G-prime and G-primary G-ideals on G-schemes

Mitsuyasu Hashimoto

Joint with Mitsuhiro Miyazaki

Nagoya University

3 October, 2008
## Notation

### Notation 1

Throughout this talk,

- $S$: scheme
- $G$: an $S$-group scheme flat of finite type
- $X$: a $G$-scheme (i.e., an $S$-scheme with a left $G$-action)

We always assume that $X$ is noetherian.

$\mu : G \times G \to G$ denotes the product, and $a : G \times X \to X$ denotes the action. Note that $a$ is flat of finite type.
**$G$-linearized $\mathcal{O}_X$-module**

**Definition 2 (Mumford)**

A $G$-linearized $\mathcal{O}_X$-module (an equivariant $(G, \mathcal{O}_X)$-module) is a pair $(\mathcal{M}, \Phi)$ such that $\mathcal{M}$ is an $\mathcal{O}_X$-module, and $\Phi : a^*\mathcal{M} \to p_2^*\mathcal{M}$ is an isomorphism of $\mathcal{O}_{G \times X}$-modules such that

$$(\mu \times 1_X)^*\Phi : (\mu \times 1_X)^*a^*\mathcal{M} \to (\mu \times 1_X)^*p_2^*\mathcal{M}$$

agrees with

$$(\mu \times 1_X)^*a^*\mathcal{M} \xrightarrow{\sim} (1_G \times a)^*a^*\mathcal{M} \xrightarrow{\Phi} (1_G \times a)^*p_2^*\mathcal{M} \xrightarrow{\sim} p_2^*a^*\mathcal{M} \xrightarrow{\Phi} p_2^*p_2^*\mathcal{M} \xrightarrow{\sim} (\mu \times 1_X)^*p_2^*\mathcal{M},$$

where $p_{23} : G \times G \times X \to G \times X$ is the projection.
Morphisms and submodules

**Definition 3**

A morphism $\varphi : (\mathcal{M}, \Phi) \rightarrow (\mathcal{N}, \Psi)$ of $G$-linearized $\mathcal{O}_X$-modules is a morphism $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ such that $\Psi \circ (a^* \varphi) = (p_2^* \varphi) \circ \Phi$.

**Definition 4**

Let $(\mathcal{M}, \Phi)$ be a $G$-linearized $\mathcal{O}_X$-module. We say that $\mathcal{N}$ is an equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ if $\mathcal{N}$ is an $\mathcal{O}_X$-submodule of $\mathcal{M}$, and $\Phi(a^* \mathcal{N}) = p_2^* \mathcal{N}$ (note that $a$ and $p_2$ are flat). If, moreover, $\mathcal{M} = \mathcal{O}_X$, then we say that $\mathcal{N}$ is a $G$-ideal of $\mathcal{O}_X$. 
The category $\text{Qch}(G, X)$

**Theorem 5 (H—)**

The category $\text{Qch}(G, X)$ of quasi-coherent $G$-linearized $\mathcal{O}_X$-modules is a locally noetherian abelian category, and $(\mathcal{M}, \Phi)$ is a noetherian object of $\text{Qch}(G, X)$ if and only if $\mathcal{M}$ is coherent. The forgetful functor $F_X : \text{Qch}(G, X) \to \text{Qch}(X)$ given by $(\mathcal{M}, \Phi) \mapsto \mathcal{M}$ is faithful exact, and admits a right adjoint.

If it is convenient and there is no danger, we omit the $\Phi$ of $(\mathcal{M}, \Phi)$, and we say that $\mathcal{M}$ is in $\text{Qch}(G, X)$. 
Operations on $\text{Qch}(G, X)$

Let $\mathcal{M}, \mathcal{N}, \mathcal{L}$ be in $\text{Qch}(G, X)$, $\mathcal{I}$ be a $G$-ideal, and $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, and $\mathcal{M}_\lambda$ be quasi-coherent equivariant $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$. Let $\mathcal{L}$ and $\mathcal{M}_3$ be coherent. Then the following modules have structures of quasi-coherent $G$-linearized $\mathcal{O}_X$-modules.

- $\Tor^\mathcal{O}_X_i(\mathcal{M}, \mathcal{N})$, $\Ext^i_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M})$,
- $H^i_\mathcal{I}(\mathcal{M}) \cong \lim_{\to} \Ext^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M})$,
- The Fitting ideal $\text{Fitt}_j(\mathcal{L})$,
- $\mathcal{M}_1 \cap \mathcal{M}_2$, $\sum_{\lambda} \mathcal{M}_\lambda$, $\mathcal{I}\mathcal{M}_1$,
- $\mathcal{M}_1 : \mathcal{M}_3$, $\mathcal{M}_1 : \mathcal{I}$, …
The star operation

Let $\mathcal{M}$ be in $\text{Qch}(G,X)$, and $m$ be an $\mathcal{O}_X$-submodule of $\mathcal{M}$. The sum of all quasi-coherent equivariant $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$ contained in $m$ is denoted by $m^*$. $m^*$ is the largest quasi-coherent equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ contained in $m$.

Remark 6
This notation goes back at least to Matijevic-Roberts paper in 1974.

Let $Y = V(\alpha)$ be a closed subscheme of $X$. Then $Y^* := V(\alpha^*)$ is the smallest $G$-stable closed subscheme of $X$ containing $Y$. 

Some formulas

From now on, all ideals and $G$-ideals are required to be coherent. All modules and $G$-linearized modules are required to be quasi-coherent.

Lemma 7

Let $\mathcal{M}$ be in $\text{Qch}(G, X)$, $m$, $n$, and $m_\lambda$ be $\mathcal{O}_X$-submodules of $\mathcal{M}$, and $\mathcal{N}$ be a coherent equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Let $\mathcal{I}$ be a $G$-ideal of $\mathcal{O}_X$. Then we have:

- $(\bigcap_\lambda m_\lambda^*)^* = (\bigcap_\lambda m_\lambda)^*$
- $m^* \cap n^* = (m \cap n)^*$
- $(m : \mathcal{N})^* = m^* : \mathcal{N}$
- $(m : \mathcal{I})^* = m^* : \mathcal{I}$
**Lemma 8**

Let \( \mathcal{P} \) be a \( G \)-ideal of \( \mathcal{O}_X \). Then the following are equivalent.

- There exists some ideal \( p \) of \( \mathcal{O}_X \) such that \( p \) is prime (i.e., \( V(p) \) is integral) and \( p^* = \mathcal{P} \).
- \( \mathcal{P} \neq \mathcal{O}_X \), and if \( I \) and \( J \) are \( G \)-ideals of \( \mathcal{O}_X \) and \( IJ \subset \mathcal{P} \), then \( I \subset \mathcal{P} \) or \( J \subset \mathcal{P} \).

**Definition 9**

If the equivalent conditions in the lemma are satisfied, we say that \( \mathcal{P} \) is a \( G \)-prime \( G \)-ideal.
The $G$-radical

**Definition 10**

Let $\mathcal{I}$ be a $G$-ideal of $\mathcal{O}_X$. Then $V_G(\mathcal{I})$ denotes the set of $G$-prime ideals containing $\mathcal{I}$. We set $\sqrt[\mathcal{G}]{\mathcal{I}} := (\bigcap_{\mathcal{P} \in V_G(\mathcal{I})} \mathcal{P})^*$, and call $\sqrt[\mathcal{G}]{\mathcal{I}}$ the $G$-radical of $\mathcal{I}$.

**Lemma 11**

Let $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{P}$ be $G$-ideals of $\mathcal{O}_X$. Then we have:

- $\mathcal{I} \subset \sqrt[\mathcal{G}]{\mathcal{I}} \subset \sqrt{\mathcal{I}}$, $\sqrt[\mathcal{G}]{\mathcal{I}} = \sqrt{\mathcal{I}}^*$
- If $\mathcal{I} \supset \mathcal{J}$, then $\sqrt[\mathcal{G}]{\mathcal{I}} \supset \sqrt[\mathcal{G}]{\mathcal{J}}$.
- $\sqrt[\mathcal{G}]{\mathcal{I}\mathcal{J}} = \sqrt[\mathcal{G}]{\mathcal{I}} \cap \sqrt[\mathcal{G}]{\mathcal{J}} = \sqrt{\mathcal{I}} \cap \sqrt{\mathcal{J}}$.
- $\sqrt[\mathcal{G}]{\sqrt[\mathcal{G}]{\mathcal{I}}} = \sqrt[\mathcal{G}]{\mathcal{I}}$.
- If $\mathcal{P}$ is a $G$-prime, then $\sqrt[\mathcal{G}]{\mathcal{P}} = \mathcal{P}$. 
**G-radical G-ideal**

**Lemma 12**
Let $\mathcal{I}$ be a $G$-ideal of $\mathcal{O}_X$. Then the following are equivalent.

- $\mathcal{I} = \sqrt[6]{\mathcal{I}}$
- $\mathcal{I}$ is the intersection of finitely many $G$-prime $G$-ideals.
- There exists some ideal $a$ of $\mathcal{O}_X$ such that $a$ is radical (i.e., $V(a)$ is reduced), and $a^* = \mathcal{I}$.

**Definition 13**
If the equivalent conditions in the lemma are satisfied, then we say that $\mathcal{I}$ is $G$-radical.

A $G$-prime $G$-ideal is $G$-radical.
From now on, until the end of the talk, let $\mathcal{M}$ be a coherent $G$-linearized $\mathcal{O}_X$-module, and $\mathcal{N}$ its coherent equivariant $(G, \mathcal{O}_X)$-submodule.

**Definition 14**

We say that $\mathcal{N}$ is $G$-primary if $\mathcal{N} \neq \mathcal{M}$, and for any coherent equivariant $(G, \mathcal{O}_X)$-submodule $\mathcal{L}$ of $\mathcal{M}$, either $\mathcal{N} : \mathcal{L} = \mathcal{O}_X$ or $\mathcal{N} : \mathcal{L} \subset \sqrt{\mathcal{N}} : \mathcal{M}$ holds.

If $\mathcal{N}$ is $G$-primary, then $\mathcal{P} = \sqrt{\mathcal{N}} : \mathcal{M}$ is $G$-prime. In this case, we say that $\mathcal{N}$ is $\mathcal{P}$-$G$-primary.
A criterion

Lemma 15

- For a prime ideal \( p \) of \( \mathcal{O}_X \), \( p^* \) is \( G \)-prime.
- For a radical ideal \( a \) of \( \mathcal{O}_X \), \( a^* \) is \( G \)-radical.
- If \( n \) is a \( p \)-primary \( \mathcal{O}_X \)-submodule of \( M \), then \( n^* \) is a \( p^*-G \)-primary submodule of \( M \).
- For a \( G \)-primary submodule \( N \) of \( M \), there exists some primary \( \mathcal{O}_X \)-submodule \( n \) of \( M \) such that \( n^* = N \).
\section*{G-primary decomposition}

\textbf{Definition 16}

An expression

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r$$

is called a \textit{G-primary decomposition} if this equation holds, and each $\mathcal{M}_i$ is a \textit{G-primary submodule} of $\mathcal{M}$. We say that the decomposition is \textit{minimal} if $\mathcal{N} \neq \bigcap_{j \neq i} \mathcal{M}_j$ for any $i$, and $\sqrt[\cap]{\mathcal{M}_i} : \mathcal{M}$ is distinct.
The existence

Proposition 17

\( \mathcal{N} \) has a minimal \( G \)-primary decomposition.

Proof.

Let

\[ \mathcal{N} = m_1 \cap \cdots \cap m_r \]

be a usual primary decomposition. Then

\[ \mathcal{N} = \mathcal{N}^* = (m_1 \cap \cdots \cap m_r)^* = m_1^* \cap \cdots \cap m_r^* \]

is a \( G \)-primary decomposition. We can make it minimal, as usual. \( \square \)
**Theorem 18**

The set

$$\text{Ass}_G(\mathcal{M}/\mathcal{N}) = \{ \sqrt[M_i]{\mathcal{M}} : \mathcal{M} \mid i = 1, \ldots, r \}$$

is independent of the choice of minimal $G$-primary decomposition

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r,$$

and depends only on $\mathcal{M}/\mathcal{N}$.

We call an element of $\text{Ass}_G(\mathcal{M}/\mathcal{N})$ a $G$-associated $G$-prime. The set of minimal elements of $\text{Ass}_G(\mathcal{M}/\mathcal{N})$ is denoted by $\text{Min}_G(\mathcal{M}/\mathcal{N})$, and its element is called a $G$-minimal $G$-prime. An element of $\text{Ass}_G(\mathcal{M}/\mathcal{N}) \setminus \text{Min}_G(\mathcal{M}/\mathcal{N})$ is called a $G$-embedded $G$-prime.
Let
\[ \mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r \]
be a minimal $G$-primary decomposition and
\[ \mathcal{M}_i = m_{i,1} \cap \cdots \cap m_{i,s_i} \]
a minimal primary decomposition. Then
\[ \mathcal{N} = \bigcap_{i=1}^{r} (m_{i,1} \cap \cdots \cap m_{i,s_i}) \]
is a minimal primary decomposition.
No embedded prime of $G$-primary submodule

**Proposition 20**

A $G$-primary submodule $N$ of $M$ does not have an embedded prime. For each minimal prime $p$ of $M/N$, we have $p^* = \sqrt[\ G \ ]{N} : M$.

**Corollary 21**

We have

$$\text{Ass}(M/N) = \bigcap_{i=1}^{s} \text{Ass}(M/M_i) = \bigcap_{P \in \text{Ass}_G(M/N)} \text{Ass}(O_X/P)$$

and

$$\text{Ass}_G(M/N) = \{p^* \mid p \in \text{Ass}(M/N)\}$$
Another corollary

Corollary 22

We have $\text{Ass}(\mathcal{M}/\mathcal{N}) = \text{Min}(\mathcal{M}/\mathcal{N})$ if and only if $\text{Ass}_G(\mathcal{M}/\mathcal{N}) = \text{Min}_G(\mathcal{M}/\mathcal{N})$. 
Lemma 23
Assume that $G$ is $S$-smooth. If $\mathfrak{a}$ is a radical ideal of $\mathcal{O}_X$, then $\mathfrak{a}^*$ is also radical. In particular, any $G$-radical $G$-ideal is radical.

Corollary 24
Assume that $G$ is $S$-smooth. If $\mathcal{I}$ is a $G$-ideal of $\mathcal{O}_X$, then $\sqrt{\mathcal{I}} = \mathcal{G}^{\mathcal{G}} \mathcal{I}$. In particular, $\sqrt{\mathcal{I}}$ is a $G$-radical $G$-ideal.
Groups with connected fibers

Lemma 25
Assume that \( G \to S \) has connected fibers. If \( q \) is a primary ideal of \( O_X \), then \( q^* \) is also primary. In particular, a \( G \)-primary \( G \)-ideal is primary.

Corollary 26
Assume that \( G \to S \) has connected fibers. If \( \mathcal{I} \) is a \( G \)-ideal, then a minimal \( G \)-primary decomposition of \( \mathcal{I} \) is also a minimal primary decomposition.
Smooth groups with connected fibers

Corollary 27

Assume that $G \to S$ is smooth with connected fibers. If $p$ is a prime, then $p^*$ is also a prime. Any $G$-prime $G$-ideal is a prime. For a $G$-ideal $\mathcal{I}$ of $\mathcal{O}_X$, any associated prime of $\mathcal{I}$ is a $G$-prime $G$-ideal.
The dimension of the fiber

**Theorem 28**

Let $0$ be $G$-primary in $\mathcal{O}_X$. Then the dimension of the fiber of $p_2 : G \times X \to X$ is constant.
**$G$-primary implies equi-dimensional**

**Theorem 29**

Let $0$ be $G$-primary in $\mathcal{O}_X$. If $X$ has an affine open covering $(\text{Spec } A_i)$ such that each $A_i$ is Hilbert, universally catenary, and for any minimal prime of $P$ of $A_i$, the heights of maximal ideals of $A_i/P$ are the same (for example, $X$ is of finite type over a field or $\mathbb{Z}$). Then the dimensions of the irreducible components of $X$ are the same.

**Remark 30**

There is an example of $G = X$ such that the dimensions of the irreducible components are different. The red assumptions are necessary.
**Theorem 31**

Let \( Q \) be a \( G \)-primary \( G \)-ideal of \( \mathcal{O}_X \). Let \( x \) and \( y \) be the generic points of irreducible components of \( V(Q) \). Then \( \dim \mathcal{M}_x = \dim \mathcal{M}_y \).
Matijevic–Roberts type theorem

Theorem 32

Let \( y \in X \) and \( Y = \bar{y} \). Let \( \eta \) be the generic point of an irreducible component of \( Y^* \). Then:

- If \( M_\eta \) is maximal Cohen–Macaulay (resp. of finite injective dimension, projective dimension \( m \), \( \dim - \operatorname{depth} = n \), torsionless, reflexive, G-dimension \( g \)), then so is \( M_y \).
- If \( O_{X, \eta} \) is a complete intersection, then so is \( O_{X, y} \).
- If \( G \) is smooth and \( O_{X, \eta} \) is regular, then \( O_{X, y} \) is regular.
- Assume that \( G \) is smooth and \( X \) is a locally excellent \( \mathbb{F}_p \)-scheme. If \( O_{X, \eta} \) is weakly \( F \)-regular (resp. \( F \)-regular, \( F \)-rational), then so is \( O_{X, y} \).
A Corollary on graded rings

Consider the case $S = \text{Spec } \mathbb{Z}$, $G = \mathbb{G}_m^n$, and $X = \text{Spec } A$ is affine. Then $A$ is a $\mathbb{Z}^n$-graded ring.

Corollary 33

Let $A$ be a locally excellent $\mathbb{Z}^n$-graded $\mathbb{F}_p$-algebra. Let $P$ be a prime ideal of $A$, and let $P^*$ be the prime ideal generated by homogeneous elements of $P$. If $A_{P^*}$ is weakly $F$-regular (resp. $F$-regular, $F$-rational), then so is $A_P$. 

M. Hashimoto (joint with M. Miyazaki)
A history of Matijevic–Roberts type theorem

Theorem 32 for graded rings (i.e., the case that $S = \text{Spec} \mathbb{Z}$, $G = \mathbb{G}_m^n$, and $X$ affine) (excluding (weak) $F$-regularity and $F$-rationality):

- Conjectured by Nagata (for the case $n = 1$, for Cohen–Macaulay property).

General case (again excluding (weak) $F$-regularity and $F$-rationality):

- The case that $S$ is noetherian affine, and $G$ is affine, smooth with connected fibers (H—)
- $G$ is smooth with connected fibers (Ohtani - H—, unpublished)
- General case: Theorem 32
$G$-artinian $G$-schemes

**Definition 34**

$X$ is said to be **$G$-artinian** if every $G$-prime of $\mathcal{O}_X$ is a $G$-minimal prime of $0$.

**Corollary 35**

A $G$-artinian $G$-scheme is Cohen–Macaulay.
Thank you. This slide is available at Hashimoto’s home page (by the next Tuesday).