $\mathrm{link}_\Delta\{i\}$ is a complete intersection by the assumption, $\mathrm{link}_\Delta(F)$ is also a complete intersection, as required.

$(2) \Longrightarrow (1)$: We use an induction on $d \geq 3$. First suppose that $d = 3$. Then for each $i \in V$, we have that $\dim \mathrm{link}_\Delta\{i\} = 1$. Hence $\mathrm{link}_\Delta\{i\}$ is a complete intersection by the assumption (3). Hence by Theorem 2.4, $K[\Delta]$ is a complete intersection.

Next suppose that $d \geq 4$. Let $i \in V$. Since $K[\Delta]$ satisfies (S_2), we have that $\Gamma = \mathrm{link}_\Delta\{i\}$ is connected and $\dim \Gamma = (d-1) - 1 = d - 2 \geq 2$. Moreover, for any face G in Γ with $\dim \mathrm{link}_\Gamma(G) = 1$, $\mathrm{link}_\Gamma(G) = \mathrm{link}_\Delta(G \cup \{i\})$ is a complete intersection by assumption. Hence, by the induction hypothesis, $K[\mathrm{link}_\Delta\{i\}]$ is a complete intersection. Therefore $K[\Delta]$ is a complete intersection by Theorem 2.4 again. $\square$

Combining Theorem 2.4 with Cowsik–Nori's theorem and Goto–Takayama's theorem, we get:

Corollary 2.9. *Let Δ be a simplicial complex with* $\dim \Delta \geq 2$*. Assume that Δ is pure and connected. Then the following conditions are equivalent*:

1. S/I_Δ^ℓ *is Cohen–Macaulay for every* $\ell \geq 1$.
2. S/I_Δ^ℓ *is Buchsbaum for every* $\ell \geq 1$.
3. S/I_Δ^ℓ *has (FLC) for every* $\ell \geq 1$.

If S/I_Δ^ℓ is (FLC) (resp. Cohen–Macaulay) for some positive integer ℓ, then S/I_Δ is Buchsbaum (resp. Cohen–Macaulay). In particular, Δ is pure. See [HTT, Theorem 2.6].

If Δ is *not* connected, then (2) and (3) are not necessarily equivalent. See below.

Example 2.10. Let $n \geq 2$ be a positive integer. Let

$$I = I_\Delta = (x_1, \ldots, x_n)(y_1 \ldots, y_n) \subseteq S = K[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

Then Δ is the disjoint union of the standard $(n-1)$-simplices. Moreover, S/I^ℓ has (FLC) for every $\ell \geq 1$ by Theorem 1.2. And one can see that S/I^ℓ is *not* Buchsbaum for every $\ell \geq 2$.

3. The case $\dim \Delta = 1$

Proposition 3.1. *Let Δ be a connected simplicial complex of* $\dim \Delta = 1$*. Then the following conditions are equivalent*:

1. $K[\Delta]$ *is a locally complete intersection.*
2. $K[\Delta]$ *is a locally Gorenstein.*
3. Δ *is either one of the following complexes*:
   - (a) *n-gon for some* $n \geq 3$;
   - (b) *n-pointed path for some* $n \geq 2$.

Proof. Suppose that $\dim \operatorname{link}_\Delta\{i\} = 0$. Then $\operatorname{link}_\Delta\{i\}$ consists of finite points. Hence if it is Gorenstein, then it is either one point or two points. Such a link is also a complete intersection. □

In the case $\dim \Delta = 1$, (1) and (3) in Corollary 2.9 is *not* equivalent in general. But we get the following result.

Proposition 3.2. *Let Δ be a simplicial complex with* $\dim \Delta = 1$*. Assume that Δ is pure and connected. Then the following conditions are equivalent*:

1. S/I_Δ^ℓ *is Cohen–Macaulay for every* $\ell \geq 1$.
2. S/I_Δ^ℓ *is Buchsbaum for every* $\ell \geq 1$.

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