U-invariants of a G-action

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Notation

Notation 1

Throughout this talk,

- k: algebraically closed field
- G: a (connected) reductive group over k
- T: a maximal torus of G
- Δ: a base of the set of roots of G
- B: the negative Borel subgroup of G
- I: a subset of Δ
- $L = L_I$: the Levi subgroup $C_G(\bigcap_{\alpha \in I}(\operatorname{Ker}\alpha)^\circ)$
- $P = P_I$: the parabolic subgroup generated by L and B

Notation (continued)

Notation 2

- $U = B_u$: the unipotent radical of B
- $V = P_u$: the unipotent radical of P
- $D := B \cap L$, $W := D_u$.
- S: a G-algebra, $X = \operatorname{Spec} S$.
- $X := X(T)$, X_G^+: the set of dominant weights of G
- For $\lambda \in X^+$, $\nabla_G(\lambda) := \operatorname{ind}_B^G \lambda$, the dual Weyl module of heighest weight λ.

Example (1)

- $G := \mathrm{GL}_n$
- $T = \{\text{diagomal matrices in } G\}$
- Then

$$X(T) = \mathrm{Hom}_{\mathrm{Alg.Grp.}}(T, \mathrm{GL}_1) = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{\lambda_1} \cdots t_n^{\lambda_n} \mid \lambda \in \mathbb{Z}^n \right\} \cong \mathbb{Z}^n$$

Example (2)

- The set of roots is

$$\{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n, \quad i \neq j\} \subset \mathbb{Z}^n \cong X(T),$$

  where $\varepsilon_i = (0, \dots, 1, \dots, 0)$ (1 is in the ith position).
- $\Delta := \{\alpha_i \mid 1 \leq i < n\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.
- Then $B = \{\text{lower triangular matrices in } G\}$.
- $I := \Delta \setminus \{\alpha_{a_1}, \dots, \alpha_{a_{s-1}}\}$ ($1 \leq a_1 < \cdots < a_{s-1} < n$).
- Then

$$L = \left\{ \begin{pmatrix} A_1 & & O \\ & \ddots & \\ O & & A_s \end{pmatrix} \in G \;\middle|\; A_l \in \mathrm{GL}_{a_l - a_{l-1}} \right\}$$

Example (3)

and

$$P = \left\{ \begin{pmatrix} A_1 & & O \\ & \ddots & \\ * & & A_s \end{pmatrix} \in G \mid A_l \in \mathrm{GL}_{a_l - a_{l-1}} \right\}$$

where $a_0 = 0$ and $a_s = n$.

- Then

$$U = B_u = \left\{ \begin{pmatrix} 1 & & O \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}$$

Example (4)

- Then

$$V = P_u = \left\{ \begin{pmatrix} A_1 & & O \\ & \ddots & \\ * & & A_s \end{pmatrix} \in P \mid A_l = E_{a_l - a_{l-1}} (1 \leq l \leq s) \right\}$$

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$$D = B \cap L = \left\{ \begin{pmatrix} A_1 & & O \\ & \ddots & \\ O & & A_s \end{pmatrix} \in G \mid A_l : \text{ lower triangular} \right\}$$

Example (5)

- Then
$$W = D_u = \left\{ \begin{pmatrix} A_1 & & O \\ & \ddots & \\ O & & A_s \end{pmatrix} \in U \cap D \right\}$$
- $X_G^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\} \subset \mathbb{Z}^n \cong X(T)$.
- The dual Weyl module is nothing but the tensor product of a Schur module and a one-dimensional representation.

Grosshans's isomorphism

Lemma 3 (Grosshans)

Let H be a closed subgroup of G. Then

$$S^H \cong (k[G]^H \otimes S)^G,$$

where for $f \in k[G] \otimes S = k[G \times X]$, $f \in \mathrm{RHS}$ means $f(hgg_1^{-1}, g_1x) = f(g,x)$ for $g, g_1 \in G$, $h \in H$, and $x \in X$.

The invariant subring S^H of a G-algebra is much better than T^H for a mere H-algebra T.

Theorem 4

$k[G]^U$ is of finite type over k. It has rational singularities if $\mathrm{char}(k) = 0$, and is F-regular if $\mathrm{char}(k) > 0$.

Proof.

This is more or less well-known. First consider the case that G is simply connected semisimple. Then $k[G]^U$ is the total coordinate ring of the flag variety G/B. It is finitely generated by Ramanan–Ramanathan. It has rational singularities by Kempf. G/B is globally F-regular, and $k[G]^U$ is F-regular. The general case follows easily. ☐

S^U has rational singularities

Theorem 5

Let $\mathrm{char}(k) = 0$. If S is of finite type and has rational singularities, then so is S^U.

Proof.

We have seen that $k[G]^U$ is finitely generated and has rational singularities. So $k[G]^U \otimes S$ is finitely generated and has rational singularities by Elkik. So

$$S^U \cong (k[G]^U \otimes S)^G$$

is also finitely generated and has rational singularities by Hilbert and Boutot. ☐

Popov's filtration

Theorem 6 (Popov)

Assume that $\mathrm{char}(k) = 0$. There is a filtration (S_t) of S (i.e., $1 \in S_0$, $S_{-1} = 0$, $S_i S_j \subset S_{i+j}$, $\bigcup_i S_i = S$) consisting of G-submodules of S such that the associated graded ring is $k[G]^U \# S^U$, where $\#$ denotes the Segre product with respect to the $X(T)$-grading.

Corollary 7

Assume that $\mathrm{char}(k) = 0$. S is finitely generated and has rational singularities if and only if S^U is finitely generated and has rational singularities.

Grosshans filtration

Grosshans considered the same filtration in characteristic $p > 0$, and proved the following.

Theorem 8 (Grosshans)

S is finitely generated if and only if S^U is finitely generated.

Good filtration

Definition 9

A finite dimensional G-module M is said to be good if there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for each i, there exists some $\lambda(i) \in X_G^+$ such that $M_i/M_{i-1} \cong \nabla_G(\lambda(i))$. In general, a G-module M is said to be good if it is a filtered inductive limit of finite dimensional good G-modules.

Theorem 10 (Grosshans)

If S is good, then S has a filtration (S_t) consisting of G-modules such that the associated graded ring is $k[G]^U \# S^U$.

A corollary

Corollary 11 (Grosshans)

If S is good and S^U is reduced (resp. normal, F-rational), then S is reduced (resp. normal, F-rational).

A question

Question 12

Assume that $\mathrm{char}(k) > 0$. Let S be good. Then, can we say that S is strongly F-regular if and only if S^U is strongly F-regular?

Good symmetric algebra

Theorem 13 (H–)

Assume that $\mathrm{char}(k) = p > 0$. Let S be good, and $S = \mathrm{Sym}\, M$ for some finite dimensional G-module M. Then S^G is strongly F-regular.

Main theorem

Theorem 14 (Main Theorem)

Assume that $\mathrm{char}(k) = p > 0$. Let S be good, and $S = \mathrm{Sym}\, M$ for some finite dimensional G-module M. Then S^U is strongly F-regular.

A lemma on good modles

Theorem 15 (Donkin)

Let w_0 (resp. w_L) denote the longest element of the Weyl group of G (resp. L). For $\lambda \in X_G^+$, we have $\nabla_G(\lambda)^V \cong \nabla_L(w_P w_0 \lambda)$.

Theorem 16 (Donkin)

Let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be a short exact sequence of G-modules. If M_1 is good, then

$$0 \longrightarrow M_1^V \longrightarrow M_2^V \longrightarrow M_3^V \longrightarrow 0$$

is exact.

A lemma

From these two theorems due to Donkin, we immediately get

Lemma 17

Let M be a good G-module. Then M^V is good as an L-module.

A corollary to the main theorem

Corollary 18

Let $S = \operatorname{Sym} M$ be as in the theorem. Then S^V is finitely generated and strongly F-regular UFD. In particular, S^V is Gorenstein.

Proof.

By the theorem, $(S^V)^W \cong S^U$ is finitely generated and strongly F-regular. So by Lemma 17 and Corollary 11, S^V is finitely generated and F-rational. On the other hand, S^V is a UFD, since S is a polynomial ring and V is unipotent. □

Steinberg module

To prove Main Theorem, replacing G appropriately, we may assume that the half sum of positive roots ρ is in $X(T)$. Set $St_r := \nabla_G((p^r - 1)\rho)$ for $r \geq 1$, and call it the rth Steinberg module.

Lemma 19

For any finite dimensional G module M, there exists some $r > 0$ such that for any subquotient N of M, any nonzero map $St_r \to St_r \otimes N$ splits.

Let U_r denote the rth Frobeniius kernel of U. Then $St_r^{U_r} \cong k$ as U/U_r-modules. In particular, for any U-module M, $(M^U)^{(r)} \to (M^{(r)} \otimes St_r)^U$ is an isomorphism.

Discussion

Take a homogeneous element $a \in A := S^U$ (with respect to both the $X(T)$-grading and the usual grading of S) such that $A[1/a]$ is regular. It suffices to show that there exists some $r > 0$ such that $aF^r : A^{(r)} \to A$ $(aF^r(x^{(r)}) = ax^{p^r})$ splits.
The degree with respect to the $X(T)$-grading of a is of the form $w_0\lambda$, with λ being dominant, where w_0 is the longest element of the Weyl group. Let $a \in S_d$. Take $r \gg 0$ so that

- $a \notin \mathfrak{m}^{(r)}S$, where $\mathfrak{m} = \bigoplus_{i>0} S_i$
- For any subquotient N of $\nabla_G(\lambda) \otimes S_d$, any nonzero map $St_r \to N \otimes St_r$ splits.

Discussion (Continued)

Lemma 20

Let a and r be as in the last page. Then

(1) $aF^r \otimes 1 : S^{(r)} \otimes St_r \to S_{<d+1} \otimes St_r$ splits as a $(U, S^{(r)})$-linear map.

(2) (H—, 2001) The inclusion $S_{<d+1} \otimes St_r \hookrightarrow S \otimes St_r$ splits as a $(G, S^{(r)})$-linear map.

Consequently, there is a $(U, S^{(r)})$-linear map

$$\psi : S \otimes St_r \to S^{(r)} \otimes St_r$$

such that $\psi a F^r = \mathrm{id}$.

Consider the commutative diagram of $(U, A^{(r)})$-modules

$$\begin{tikzcd}
A^{(r)} \otimes St_r \arrow[r, "i^{(r)}\otimes 1"] \arrow[d, "aF^r\otimes 1"] & S^{(r)} \otimes St_r \arrow[r, "\mathrm{id}"] \arrow[d, "aF^r\otimes 1"] & S^{(r)} \otimes St_r \, , \\
A \otimes St_r \arrow[r, "i\otimes 1"] & S \otimes St_r \arrow[ur, "\psi"] &
\end{tikzcd}$$

where $i : A \hookrightarrow S$ is the inclusion. Applying the functor $(?)^U$, we get

$$\begin{tikzcd}
A^{(r)} \arrow[r, "\mathrm{id}"] \arrow[d, "aF^r"] & A^{(r)} \arrow[r, "\mathrm{id}"] \arrow[d] & A^{(r)} \, . \quad \square \\
A \arrow[r] & (S \otimes St_r)^U \arrow[ur] &
\end{tikzcd}$$

An application

Let (Q_0, Q_1, s, t) be a finite quiver, and $d : Q_0 \to \mathbb{N}$ a dimension vector. Set $V_i := k^{d(i)}$, and let $G_i \subset \mathrm{GL}(V_i)$ be any of the following:

(1) $\mathrm{GL}(V_i)$, $\mathrm{SL}(V_i)$, $\mathrm{PSL}(V_i)$
(2) $\mathrm{Sp}_{d(i)}$ ($d(i)$: even)
(3) $\mathrm{SO}_{d(i)}$ ($\mathrm{char}(k) \neq 2$)
(4) Levi subgroup of any of (1)–(3)
(5) Derived subgroup of (1)–(4)
(6) Unipotent radical of a parabolic subgroup of (1)–(5).

Set $G := \prod_{i \in Q_0} G_i$ and $V := \prod_{\alpha \in Q_1} \mathrm{Hom}(V_{s(\alpha)}, V_{t(\alpha)})$. Then $(\mathrm{Sym}\, V^*)^G$ is strongly F-regular.

An example (1)

Let $M := k^m$, and $N := k^n$. Let $0 = a_0 < a_1 < \cdots < a_s = n$, and

$$H := \begin{pmatrix} H_1 & & & \\ & H_2 & & * \\ & & \ddots & \\ & O & & H_s \end{pmatrix} \subset \mathrm{GL}_n(k) \cong \mathrm{GL}(N),$$

where H_i is either $\mathrm{GL}_{a_i - a_{i-1}}$, $\mathrm{SL}_{a_i - a_{i-1}}$, or $\{E_{a_i - a_{i-1}}\}$.

An example (2)

Set $h := \inf\{l \mid H_l = \mathrm{GL}_{a_l - a_{l-1}}\}$. For $l < h$, set

$$\Gamma_l := \{[c_1, \dots, c_{a_l} \mid 1, \dots, a_l] \mid 1 \leq c_1 < \cdots < c_{a_l} \leq m\}$$

if $H_i = \mathrm{SL}_{a_i - a_{i-1}}$, and

$$\begin{aligned} \Gamma_l := \{[c_1, \dots, c_u \mid d_1, \dots, d_u] \mid a_{l-1} < u \leq a_l, \\ 1 \leq c_1 < \cdots < c_u \leq m,\ 1 \leq d_1 < \cdots < d_u \leq a_l, \\ d_t = t\ (t \leq a_{l-1})\} \end{aligned}$$

if $H_i = \{E_{a_i - a_{i-1}}\}$. Set $\Gamma := \bigcup_{l<h} \Gamma_l$. Note that Γ is a distributive lattice.

An example (3)

Theorem 21

Let $S := \mathrm{Sym}(M \otimes N)$, and let H act on S via $h \cdot (m \otimes n) = m \otimes h(n)$. Set $A := S^H$. Then

- Consider that Γ as a subset of S via

  $$[c_1, \ldots, c_u \mid d_1, \ldots, d_u] \mapsto \det(e_{c_i} \otimes f_{d_j}),$$

  where $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ are standard bases of $M = k^m$ and $N = k^n$, respectively. Then A is generated by Γ, and is an ASL over Γ.
- A is a strongly F-regular UFD.

Remarks

Remark 22

The theorem for the crucial case $s = 2$, $a_1 = l$, $H_1 = \{E_l\}$, and $H_2 = \mathrm{SL}_{m-l}$ is due to Goto–Hayasaka–Kurano–Nakamura.

Remark 23

The theorem for the case $s = n$, and $a_i = i$, $H_i = \{1\}$ $(i = 1, \dots, n)$ is contained in M. Miyazaki's result (talked at the 29th Symposium on Commutative Algebra in Nagoya) as a very special case.

SubASL

We sketch the proof of Theorem 21.

Lemma 24

Let S be a graded ASL on a poset Σ over k. Let Γ be a subset of Σ such that for any two incomparable elements $\sigma, \tau \in \Gamma$,

$$\sigma\tau = \sum c_i m_i$$

in S with each m_i in the sum being a monomial of Γ divisible by an element ξ_i in Γ smaller than both σ and τ. Then $k[\Gamma] \subset S$ is a graded ASL on Γ.

If this is the case, we say that $k[\Gamma]$ is a subASL of S.

Proof of Theorem 21 (1)

It is easy to see that $k[\Gamma] \subset S^H = A$ and $k[\Gamma] \subset S = \mathrm{Sym}(M \otimes N)$ is a subASL. It remains to show that $k[\Gamma] = A$. To verify this, it suffices to show that $\dim k[\Gamma]_d = \dim A_d$ for $d \geq 0$. By Akin–Buchsbaum–Weyman's Cauchy formula, there exists a good filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{r(d)}$$

and a bijection ν from $\{1, \ldots, r(d)\}$ to the set $\{\lambda = (\lambda_1, \ldots, \lambda_h) \mid \lambda_1 \geq \cdots \geq \lambda_h \geq 0\}$ such that $F_i/F_{i-1} \cong \nabla_{\mathrm{GL}(M)}(\nu(i)) \otimes \nabla_{\mathrm{GL}(N)}(\nu(i))$, where $h = \min(m, n)$.

Proof of Theorem 21 (2)

By Donkin's theorem (Theorem 16) and Lemma 17,

Lemma 25

For a short exact sequence of $\mathrm{GL}(M) \times \mathrm{GL}(N)$-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

with M_1 good,

$$0 \to M_1^H \to M_2^H \to M_3^H \to 0$$

is exact.

Moreover, utilizing Donkin's theorem (Theorem 15), we have

Proof of Theorem 21 (3)

Lemma 26

For $\lambda = (\lambda_1, \dots, \lambda_n) \in X^+_{\mathrm{GL}(N)}$,

$$\nabla_{\mathrm{GL}(N)}(\lambda)^H \cong \begin{cases} \nabla_{\mathrm{GL}_{a_1}}(\lambda(1)) \otimes \cdots \otimes \nabla_{\mathrm{GL}_{a_s - a_{s-1}}}(\lambda(s)) & (*)(\lambda) \text{ holds} \\ 0 & \textit{otherwise} \end{cases}$$

as P/H modules, where $P = \{(a_{ij}) \in \mathrm{GL}_n \mid \exists l \; i > a_l \geq j \Rightarrow a_{ij} = 0\}$, $\lambda(l) = (\lambda_{a_{l-1}+1}, \dots, \lambda_{a_l})$ for each l, and we say that $(*)(\lambda)$ holds if for each l, the following hold:

- $\lambda(l) = (0, 0, \dots, 0)$ if $H_l = \mathrm{GL}$;
- $\lambda(l) = (t, t, \dots, t)$ for some $t \geq 0$ if $H_l = \mathrm{SL}$;
- $\lambda(l)$ may be arbitrary if H_l is trivial.

Proof of Theorem 21 (4)

Lemma 27

$$\dim S_d^H = \sum_{\substack{\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0), \\ |\lambda| = d,\ (*)(\lambda) \text{ holds}}} \dim \nabla_{\mathrm{GL}(M)}(\lambda) \cdot \prod_{l} \dim \nabla(\lambda(l)),$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_n$.

Proof of Theorem 21 (5)

Next we count the dimension of $k[\Gamma]_d$. It is the number of standard monomials of degree d of Γ. For a standard monomial

$$m = [c_{1,1}, \ldots, c_{1,\mu_1} \mid d_{1,1}, \ldots, d_{1,\mu_1}] \cdot [c_{2,1}, \ldots, c_{2,\mu_2} \mid d_{2,1}, \ldots, d_{2,\mu_2}] \cdots [c_{r,1}, \ldots, c_{r,\mu_r} \mid d_{r,1}, \ldots, d_{r,\mu_r}],$$

we define $\mu(m) := (\mu_1, \ldots, \mu_r)$ and $\lambda(m) = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = \#\{j \mid \mu_j \geq i\}$. Then $m \in S_d$ if and only if $|\mu| = |\lambda| = d$. Such a monomial m of Γ exists if and only if $(*)(\lambda)$ holds. If this is the case, it is not so difficult to show that the number of standard monomials m of degree d of Γ such that $\lambda(m) = \lambda$ is the product $\dim \nabla_{\mathrm{GL}(M)}(\lambda) \cdot \prod_l \dim \nabla(\lambda(l))$ by famous

Proof of Theorem 21 (6)

Lemma 28

Let $\mu = (\mu_1, \dots, \mu_r)$ be a partition with $\mu_1 \leq n$, and $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i = \#\{j \mid \mu_j \geq i\}$. Then the number of standard monomials m with $\mu(m) = \mu$ agrees with $\dim \nabla_{\mathrm{GL}_n}(\lambda)$.

This completes the proof of the theorem.

This is the end of the talk

Thank you. This slide is available at Hashimoto's web page.