Schur algebras

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1 Polynomial representations of GL_n

(1.1) Schur algebras, found by I. Schur at the begining of the 20th century, is a powerful tool to study polynomial representations of general linear group. The purpose of this section is to study the relationship of Schur algebras and the polynomial representations of GL_n.

(1.2) Let k be an algebraically closed field of arbitrary characteristic.

For a ring A, an A-module means a left A-module, unless otherwise specified. However, an ideal of A means a two-sided ideal, not a left ideal. $A\,\mathrm{mod}$ denotes the category of finitely generated A-modules.

For a group G, a G-module means a kG-module, where kG is the group algebra of G over k. If V is a finite dimensional vector space, then giving a G-module structure to V is the same thing as giving a group homomorphism $\rho : G \rightarrow GL(V)$.

A finite dimensional $GL_n(k)$-module $V \cong k^m$ is said to be a polynomial (resp. rational) representation if the corresponding group homomorphism $\rho : GL_n(k) \rightarrow GL(V) \cong GL_m(k)$ satisfies the following. For each $(a_{ij}) \in GL_n(k)$, when we write $\rho(a_{ij}) = (\rho_{st}(a_{ij}))$, then each $\rho_{st}(a_{ij})$ is a polynomial function (resp. rational function everywhere defined on GL_n) in a_{ij}. We may also say that ρ is a polynomial (resp. rational) representation. Note that this condition is independent of the choice of the basis of V.

(1.3) Let V be a $GL_n(k)$-module which may not be finite dimensional. We say that V is a polynomial (resp. rational) representation of GL_n if $V = \bigcup_W W$, where W runs through all the finite dimensional GL_n-submodules of V which are polynomial (resp. rational) representations.

(1.4) If $\rho : GL_n(k) \to GL_m(k)$ is a polynomial representation, and if there exists some $r \geq 0$ such that for any s, t, ρ_{st} is a homogeneous polynomial of degree r, then we say that ρ is a polynomial representation of degree r. This notion is also independent of the choice of basis.

(1.5) We give some examples. The one-dimensional representation

$$\det^m : GL_n(k) \to GL_1(k) = k^\times$$

given by $A \mapsto \det(A)^m$ is a polynomial representation of degree mn for $m \geq 0$.

(1.6) The map $\rho : GL_2(k) \to GL_3(k)$ given by

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x^2 & xy & y^2 \\ 2xz & xw + yz & 2yw \\ z^2 & zw & w^2 \end{pmatrix}$$

is a polynomial representation of degree two.

1.7 Exercise. Show (1.6).

(1.8) $\rho : GL_n \to GL_m$ is a rational representation if and only if $A \mapsto \rho(A) \cdot \det(A)^s$ is a polynomial representation for some $s \geq 0$. Thus there is not much difference between rational representations and polynomial representations, and most problems for rational representations are reduced to those for polynomial representations.

(1.9) The identity map $GL(V) \to GL(V)$ is obviously a polynomial representation of degree one. This representation is called the vector representation of $GL(V)$.

(1.10) If V is a polynomial representation of GL_n and W is a GL_n-submodule of V (that is, W is a k-subspace of V, and $Aw \in W$ for any $A \in GL_n$ and $w \in W$), then W and V/W are polynomial representations. If, moreover, V is of degree r, then W and V/W are of degree r.

1.11 Exercise. Show (1.10).

1.12 Exercise. Let V be a finite dimensional $GL_n(k)$-module and W be its $GL_n(k)$-submodule. Show by an example that even if W and V/W are polynomial representations, V may not be so.

(1.13) For two polynomial representations V and W of GL_n, the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are polynomial representation. $A(v+w) = Av + Aw$ in $V \oplus W$, and $A(v \otimes w) = Av \otimes Aw$ in $V \otimes W$. If V and W are of degree r, then so is $V \oplus W$. If V and W are of degree r and r' respectively, then $V \otimes W$ is of degree $r + r'$. It is easy to see that an infinite direct sum of polynomial representations of GL_n is a polynomial representation.

(1.14) Let V be a finite dimensional rational representation of GL_n, and W be a rational representation of GL_n. Then $\operatorname{Hom}(V, W)$ is a rational representation of GL_n again. The action is given by $(g\varphi)(v) = g(\varphi(g^{-1}(v)))$ for $g \in GL_n(k)$, $\varphi \in \operatorname{Hom}(V, W)$, and $v \in V$. In particular, $V^* = \operatorname{Hom}(V, k)$ is a rational representation. As g^{-1} is involved, even if both V and W are polynomial representations, $\operatorname{Hom}(V, W)$ may not be so. Note that $\operatorname{Hom}(V, W) \cong W \otimes V^*$ as a $GL_n(k)$-module. In a functorial notation, the action of $g \in GL_n$ on V^* is given by the action of $(g^*)^{-1} = \operatorname{Hom}(g, k)^{-1} = \operatorname{Hom}(g^{-1}, k)$.

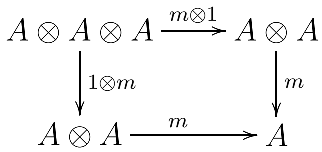
(1.15) Let V be a polynomial representation of GL_n. Then $V^{\otimes d}$ is so. Let $TV := \bigoplus_{d \geq 0} V^{\otimes d}$ be the tensor algebra. Then GL_n acts on it, and the two sided ideals $TV(v \otimes w - w \otimes v \mid v, w \in V)TV$ and $TV(v \otimes v \mid v \in V)TV$ are GL_n-submodules of TV. So the quotient algebras $\operatorname{Sym} V$ and $\bigwedge V$ admit GL_n-algebra structure such that $TV \to \operatorname{Sym} V$ and $TV \to \bigwedge V$ preserve degree. Being quotients of $V^{\otimes d}$, $\operatorname{Sym}_d V$ and $\bigwedge^d V$ are also polynomial representations. If V is of degree r, then $V^{\otimes d}$, $\operatorname{Sym}_d V$, and $\bigwedge^d V$ are of degree rd.

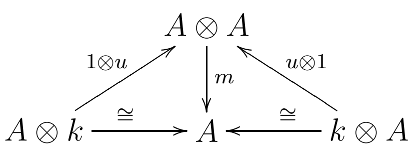
(1.16) For a k-vector space V, we define $D_d V := (\operatorname{Sym}_d V^*)^*$. We call $D_d V$ the dth divided power of V. If V is a polynomial representation of GL_n, then so is $D_d V$. Indeed, in a functorial language, $g \in GL_n(k)$ acts on $D_d V$ by

$$((\operatorname{Sym}(g^*)^{-1})^{-1})^* = (((\operatorname{Sym} g^*)^{-1})^{-1})^* = (\operatorname{Sym} g^*)^*.$$

If the matrix of g is A, then the matrix of g^* with respect to the dual basis is the transpose ${}^tA$. So $D_d V$ is a polynomial representation.

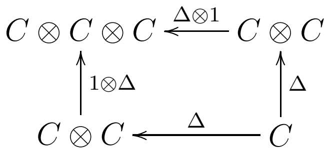
(1.17) Let B be a k-algebra. Then the product map $m_B : B \otimes B \to B$ and the unit map $u : k \to B$ are defined by $m_B(b \otimes b') = bb'$ and $u(a) = a$, respectively, and the diagrams

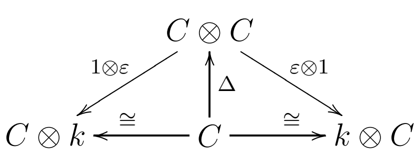




are commutative, because of the associativity law and the unit law.

Reversing the directions of arrows, we get the definition of coalgebras. We say that $C = (C, \Delta, \varepsilon)$ is a k-coalgebra if k-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ are given, and the diagrams





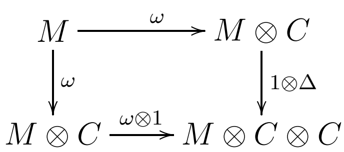
are commutative. The commutativity of the first diagram is called the coassociativity law, while the commutativity of the second diagram is called the counit law.

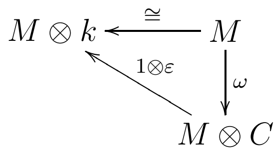
(1.18) If C is a k-coalgebra and $c \in C$, then $\Delta(c)$ is sometimes denoted by $\sum_{(c)} c_{(1)} \otimes c_{(2)}$ (Sweedler's notation). $(\Delta \otimes 1)\Delta(c) = (1 \otimes \Delta)\Delta(c)$ is denoted by $\sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, and so on. The counit law is expressed as

$$\sum_{(c)} \varepsilon(c_{(2)})c_{(1)} = \sum_{(c)} \varepsilon(c_{(1)})c_{(2)} = c$$

for any $c \in C$. For more about coalgebras and related notion, see [Sw].

(1.19) A right C-comodule is a k-vector space M with a map $\omega_M : M \to M \otimes C$ such that the diagrams





are commutative. The commutativity of the first diagram is called the coassociativity law, and the second one is called the counit law. For $m \in M$, $\omega(m)$ is denoted by $\sum_{(m)} m_{(0)} \otimes m_{(1)} \in M \otimes C$. $(1 \otimes \Delta)\omega(m) = (\omega \otimes 1)\omega(m)$ is denoted by $\sum_{(m)} m_{(0)} \otimes m_{(1)} \otimes m_{(2)} \in M \otimes C \otimes C$, and so on.

(1.20) A map $f : D \to C$ between two k-coalgebras is called a coalgebra map if it is k-linear, $\Delta_C f = (f \otimes f)\Delta_D$, and $\varepsilon_C f = \varepsilon_D$.

For a k-coalgebra C, right C-comodules M and N, and a map $f : M \to N$, we say that f is a comodule map if f is k-linear, and $\omega_N f = (f \otimes 1_C)\omega_M$. The identity map and the composite of two comodule maps are comodule maps, and the category of right C-comodules $\operatorname{Comod} C$ is obtained. Note that $\operatorname{Comod} C$ is an abelian k-category.

(1.21) If C is a k-coalgebra, then the dual C^* is a k-algebra with the product given by

$$(\varphi\psi)(c) = \sum_{(c)} (\varphi(c_{(1)}))(\psi(c_{(2)}))$$

for $\varphi, \psi \in C^*$. The k-algebra C^* is called the dual algebra of C. If M is a right C-comodule, then M is a left C^*-module with the structure given by

$$\varphi m = \sum_{(m)} (\varphi m_{(1)}) m_{(0)}.$$

This gives a functor $\operatorname{Comod} C \to C \operatorname{Mod}$ $(M \mapsto M)$. It is obviously exact, and known to be fully faithful. If C is finite dimensional, it is an equivalence.

1.22 Exercise. Check (1.21).

(1.23) Given a polynomial representation $\rho : GL_n(k) \to GL_m(k)$, we can write $\rho(a_{ij}) = (\rho_{st}(a_{ij}))$ for some polynomials ρ_{st}. Then $\rho(a_{ij})$ makes sense for any $(a_{ij}) \in M_n(k)$, and we get an extended morphism $\rho' : M_n(k) \to M_m(k)$ which is a semigroup homomorphism.

1.24 Exercise. Prove that ρ' is a semigroup homomorphism.

Conversely, if $\rho' : M_n(k) \to M_m(k)$ is a k-morphism which is a semigroup homomorphism, then the restriction $\rho = \rho'|_{GL_n}$ of ρ' to GL_n is a polynomial representation.

Thus, a finite dimensional polynomial representation of GL_n is canonically identified with a morphism $M_n(k) \to M_m(k)$ which is also a semigroup homomorphism.

(1.25) Let us denote the coordinate ring $k[M_n(k)]$ of the affine space $M_n(k)$ by S. It is the polynomial ring $k[x_{ij}]$ in n^2-variables over k. An element $f \in S$ is a function $M_n(k) \to \mathbb{A}^1$, where $\mathbb{A}^1 = k$ is the affine line. That is, $f : (a_{ij}) \mapsto f(a_{ij}) \in k$ is a function. The product $\mu : M_n(k) \times M_n(k) \to M_n(k)$ induces a k-algebra map $\Delta : k[M_n(k)] \to k[M_n(k) \times M_n(k)]$ defined by $(\Delta f)(A, B) = f\mu(A, B) = f(AB)$. Identifying $k[M_n(k) \times M_n(k)]$ with $S \otimes S$ via $(f \otimes f')(A, B) = f(A)f'(B)$, Δ is a k-algebra map from S to $S \otimes S$. The associativity of the product $(AB)C = A(BC)$ for $A, B, C \in M_n(k)$ yields the coassociativity $(\Delta \otimes 1_S) \circ \Delta = (1_S \otimes \Delta) \circ \Delta$. Let us denote the evaluation at the unit element by $\varepsilon : S \to k$. That is, $\varepsilon(f) = f(E)$, where E is the unit matrix. Then the coassociativity law follows from the fact that E is a unit element of the semigroup S. Thus S together with Δ and ε is a k-coalgebra.

1.26 Exercise. $S = k[x_{ij}]$ is a polynomial ring. Give $\Delta(x_{ij})$ and $\varepsilon(x_{ij})$ explicitly, and prove directly that the coassociativity and the counit laws hold.

(1.27) Let C and D be coalgebras and $f : D \to C$ a coalgebra map. Let M be a D-comodule. Then letting the composite map

$$M \xrightarrow{\omega_M} M \otimes D \xrightarrow{1_M \otimes f} M \otimes C$$

the structure map, M is a C-comodule. This gives the restriction functor $\operatorname{res}^D_C : \operatorname{Comod} D \to \operatorname{Comod} C$. Obviously, it is an exact functor.

(1.28) Let V be an m-dimensional polynomial representation of GL_n. Let $v_1, \ldots, v_m$ be a basis of V, and let us identify $\operatorname{End}(V)$ by $M_m(k)$ via the basis. Let us identify $k[M_m(k)]$ with the polynomial algebra $k[y_{st}]$ in a natural way. Then V is a (right) $k[M_m(k)]$-comodule by $\omega(v_t) = \sum_s v_s \otimes y_{st}$.

Let $\rho : M_n(k) \to M_m(k)$ be the map coming from the representation. Then ρ is a semigroup homomorphism. Let $\rho^* : k[M_m(k)] \to k[M_n(k)]$ be the k-algebra map given by $(\rho^*(f))(A) = f(\rho(A))$. As ρ is a semigroup homomorphism, it is easy to check that ρ^* is a k-coalgebra map. So via the restriction $\operatorname{Comod} k[M_m(k)] \to \operatorname{Comod} k[M_n(k)]$, V is a right $k[M_n(k)]$-comodule. Note that the coaction of V as a $k[M_n(k)]$-comodule is given by $\omega(v_t) = \sum_s v_s \otimes \rho^*(y_{st}) = \sum_s v_s \otimes \rho_{st}$.

(1.29) Conversely, assume that V is a finite dimensional right $k[M_n(k)]$-comodule. Then defining $\rho_{st} \in k[M_n(k)]$ by $\omega(v_t) = \sum_s v_s \otimes \rho_{st}$, we get a polynomial representation given by $\rho(A) = (\rho_{st}(A))$. Thus a finite dimensional polynomial representation of GL_n and a right $k[GL_n]$-comodule are one and the same thing. More generally, it is not so difficult to show that (possibly infinite dimensional) polynomial representation of GL_n and a right $k[GL_n]$-comodule are the same thing.

(1.30) Let C be a k-coalgebra, and $D \subset C$. We say that D is a subcoalgebra of C if D is a k-subspace of C, and $\Delta(D) \subset D \otimes D$, where Δ is the coproduct of C. Or equivalently, D is a subcoalgebra if D has a k-coalgebra structure (uniquely) such that the inclusion $D \hookrightarrow C$ is a k-coalgebra map.

1.31 Exercise. Prove that if D is a subcoalgebra of C, then the restriction functor $\operatorname{res}_C^D : \operatorname{Comod}(D) \to \operatorname{Comod}(C)$ is full, faithful, and exact. A C-comodule M is of the form $\operatorname{res}_C^D V$ if and only if $\omega_M(M) \subset M \otimes D$. If this is the case, M is a D-comodule in an obvious way, and letting $V = M$, $M = \operatorname{res}_C^D V$. Thus a D-comodule is identified with a C-comodule M such that $\omega_M(M) \subset M \otimes D$.

(1.32) Let $C = \bigoplus_{i \in I} C_i$ be a k-coalgebra such that each C_i is a subcoalgebra of C. In this case, we say that C is the direct sum of C_i. Let (M_i) be a collection such that each M_i is a C_i-comodule. Then M_i is a C-comodule by restriction, and hence $\bigoplus_i M_i$ is also a C-comodule. This gives a functor $F : (M_i) \mapsto \bigoplus_i M_i$ from $\prod_i \operatorname{Comod} C_i$ to $\operatorname{Comod} C$.

Let M be a C-comodule. Define M_i to be $\omega_M^{-1}(M \otimes C_i)$. Then it is easy to check that M_i is a C_i-comodule and $M = \bigoplus M_i$. The functor $G : M \mapsto (M_i)$ from $\operatorname{Comod} C$ to $\prod_i \operatorname{Comod} C_i$ is a quasi-inverse of F, and hence F and G are equivalence.

1.33 Exercise. Prove (1.32).

(1.34) Let $V = k^n$. Then a polynomial representation of $GL(V) = GL_n$ is nothing but a $S = k[M_n(k)]$-comodule. Note that $S = \bigoplus_i S_i$ is a graded k-algebra, and each S_i is a subcoalgebra of S. An S-comodule V is of degree r if and only if V is an S_r-comodule, that is to say, $\omega_V(V) \subset V \otimes S_r$. Thus the category $\operatorname{Comod} S$ of the polynomial representations of $GL(V)$ is equivalent to $\prod_i \operatorname{Comod} S_i$, and the study of polynomial representations of $GL(V)$ is reduced to the study of S_r-comodules of various r.

(1.35) $\operatorname{Comod} S_r$ is equivalent to the category $S_r^* \operatorname{Mod}$, the category of left S_r^*-modules. Thus the study of polynomial representations of $GL(V)$ is reduced to the study of S_r^*-modules. We define the Schur algebra $S(n, r)$ to be S_r^*. Note that $S(n,r)$ is $\binom{n^2+r-1}{r}$-dimensional. In particular, $S(n,r)$ is a finite dimensional k-algebra.

For a finite dimensional polynomial representation (V,ρ) of GL_n of degree r, V is an $S(n,r)$ module via $\xi v_t = \sum_s(\xi(\rho_{st}))v_s$ for $\xi \in S(n,r)$, where $v_1,\dots,v_m$ is a basis of V, and $\rho((a_{ij})) = (\rho_{st}((a_{ij})))$ for $(a_{ij}) \in GL_n = GL(V)$.

(1.36) Let E be a finite dimensional k-vector space. Then we define $H : (E^*)^{\otimes r} \to (E^{\otimes r})^*$ by

$$(H(\xi_1 \otimes \cdots \otimes \xi_r))(x_1 \otimes \cdots \otimes x_r) = (\xi_1 x_1)\cdots(\xi_r x_r)$$

for $\xi_1,\dots,\xi_r \in E^*$ and $x_1,\dots,x_r \in E$. Note that H is an isomorphism. We identify $(E^*)^{\otimes r}$ and $(E^{\otimes r})^*$ via H.

(1.37) Let E be a finite dimensional k-vector space. The sequence

$$\bigoplus_{i=1}^{r-1}(E^*)^{\otimes r} \xrightarrow{\sum_i(1-\tau_i)} (E^*)^{\otimes r} \to \operatorname{Sym}_r E^* \to 0$$

is exact, where $\tau_i(\xi_1 \otimes \cdots \otimes \xi_r) = \xi_1 \otimes \cdots \otimes \xi_{i+1} \otimes \xi_i \otimes \cdots \otimes \xi_r$. Taking the dual,

$$0 \to D_r E \to E^{\otimes r} \xrightarrow{\sum_i(1-\sigma_i)} \bigoplus_{i=1}^{r-1} E^{\otimes r}$$

is also exact, where the symmetric group $\mathfrak{S}_r$ acts on $E^{\otimes r}$ via

$$\sigma(x_1 \otimes \cdots \otimes x_r) = x_{\sigma^{-1}1} \otimes \cdots \otimes x_{\sigma^{-1}r},$$

and σ_i is the transposition $(i, i+1)$. As the symmetric group is generated by $\sigma_1,\dots,\sigma_{r-1}$, we have that $D_r E$ is identified with $(E^{\otimes r})^{\mathfrak{S}_r}$.

(1.38) Let $V = k^n$, and $E = \operatorname{End}(V) \cong \operatorname{Mat}_n(k)$. Then the Schur algebra $S(n,r)$ is identified with $D_r E$. Note that the diagonalization $E \to E \times \cdots \times E$ ($x \mapsto (x,x,\dots,x)$) is a semigroup homomorphism. So the corresponding map $S \otimes \cdots \otimes S \to S$, which is nothing but the product map, is a bialgebra map (that is, a k-algebra map which is also a coalgebra map), where $S = \operatorname{Sym} E^*$. Thus the restriction of the product

$$(E^*)^{\otimes r} \to \operatorname{Sym}_r E^*$$

is also a coalgebra map. This shows that $S(n,r) = D_r E \to E^{\otimes r}$ is an algebra map. Note that $\Phi : E^{\otimes r} \to \operatorname{End}(V^{\otimes r})$ given by

$$(\Phi(\phi_1 \otimes \cdots \otimes \phi_r))(v_1 \otimes \cdots \otimes v_r) = \phi_1(v_1) \otimes \cdots \otimes \phi_r(v_r)$$

is a $\mathfrak{S}_r$-algebra isomorphism. Identifying $E^{\otimes r}$ by $\operatorname{End}(V^{\otimes r})$ via Φ, The subalgebra $S(n,r) = (E^{\otimes r})^{\mathfrak{S}_r}$ is identified with $(\operatorname{End} V^{\otimes r})^{\mathfrak{S}_r} = \operatorname{End}_{\mathfrak{S}_r} V^{\otimes r}$. Thus we have

1.39 Theorem. $S(n,r)$ is k-isomorphic to $\operatorname{End}_{\mathfrak{S}_r} V^{\otimes r}$.

By Maschke's theorem, $k\mathfrak{S}_r$ is semisimple if the characteristic of k is zero or larger than r. If this is the case, $V^{\otimes r}$ is a semisimple $k\mathfrak{S}_r$-module, and hence $S(n,r) \cong \operatorname{End}_{\mathfrak{S}_r} V^{\otimes r}$ is also semisimple.

1.40 Corollary. If the characteristic of k is zero or larger than r, then $S(n,r)$ is semisimple.

(1.41) Notes and references. Quite a similar discussion can be found in [Gr]. This book is recommended as a good reading.

References

[Sw] M. Sweedler, *Hopf Algebras,* Benjamin (1969).

[Gr] J. A. Green, *Polynomial Representations of GL_n,* Lecture Notes in Math. **830**, Springer (1980).

2 Weyl modules

(2.1) Let W be an m-dimensional k-vector space with the basis $w_1, \ldots, w_m$. Let $\eta_1, \ldots, \eta_m$ be the dual basis of W^*. Then the symmetric algebra $S = \operatorname{Sym} W^*$ is the polynomial ring $k[\eta_1, \ldots, \eta_m]$. We define $\Delta : W \to S \otimes S$ by $\Delta(w) = w \otimes 1 + 1 \otimes w \in S_1 \otimes S_0 \oplus S_0 \otimes S_1$. Δ is extended to a k-algebra map $\Delta : S \to S \otimes S$ uniquely. It is easy to see that Δ makes S a graded k-bialgebra. We define DW to be the graded dual $\bigoplus_{r \geq 0} D_r W = \bigoplus_{r \geq 0} S_r^*$ of $S = \operatorname{Sym} W^*$. Note that DW is also a graded k-bialgebra. The algebra structure of DW is defined to be that of the subalgebra of the dual algebra S^* of S. The coproduct $\Delta : D_{a+b} W \to D_a W \otimes D_b W$ is given by $(\Delta x)(\alpha \otimes \beta) = x(\alpha\beta)$ for $x \in D_{a+b} W$, $\alpha \in S_a$, and $\beta \in S_b$. Note that

$$(W^{\otimes(a+b)})^{\mathfrak{S}_{a+b}} = D_{a+b} W \xrightarrow{\Delta} D_a W \otimes D_b W = (W^{\otimes a})^{\mathfrak{S}_a} \otimes (W^{\otimes b})^{\mathfrak{S}_b} = (W^{\otimes(a+b)})^{\mathfrak{S}_a \times \mathfrak{S}_b}$$

is nothing but the inclusion. As S is commutative and cocommutative, DW is commutative and cocommutative. Note that if W is a polynomial representation of GL_n, then DW is a polynomial representation of GL_n, and the structure maps of DW as a k-bialgebra are GL_n-linear. In particular, DW is a polynomial representation of $GL(W)$.

(2.2) Note that $B_r = \{\eta_\lambda = \eta_1^{\lambda_1} \cdots \eta_m^{\lambda_m} \mid |\lambda| = r\}$ is a basis of S_r, where $|\lambda| = \lambda_1 + \cdots + \lambda_m$. Let $C_r = \{w^{(\lambda)} \mid |\lambda| = r\}$ be the dual basis, where $w^{(\lambda)}$ is dual to η_λ. The basis element $w^{((0,\ldots,0,r,0\ldots,0))}$ dual to η_j^r is denoted by $w_j^{(r)}$. It is easy to check that $w^{(\lambda)} = w_1^{(\lambda_1)} \cdots w_m^{(\lambda_m)}$. By the unique k-algebra map $\Theta : \operatorname{Sym} W \to DW$ which is the identity map on degree one, $w^\lambda = w_1^{\lambda_1} \cdots w_m^{\lambda_m}$ is mapped to $(\lambda_1)! \cdots (\lambda_m)! w^{(\lambda)}$. In particular, $\operatorname{Sym}_r W \cong D_r W$ as a $GL(W)$-module if the characteristic of k is zero or larger than r.

(2.3) Let $V = k^n$ be an n-dimensional k-vector space with the basis $v_1, \ldots, v_n$. Set $E := \operatorname{End}(V)$, and define $\xi_{ij} \in E$ by $\xi_{ij} v_l = \delta_{jl} v_i$ for $i, j \in [1, n]$, where δ_{jl} is Kronecker's delta. It is easy to see that $\xi_{ij}\xi_{st} = \delta_{js}\xi_{it}$.

E^* has the dual basis $\{c_{ij} \mid i, j \in [1, n]\}$, where $c_{ij}(\xi_{st}) = \delta_{is}\delta_{jt}$. Then the coalgebra structure of E^* is given by

$$\Delta(c_{ij}) = \sum_l c_{il} \otimes c_{lj}.$$

Indeed,

$$(\Delta(c_{ij}))(\xi_{st} \otimes \xi_{uv}) = c_{ij}(\xi_{st}\xi_{uv}) = c_{ij}(\delta_{tu}\xi_{sv}) = \delta_{tu}\delta_{is}\delta_{jv},$$

and

$$(\sum_l c_{il} \otimes c_{lj})(\xi_{st} \otimes \xi_{uv}) = \sum_l \delta_{is}\delta_{lt}\delta_{lu}\delta_{jv} = \delta_{is}\delta_{tu}\delta_{jv}.$$

(2.4) Let $I(n,r)$ denote the set $\mathrm{Map}([1,r],[1,n])$, the set of maps from $[1,r] = \{1,\ldots,r\}$ to $[1,n] = \{1,\ldots,n\}$. Such a map is identified with a sequence $i = (i_1,\ldots,i_n)$ of elements of $[1,n]$. As $\mathfrak{S}_r$ acts on $[1,r]$, it also acts on $I(n,r)$ by $(\sigma i)(l) = i(\sigma^{-1}(l))$. In other words, $\sigma(i_1,\ldots,i_n) = (i_{\sigma^{-1}(1)},\ldots,i_{\sigma^{-1}(n)})$. $\mathfrak{S}_r$ also acts on $I(n,r)^2$ by $\sigma(i,j) = (\sigma i, \sigma j)$. We say that $(i,j) \sim (i',j')$ if (i,j) and (i',j') lie on the same orbit with respect to the action of $\mathfrak{S}_r$.

Let $r \geq 1$. Note that $S_r = \mathrm{Sym}_r E^*$ has a basis $\{c_{ij} = c_{i_1j_1}c_{i_2j_2}\cdots c_{i_rj_r} \mid (i,j) \in I(n,r)^2/\mathfrak{S}_r\}$. The dual basis of $S(n,r)$ is denoted by $\{\xi_{ij} \mid (i,j) \in I(n,r)^2/\mathfrak{S}_r\}$. Note that

$$\Delta(c_{ij}) = \sum_{s \in I(n,r)} c_{is} \otimes c_{sj}.$$

So

$$\xi_{ij}\xi_{uv} = \sum_{pq} Z(i,j,u,v,p,q)\xi_{pq}$$

in the Schur algebra $S(n,r)$, where $Z(i,j,u,v,p,q)$ is the number of $s \in I(n,r)$ such that $(i,j) \sim (p,s)$ and $(u,v) \sim (s,q)$.

(2.5) In particular, if $\xi_{ij}\xi_{uv} \neq 0$, then $j \sim u$. Note that $\xi_{ii}\xi_{ij} = \xi_{ij}$ and $\xi_{ij}\xi_{jj} = \xi_{ij}$ for $i,j \in I(n,r)$. So $\{\xi_{ii}\}$, where i runs through $I(n,r)/\mathfrak{S}_r$, is a set of mutually orthogonal idempotents of $S(n,r)$, and $\sum_i \xi_{ii} = 1_{S(n,r)}$.

(2.6) Set $T(n,r)$ to be the k-span of $\{\xi_{ii} \mid i \in I(n,r)/\mathfrak{S}_r\}$. It is a k-subalgebra of $S(n,r)$, and $T(n,r)$ is the direct product of $k\xi_{ii} \cong k$ for various i as a k-algebra. We define

$$\Lambda(n,r) = \{\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n_{\geq 0} \mid |\lambda| = r\}.$$

For $i \in I(n,r)/\mathfrak{S}_r$, we define $\nu(i) \in \Lambda(n,r)$ by $\nu(i)_j = \#\{l \mid i_l = j\}$. Note that $\nu : I(n,r)/\mathfrak{S}_r \to \Lambda(n,r)$ is a bijection. We denote ξ_{ii} by $\xi_{\nu(i)}$.

(2.7) For a $T(n,r)$-module M and $\lambda \in \Lambda(n,r)$, we define M_λ to be $\xi_\lambda M$. As $\{\xi_\lambda \mid \lambda \in \Lambda(n,r)\}$ is a set of mutually orthogonal idempotents of $T(n,r)$ with $\sum_\lambda \xi_\lambda = 1$, we have that $M = \bigoplus_\lambda M_\lambda$. We say that $\lambda \in \Lambda(n,r)$ is a weight of M if $M_\lambda \neq 0$. For a finite dimensional $T(n,r)$-module M, we define

$$\chi(M) := \sum_\lambda (\dim_k M) t_1^{\lambda_1} \cdots t_n^{\lambda_n} \in \mathbb{Z}[t_1, \ldots, t_n].$$

We use this convention to an $S(n,r)$-module M. Plainly, an $S(n,r)$-module is a $T(n,r)$-module.

(2.8) For a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers, we define $\bigwedge_\lambda V := \bigwedge^{\lambda_1} V \otimes \bigwedge^{\lambda_2} V \otimes \cdots$, $\mathrm{Sym}_\lambda V := \mathrm{Sym}_{\lambda_1} V \otimes \mathrm{Sym}_{\lambda_2} V \otimes \cdots$, and $D_\lambda V := D_{\lambda_1} V \otimes D_{\lambda_2} V \otimes \cdots$. If $|\lambda| = r$, then $\bigwedge_\lambda V$, $\mathrm{Sym}_\lambda V$, and $D_\lambda V$ are $S(n,r)$-modules. For $\lambda \in \Lambda(n,r)$, we define

$$f_\lambda : D_\lambda V \to S(n,r)\xi_\lambda$$

by

$$f_\lambda(v_1^{(a_{11})} \cdots v_n^{(a_{n1})} \otimes \cdots \otimes v_1^{(a_{1n})} \cdots v_n^{(a_{nn})}) = \xi_{ij} = \xi_{11}^{(a_{11})} \cdots \xi_{n1}^{(a_{n1})} \cdots \xi_{1n}^{(a_{1n})} \cdots \xi_{nn}^{(a_{nn})},$$

where $i = (1^{a_{11}}, \ldots, n^{a_{n1}}, 1^{a_{12}}, \ldots, n^{a_{n2}}, \ldots, 1^{a_{1n}}, \ldots, n^{a_{nn}})$ and $j = (1^{\lambda_1}, \ldots, n^{\lambda_n})$. It is easy to see that f_λ is a $GL_n(k)$-isomorphism. As ξ_λ is an idempotent of $S(n,r)$ and $\sum_\lambda \xi_\lambda = 1$, we have

2.9 Lemma. $D_\lambda V$ for $\lambda \in \Lambda(n,r)$ is a projective $S(n,r)$-module.

$$\mathrm{add}(\{D_\lambda V \mid \lambda \in \Lambda(n,r)\}) = \mathrm{add}(\{S(n,r)\}),$$

where for a ring A and a set X of A-modules, $\mathrm{add}\, X$ denotes the set of A-modules which is isomorphic to a direct summand of a finite direct sum of elements of X.

(2.10) For $\lambda \in \Lambda(n,r)$ and an $S(n,r)$-module M, we have

$$\mathrm{Hom}_{S(n,r)}(D_\lambda V, M) \cong \mathrm{Hom}_{S(n,r)}(S(n,r)\xi_\lambda, M) \cong \xi_\lambda M.$$

Note that $\varphi \in \mathrm{Hom}_{S(n,r)}(D_\lambda V, M)$ corresponds to $\varphi(v_1^{(\lambda_1)} \otimes \cdots \otimes v_n^{(\lambda_n)}) \in \xi_\lambda M$. In particular, λ is a weight of M if and only if $\mathrm{Hom}_{S(n,r)}(D_\lambda V, M) \neq 0$.

(2.11) We define $\varepsilon_i := (0, \ldots, 0, 1, 0, \ldots)$, where 1 is at the ith position. We also define $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. For $\lambda, \mu \in \Lambda(n,r)$, we say that $\lambda \geq \mu$ if there exist $c_1, \ldots, c_{n-1} \geq 0$ such that $\lambda - \mu = \sum_i c_i \alpha_i$. This gives an ordering of $\Lambda(n,r)$, called the dominant order.

(2.12) Let A be a ring, M a (left) A-module, and X a set of A-modules. Then we define the X-trace of M, denoted by $\operatorname{tr}_X M$ the sum of all A-submodules of M which is a homomorphic image of elements of X.

$$\operatorname{tr}_X M = \sum_{N \in X} \sum_{\phi \in \operatorname{Hom}_A(N,M)} \operatorname{Im} \phi.$$

Obviously, for $N \in X$, $\operatorname{Hom}_A(N, \operatorname{tr}_X M) \to \operatorname{Hom}_A(N, M)$ is an isomorphism. In particular, if N is projective, then $\operatorname{Hom}_A(N, M/\operatorname{tr}_X M) = 0$.

(2.13) Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a sequence of nonnegative integers, and $\sigma \in \mathfrak{S}_n$. Let $\sigma\lambda$ denote $(\lambda_{\sigma^{-1}(1)}, \ldots, \lambda_{\sigma^{-1}(n)})$ as before. Then

$$\tau : D_\lambda V \to D_{\sigma\lambda} V$$

given by $a_1 \otimes \cdots \otimes a_n \mapsto a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$ is an isomorphism $S(n,r)$-modules. In particular, for a finite dimensional $S(n,r)$-module M, we have $M_\lambda \cong M_{\sigma\lambda}$. It follows that $\chi(M)$ is a symmetric polynomial.

(2.14) $\chi(\bigwedge^r V) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} t_{i_1} \cdots t_{i_r}$ is the elementary symmetric polynomial. $\chi(\operatorname{Sym}_r V) = \chi(D_r V) = \sum_{\lambda \in \Lambda(n,r)} t_1^{\lambda_1} \cdots t_n^{\lambda_n}$ is the complete symmetric polynomial.

(2.15) Let $\lambda = (\lambda_1, \lambda_2)$, and $1 \leq j \leq \lambda_2$. We define the box map to be the composite

$$\square : D_{\lambda + j\alpha_1} V = D_{\lambda_1 + j} V \otimes D_{\lambda_2 - j} V \xrightarrow{\Delta \otimes 1} D_{\lambda_1} V \otimes D_j V \otimes D_{\lambda_2 - j} V \xrightarrow{1 \otimes m} D_{\lambda_1} V \otimes D_{\lambda_2} V,$$

where Δ and m denote the coproduct and the product of DV, respectively.

(2.16) We define $\Lambda(n,r)^+ = \{\lambda \in \Lambda(n,r) \mid \lambda_1 \geq \cdots \geq \lambda_n\}$. $\Lambda(n,r)^+$ is an ordered set with respect to the dominant order. For $\lambda \in \Lambda(n,r)^+$, we define

$$\square_\lambda : \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{\lambda_{i+1}} D_{\lambda + j\alpha_i} V \xrightarrow{\sum \square} D_\lambda V,$$

where $\square : D_{\lambda+j\alpha_i}V \to D_\lambda V$ is given by

$$D_{\lambda+j\alpha_i}V = D_{\lambda_1}V \otimes \cdots D_{\lambda_{i-1}}V \otimes D_{\lambda_i+j}V \otimes D_{\lambda_{i+1}-j}V \otimes \cdots \xrightarrow{1\otimes\cdots\otimes 1\otimes\square\otimes\cdots} D_{\lambda_1}V \otimes \cdots D_{\lambda_{i-1}}V \otimes D_{\lambda_i}V \otimes D_{\lambda_{i+1}}V \otimes \cdots = D_\lambda V.$$

We define $\Delta(\lambda) := D_\lambda V/\operatorname{Im}(\square_\lambda)$, and call $\Delta(\lambda)$ the Weyl module of V. If we want to emphasize V, then $\Delta(\lambda)$ is also denoted by $K_\lambda V$.

(2.17) Let $\lambda \in \Lambda(n,r)$. We define the Young diagram $Y(\lambda)$ of λ to be $\{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le n,\ 1 \le j \le \lambda_i\}$. An element of $\operatorname{Tab}(\lambda) := \operatorname{Map}(Y(\lambda),[1,n])$ is called a tableau of shape λ. Let $T \in \operatorname{Tab}(\lambda)$. T is called co-row-standard if $T(i,j) \le T(i,j')$ for any i,j,j' with $j < j'$. The set of co-row-standard tableaux is denoted by $\operatorname{CoRow}(\lambda)$. Associated with a co-row-standard tableau T, we have

$$p(T) = v_1^{(a(1,1))} \cdots v_n^{(a(1,n))} \otimes v_1^{(a(2,1))} \cdots v_n^{(a(2,n))} \otimes \cdots \otimes v_n^{(a(n,1))} \cdots v_n^{(a(n,n))} \in D_{\lambda_1}V \otimes D_{\lambda_2}V \otimes \cdots \otimes D_{\lambda_n}V = D_\lambda V,$$

where $a(i,j) = \#\{l \mid T(i,l) = j\}$.

2.18 Example.

$$p\begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 4 & \end{pmatrix} = v_1^{(2)}v_2v_3v_4 \otimes v_2^{(3)}v_4.$$

Assume that $\lambda \in \Lambda(n,r)^+$. T is called co-column-standard if $T(i,j) < T(i',j)$ for $i < i'$. T is called co-standard if it is both co-row-standard and co-column standard.

What is important is the following.

2.19 Theorem (Akin–Buchsbaum–Weyman [ABW, (II.3.16)]). $\{p(T) \mid T \text{ is co-standard}\}$ is a basis of $\Delta(\lambda)$.

2.20 Exercise. Express the tableau

$$p\begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 4 & \end{pmatrix}$$

as a linear combination of co-standard tableaux in $K_{(5,4)}V$.

Let $\lambda \in \Lambda(n,r)$ and $T \in \mathrm{CoRow}(\lambda)$. Define $\mathrm{Cont}(T)$ to be the sequence $(\mu_1, \ldots, \mu_n)$, where $\mu_l := \#\{(i,j) \in Y(\lambda) \mid T(i,j) = l\}$. Note that $\mathrm{Cont}(T) \in \Lambda(n,r)$. Then $p(T) \in D_\lambda V$ is actually in the weight $\mathrm{Cont}(T)$ space $(D_\lambda V)_{\mathrm{Cont}(T)}$ of $D_\lambda V$.

2.21 Lemma. If $\lambda \in \Lambda(n,r)^+$ and T is a standard tableau of shape λ, then $\mathrm{Cont}(T) \leq \lambda$. The only standard tableau T of shape λ such that $\mathrm{Cont}(T) = \lambda$ is the tableau T given by $T(i,j) = i$ (the canonical tableau).

2.22 Exercise. Prove Lemma 2.21.

By Theorem 2.19 and Lemma 2.21, we immediately have

2.23 Lemma. Let $\lambda \in \Lambda(n,r)^+$ and $\mu \in \Lambda(n,r)$. If $\Delta(\lambda)_\mu \neq 0$, then $\mu \leq \lambda$. $\Delta(\lambda)_\lambda$ is one-dimensional, and is spanned by the canonical tableau.

(2.24) Let A be a ring and M a left A-module. We denote $M/\operatorname{rad} M$ by $\operatorname{top} M$, and call it the top of M.

2.25 Proposition. Let $\lambda \in \Lambda(n,r)^+$. The $S(n,r)$-module $\Delta(\lambda)$ has the simple top.

Proof. Let W be the sum of all $S(n,r)$-submodules V of $\Delta(\lambda)$ such that $V_\lambda = 0$. Clearly, $W_\lambda = 0$, and hence $W \neq \Delta(\lambda)$. If U is an $S(n,r)$-submodule of $\Delta(\lambda)$ such that $U \not\subset W$, then $U_\lambda \neq 0$. As $U_\lambda \subset \Delta(\lambda)_\lambda$ and $\Delta(\lambda)_\lambda$ is one-dimensional and generated by the canonical tableau, U contains the canonical tableau T. On the other hand, $\Delta(\lambda) = S(n,r)T$, since $D_\lambda V = S(n,r)T$. So $U = \Delta(\lambda)$. This means that W is the unique maximal submodule of $\Delta(\lambda)$, and hence $\operatorname{top} \Delta(\lambda) = \Delta(\lambda)/W$ is simple. □

(2.26) We denote $\mathrm{top}(\Delta(\lambda))$ by $L(\lambda)$. Note that $L(\lambda)_\lambda$ is one-dimensional and generated by the canonical tableau, and $L(\lambda)_\mu \neq 0$ implies $\mu \leq \lambda$. Let $P(\lambda)$ denote the projective cover of $L(\lambda)$.

2.27 Lemma. Let $\lambda, \mu \in \Lambda(n,r)^+$, and $\lambda \neq \mu$. Then $L(\lambda) \not\cong L(\mu)$.

Proof. Assume that $L(\lambda) \cong L(\mu)$. Then

$$\lambda = \max\{\nu \in \Lambda(n,r) \mid L(\lambda)_\nu \neq 0\} = \max\{\nu \in \Lambda(n,r) \mid L(\mu)_\nu \neq 0\} = \mu.$$

□

2.28 Lemma. $D_\lambda V$ is of the form $P(\lambda) \oplus \bigoplus_{\mu > \lambda} P(\mu)^{\oplus c(\lambda,\mu)}$. For any order filter I of $\Lambda(n,r)^+$, $\operatorname{add}(P(\lambda) \mid \lambda \in I) = \operatorname{add}(D_\lambda V \mid \lambda \in I)$.

Proof. We prove the first assertion. Assume the contrary, and let λ be a maximal element such that $D_\lambda V$ is not of the form $P(\lambda) \oplus \bigoplus_{\mu>\lambda} P(\mu)^{\oplus c(\lambda,\mu)}$. As $D_\lambda V$ has $L(\lambda)$ as a quotient, $P(\lambda)$ is a direct summand of $D_\lambda V$. By assumption, $D_\lambda V$ has a semisimple quotient M such that $M_\mu = 0$ for any $\mu \in \Lambda(n,r)^+$ which satisfies $\mu > \lambda$, and that M is not simple. Then by the definition of $\Delta(\lambda)$, M is a quotient of $\Delta(\lambda)$. This contradicts the fact that $\Delta(\lambda)$ has a simple top.

The second assertion follows immediately from the first. □

2.29 Corollary. The set $\{L(\lambda) \mid \lambda \in \Lambda(n,r)^+\}$ is a complete set of representatives of the isomorphism classes of the simples of $S(n,r)$. For $\lambda \in \Lambda(n,r)^+$, $\Delta(\lambda) \cong P(\lambda)/\operatorname{tr}_{Z(\lambda)}(P(\lambda))$, where $Z(\lambda) = \{P(\mu) \mid \mu \in \Lambda(n,r)^+,\ \mu > \lambda\}$. If $\operatorname{Hom}_{S(n,r)}(P(\nu), \Delta(\lambda)) \neq 0$, then $\nu \leq \lambda$. $\operatorname{End}_{S(n,r)} \Delta(\lambda) \cong k$.

Proof. Note that $\operatorname{add}\{P(\lambda) \mid \lambda \in \Lambda(n,r)^+\} = \operatorname{add} S(n,r)$ by Lemma 2.28, (2.13), and Lemma 2.9. The first assertion follows from this and Lemma 2.27. The second assertion is a consequence of Lemma 2.28. The third and the fourth assertions follow from Lemma 2.23. □

(2.30) Let V and W be k-vector spaces, and $r \geq 0$. Consider the map

$$\theta'_r : D_r V \otimes D_r W \xrightarrow{\Delta\otimes\Delta} V^{\otimes r} \otimes W^{\otimes r} \xrightarrow{\tau} (V \otimes W)^{\otimes r},$$

where $\tau(a_1 \otimes \cdots \otimes a_r \otimes b_1 \otimes \cdots \otimes b_r) = a_1 \otimes b_1 \otimes \cdots \otimes a_r \otimes b_r$. It is easy to see that θ'_r factors through $D_r(V \otimes W) = (V \otimes W)^{\mathfrak{S}_r}$, and induces $\theta_r : D_r V \otimes D_r W \to D_r(V \otimes W)$. Note that the diagram

$$\begin{array}{ccc} D_r V \otimes D_r W & \xrightarrow{\theta_r} & D_r(V \otimes W) \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & \downarrow{\scriptstyle \Delta} \\ V^{\otimes r} \otimes W^{\otimes r} & \xrightarrow{\tau} & (V \otimes W)^{\otimes r} \end{array}$$

is commutative, and θ_r commutes with the action of $GL(V) \times GL(W)$.

(2.31) Let $\lambda \in \Lambda(n,r)$. Then we define $\theta_\lambda : D_\lambda V \otimes D_\lambda W \to D_r(V \otimes W)$ to be the composite

$$D_\lambda V \otimes D_\lambda W \xrightarrow{\tau} D_{\lambda_1} V \otimes D_{\lambda_1} W \otimes \cdots \otimes D_{\lambda_n} V \otimes D_{\lambda_n} W \xrightarrow{\theta_{\lambda_1}\otimes\cdots\otimes\theta_{\lambda_n}}$$
$$D_{\lambda_1}(V \otimes W) \otimes \cdots \otimes D_{\lambda_n}(V \otimes W) \xrightarrow{m} D_r(V \otimes W).$$

We define $M(\lambda) = \sum_{\mu \geq_{\text{lex}} \lambda} \operatorname{Im} \theta_\mu$ and $\dot{M}(\lambda) = \sum_{\mu >_{\text{lex}} \lambda} \operatorname{Im} \theta_\mu$, where $\geq_{\text{lex}}$ denotes the lexicographic order.

2.32 Theorem (Cauchy formula for the divided power algebra, [HK, (III.2.9)]). For each $\lambda \in \Lambda(n,r)^+$, there is a unique isomorphism $\Theta_\lambda : K_\lambda V \otimes K_\lambda W \to M(\lambda)/\dot{M}(\lambda)$ such that the diagram

$$\begin{array}{ccc} D_\lambda V \otimes D_\lambda W & \xrightarrow{\theta_\lambda} & M(\lambda) \\ \downarrow & & \downarrow \\ K_\lambda V \otimes K_\lambda W & \xrightarrow{\Theta_\lambda} & M(\lambda)/\dot{M}(\lambda) \end{array}$$

is commutative.

(2.33) Let V be a finite dimensional $S(n,r)$-module. A filtration of $S(n,r)$-modules

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V$$

is said to be a Weyl module filtration if $V_i/V_{i-1} \cong \Delta(\lambda(i))$ for some $\lambda(i) \in \Lambda(n,r)^+$.

(2.34) The left regular representation ${}_{S(n,r)}S(n,r) = D_r(V \otimes V^*)$ is identified with the following representation. $V \otimes V^*$ is a $GL(V)$-module by $g(v \otimes \varphi) = gv \otimes \varphi$. D_r is a functor from the category of $S(n,1)$-modules to the category of $S(n,r)$-modules, and we have that $D_r(V \otimes V^*)$ is an $S(n,r)$-module. Note that $K_\lambda V \otimes K_\lambda V^*$ is a direct sum of copies of $K_\lambda V = \Delta(\lambda)$. By Theorem 2.32, we have

2.35 Corollary. ${}_{S(n,r)}S(n,r)$ has a Weyl module filtration.

(2.36) Note that the k-dual $(?)^* = \operatorname{Hom}_k(?,k)$ is an equivalence $S(n,r)^{\text{op}} \to S(n,r)\,\text{mod}$. On the other hand, the transpose map $t : S(n,r) \to S(n,r)^{\text{op}}$ given by $t(\xi_{ij}) = \xi_{ji}$ (it corresponds to the transpose of matrices) is an isomorphism. Through t, a right module changes to a left module. Thus we get a transposed dual functor ${}^t(?) : S(n,r)\,\text{mod} \to S(n,r)$. It is a contravariant autoequivalence of $S(n,r)\,\text{mod}$. It is easy to see that ${}^t(V \otimes W) \cong {}^tV \otimes {}^tW$. So ${}^t(S_\lambda V) \cong D_\lambda V$. It follows that $S_\lambda V$ is an injective $S(n,r)$-module for $\lambda \in \Lambda(n,r)$.

Note also that the transposed dual does not change the formal character. As the formal character determines the simples, ${}^t(L(\lambda)) = L(\lambda)$. This shows a very important

2.37 Lemma. For λ, $\mu \in \Lambda(n,r)^+$,

$$\mathrm{Ext}^i_{S(n,r)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^i_{S(n,r)}(L(\mu), L(\lambda)).$$

2.38 Example. We show the simplest example. Let k be of characteristic two, $n = \dim V = 2$, and $r = 2$. The map $i : \bigwedge^2 V \to D_2 V$ given by $i(w_1 \wedge w_2) = w_1 w_2$ is nonzero, and hence is injective, since $\bigwedge^2 V$ is one-dimensional and hence is simple. The sequence

$$0 \to \bigwedge^2 V \to D_2 V \to D_2 V / \bigwedge^2 V \to 0$$

is exact, and is non-split, since $D_2 V = \Delta((2,0))$ has a simple top. It follows that $\bigwedge^2 V$ is not injective. It is easy to see that $D_2 V / \bigwedge^2 V$ is simple and agrees with $L(2,0)$. Note that the sequence

$$0 \to D_2 V \xrightarrow{\Delta} V \otimes V \to \bigwedge^2 V \to 0$$

is non-split, since $V \otimes V = \mathrm{Sym}_{(1,1)} V$ is projective injective, and $\bigwedge^2 V$ is not injective. This shows that $D_2 V \subset \mathrm{rad}(V \otimes V)$, since $D_2 V$ is indecomposable projective. Thus $V \otimes V$ has the simple top $\bigwedge^2 V$, and $V \otimes V = P(1,1)$. Thus we have

$$P(1,1) = \begin{matrix} L(1,1) \\ L(2,0) \\ L(1,1) \end{matrix} \quad P(2,0) = \begin{matrix} L(2,0) \\ L(1,1) \end{matrix}.$$

Note that $\Delta(1,1) = L(1,1)$. Note also that $D_2 V / \bigwedge^2 V$ is isomorphic to the first Frobenius twist $V^{(1)}$ of the vector representation.

(2.39) Notes and References. As we will see later, Corollary 2.29 and Corollary 2.35 show that $S(n,r)$ is a quasi-hereditary algebra. The notion of Schur algebra is generalized by S. Donkin [D1, D2]. This generaized Schur algebras are also quasi-hereditary. The proof usually requires the standard course in representation theory of algebraic groups [J], including Kempf's vanishing. Our argument is good only for $S(n,r)$, but is elementary in the sense that it only requires multilinear algebra.

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3 Tilting modules of GL_n

(3.1) For sure, we start with the definition of quasi-hereditary algebra. For more, see [DR] and references therein. Consider a triple (A, Λ, L) such that A is a finite dimensional k-algebra, Λ a finite ordered set, and L a bijection from Λ to the set of isomorphism classes of simple A-modules. For $\lambda \in \Lambda$, we denote the projective cover and the injective hull of $L(\lambda)$ by $P(\lambda)$ and $Q(\lambda)$, respectively. For $\lambda \in \Lambda$, define $Z(\lambda) := \{\mu \in \Lambda \mid \mu > \lambda\}$, and $Z'(\lambda) := \{\mu \in \Lambda \mid \mu \not\leq \lambda\}$. We say that A (or better, (A, Λ, L)) is adapted if $\operatorname{tr}_{Z(\lambda)} P(\lambda) = \operatorname{tr}_{Z'(\lambda)} P(\lambda)$ for any $\lambda \in \Lambda$.

3.2 Lemma. Let (A, Λ, L) be as above. Then the following are equivalent.

1. (A, Λ, L) is adapted.
2. For incomparable elements $\lambda, \mu \in \Lambda$ and a finite dimensional A-module V such that $\operatorname{top} V \cong L(\lambda)$ and $\operatorname{soc} V \cong L(\mu)$, there exists some $\nu \in \Lambda$ such that $\nu > \lambda$, $\nu > \mu$, and $L(\nu)$ is a subquotient of V, where soc denotes the socle of a module.
3. $(A^{\mathrm{op}}, \Lambda, L^*)$ is adapted, where A^{op} is the opposite k-algebra of A, and $L^*(\lambda) := L(\lambda)^*$.

(3.3) Let (A, Λ, L) be as above. For $\lambda \in \Lambda$, we define the Weyl module $\Delta(\lambda) = \Delta_A(\lambda)$ to be $P(\lambda)/\operatorname{tr}_{Z'(\lambda)} P(\lambda)$. We define the dual Weyl module $\nabla(\lambda) = \nabla_A(\lambda)$ to be $\Delta_{A^{\mathrm{op}}}(\lambda)^*$. Or equivalently, $\nabla(\lambda)$ is defined to be the largest submodule of $Q(\lambda)$ whose simple subquotient is isomorphic to $L(\mu)$ for some $\mu \leq \lambda$.

An A-module V is said to be Schurian if $\operatorname{End}_A V$ is a division ring. If V is finite dimensional, then this is equivalent to saying that $k \to \operatorname{End}_A V$ is an isomorphism, since k is algebraically closed.

3.4 Lemma. For $\lambda \in \Lambda$, the following are equivalent.

1. $\Delta(\lambda)$ is Schurian.
2. $[\Delta(\lambda) : L(\lambda)] = 1$.
3. If V is a finite dimensional A-module, $[V : L(\mu)] \neq 0$ implies $\mu \leq \lambda$, and $\operatorname{top} V \cong \operatorname{soc} V \cong L(\lambda)$, then $V \cong L(\lambda)$.
4. $[\nabla(\lambda) : L(\lambda)] = 1$.
5. $\nabla(\lambda)$ is Schurian.

(3.5) Let $\mathcal{A}$ be an abelian category, and $\mathcal{C}$ be a set of its objects. We define $\mathcal{F}(\mathcal{C})$ to be the full subcategory of $\mathcal{A}$ consisting of objects A of $\mathcal{A}$ such that there is a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = A$$

such that each V_i/V_{i-1} is isomorphic to an element of $\mathcal{C}$. Let (A, Λ, L) be as above. Then we define $\Delta = \{\Delta(\lambda) \mid \lambda \in \Lambda\}$, and $\nabla = \{\nabla(\lambda) \mid \lambda \in \Lambda\}$.

(3.6) Let $\mathcal{A}$ be an abelaian category, and $\mathcal{C}$ a set of objects or a full subcategory. We define ${}^\perp\mathcal{C}$ to be the full subcategory of $\mathcal{A}$ consisting of $A \in \mathcal{A}$ such that $\mathrm{Ext}^i_{\mathcal{A}}(A, C) = 0$ for any $C \in \mathcal{C}$ and $i > 0$. Similarly, we define $\mathcal{C}^\perp$ to be the full subcategory of $\mathcal{A}$ consisting of $B \in \mathcal{A}$ such that $\mathrm{Ext}^i_{\mathcal{A}}(C, B) = 0$ for any $C \in \mathcal{C}$ and $i > 0$.

Let A be a finite dimensional k-algebra. Set $\mathcal{A} = A\,\mathrm{mod}$. Then a full subcategory of the form $\mathcal{X} = {}^\perp\mathcal{C}$ for some subset $\mathcal{C}$ of the object set of $\mathcal{A}$ is resolving (that is, closed under extensions and epikernels, and contains all projective modules), and is closed under direct summands. Similarly, a full subcategory of the form $\mathcal{Y} = \mathcal{C}^\perp$ for some subset $\mathcal{C}$ of the object set of $\mathcal{A}$ is coresolving (that is, closed under extensions and monocokernels, and contains all injective modules), and is closed under direct summand.

3.7 Proposition. Let (A, Λ, L) be a triple such that A is a fiite dimensional k-algebra, Λ is a finite partially ordered set, and L is a bijection from Λ to the set of isomorphism classes of simples of A. Assume that A is adapted, and all Weyl modules $\Delta(\lambda)$ are Schurian. Then the following conditions are equivalent.

1. ${}_AA \in \mathcal{F}(\Delta)$.
2. If $X \in A\,\mathrm{mod}$ and $\mathrm{Ext}^1_A(X, \nabla(\lambda)) = 0$ for any $\lambda \in \Lambda$, then $X \in \mathcal{F}(\Delta)$.
3. $\mathcal{F}(\Delta) = {}^\perp\mathcal{F}(\nabla)$.
4. $\mathcal{F}(\nabla) = \mathcal{F}(\Delta)^\perp$.
5. $\mathrm{Ext}^2_A(\Delta(\lambda), \nabla(\mu)) = 0$ for $\lambda, \mu \in \Lambda$.

3.8 Definition. We say that A, or better, (A, Λ, L) is a quasi-hereditary algebra if A is adapted, $\Delta(\lambda)$ is Schurian for any $\lambda \in \Lambda$, and ${}_AA \in \mathcal{X}(\Delta)$.

Note that (A, Λ, L) is a quasi-hereditary algebra if and only if $(A^{\mathrm{op}}, \Lambda, L^*)$ is quasi-hereditary.

By Corollary 2.29 and Corollary 2.35, we immediately have that the Schur algebra $S(n,r)$ (or better, $(S(n,r), \Lambda^+(n,r), L)$) is a quasi-hereditary algebra, and $\Delta(\lambda)$ defined in the last section agrees with that in this section.

(3.9) Let (A, Λ, L) be a quasi-hereditary algebra. A finite dimensional A-module V is said to be good if $V \in \mathcal{F}(\nabla)$. V is said to be cogood if $V \in \mathcal{F}(\Delta)$.

Set $\omega = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

3.10 Theorem (Ringel [Rin]). Let A be a quasi-hereditary algebra, and $M \in A\,\mathrm{mod}$. Then there exists a unique (up to isomorphisms) short exact sequence

$$0 \to Y_M \xrightarrow{i} X_M \xrightarrow{p} M \to 0$$

such that $X_M \in \mathcal{F}(\Delta)$, $Y_M \in \mathcal{F}(\nabla)$, and p is right minimal (i.e., $\varphi \in \mathrm{End}_A(X_M)$, $p\varphi = p$ imply that φ is an isomorphism), and there exists a unique (up to isomorphisms) short exact sequence

$$0 \to M \xrightarrow{j} Y'_M \xrightarrow{q} X'_M \to 0$$

such that $Y'_M \in \mathcal{F}(\nabla)$, $X'_M \in \mathcal{F}(\Delta)$, and j is left minimal (i.e., $\psi \in \mathrm{End}_A Y'_M$, $\psi j = j$ imply ψ is an isomorphism).

We denote $X_{\nabla(M)}$ by $T(\lambda)$, and call it the indecomposable tilting module of highest weight λ. Note that $T(\lambda) \in \omega$, $T(\lambda)$ is indecomposable, and $Y'_{\Delta(M)} \cong T(\lambda)$. $T = \bigoplus_{\lambda \in \Lambda} T(\lambda)$ is called the (full) tilting module (the characteristic module) of the quasi-hereditary algebra A. Note that $\omega = \mathrm{add}\, T$. Note also that T is both tilting and cotilting module in the usual sense. There would be no problem if we call an A-module T' such that $\mathrm{add}\, T' = \mathrm{add}\, T$ a characteristic module of A, as we shall do so later. We call an object of ω a partial tilting module.

If λ is a minimal element of Λ then we have that $\Delta(\lambda) \cong L(\lambda) \cong \nabla(\lambda)$. Thus we have $L(\lambda)$ is partial tilting, and hence $L(\lambda) = T(\lambda)$.

(3.11) Now consider $GL_n = GL(V)$, where $V = k^n$ is an n-dimensional k-vector space with a basis $e_1, \ldots, e_n$. A finite dimensional polynomial representation $W = \bigoplus_r W_r$, where W_r is an $S(n,r)$-module, is said to be good (resp. cogood, partial tilting), if each W_r is so.

(3.12) As can be checked directly, for $0 \leq r \leq n$, $\bigwedge^r V$ is a simple $S(n,r)$-module whose highest weight is $\omega_r = (1, 1, \ldots, 1, 0, 0 \ldots, 0)$. As ω_r is a minimal element of $\Lambda^+(n,r)$, We have that

$$\Delta(\omega_r) \cong \nabla(\omega_r) \cong T(\omega_r) \cong L(\omega_r) \cong \bigwedge^r V.$$

The following theorem is useful in determining the tilting module of GL_n.

3.13 Theorem (Boffi–Donkin–Mathieu [Bof], [Don], [Mat]). If $M \in S(n,r)\,\mathrm{mod}$ and $N \in S(n,r')\,\mathrm{mod}$ are good (resp. cogood, partial tilting), then the tensor product $M \otimes N$ is good (resp. cogood, partial tilting) as an $S(n, r+r')$-module.

Thus for a sequence $\lambda = (\lambda_1, \ldots, \lambda_s)$ with $0 \leq \lambda_i \leq n$, the tensor product

$$\bigwedge\nolimits_\lambda V := \bigwedge^{\lambda_1} V \otimes \cdots \otimes \bigwedge^{\lambda_s} V$$

is partial tilting. Note that $e_1 \wedge \cdots \wedge e_{\lambda_1} \otimes \cdots \otimes e_1 \wedge \cdots \wedge e_{\lambda_s}$ is a highest weight vector of weight $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$, where $\tilde{\lambda}_i = \#\{j \mid \lambda_j \geq i\}$. As $\dim_k(\bigwedge^\lambda V)_{\tilde{\lambda}} = 1$, we have

3.14 Lemma. For each $\lambda \in \Lambda^+(n,r)$, there is an isomorphism of the form

$$\bigwedge\nolimits_{\tilde{\lambda}} V \cong T(\lambda) \oplus \bigoplus_{\mu < \lambda} T(\mu)^{\oplus c'(\lambda,\mu)}$$

Note that $\tilde{\tilde{\lambda}} = \lambda$ for $\lambda \in \Lambda^+(n,r)$. Note also that $c'(\lambda,\mu)$ depends on the characteristic of the base field k in general.

(3.15) Let $V = k^n$ with the basis $e_1, \ldots, e_n$. Assume that $n \geq r$. Then $D_{\omega_r} V = V^{\otimes r}$, where $\omega_r = (1, 1, \ldots, 1, 0, \ldots, 0) \in \Lambda^+(n,r)$, so

$$\mathrm{End}_{S(n,r)}(V^{\otimes r}) = (V^{\otimes r})_{\omega_r},$$

which has $\{\sigma(e_1 \otimes \cdots \otimes e_r)\}$ as its k-basis. Thus the map $k\mathfrak{S}_r \to \mathrm{End}_{S(n,r)}(V^{\otimes r})$ is an isomorphism.

(3.16) We define the autorphism of k-algebra $\Psi : k\mathfrak{S}_r \to k\mathfrak{S}_r$ by $\Psi(\sigma) = (-1)^\sigma \sigma$. So it induces the automorphism $\Psi : \mathrm{End}_{S(n,r)} V^{\otimes r} \to \mathrm{End}_{S(n,r)} V^{\otimes r}$.

3.17 Lemma. Let $\lambda, \mu \in \Lambda(n,r)$. Then there exists a unique isomorphism $\Psi : \mathrm{Hom}_{S(n,r)}(D_\lambda V, D_\mu V) \to \mathrm{Hom}_{S(n,r)}(\bigwedge_\lambda V, \bigwedge_\mu V)$ such that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{S(n,r)}(D_\lambda V, D_\mu V) & \xrightarrow{\Psi} & \mathrm{Hom}_{S(n,r)}(\bigwedge_\lambda V, \bigwedge_\mu V) \\ \downarrow{\scriptstyle \Delta_* m^*} & & \downarrow{\scriptstyle \Delta_* m^*} \\ \mathrm{Hom}_{S(n,r)}(V^{\otimes r}, V^{\otimes r}) & \xrightarrow{\Psi} & \mathrm{Hom}_{S(n,r)}(V^{\otimes r}, V^{\otimes r}) \end{array} \tag{3.17.1}$$

is commutative, and is compatible with the change of rings.

Proof (sketch). We work over arbitrary base ring R, rather than an algebraically closed field. Let $V_\mathbb{Z} = \mathbb{Z}^n$, and $V_R := R \otimes_\mathbb{Z} V_\mathbb{Z}$. $S(n,r)_\mathbb{Z} := D_r(\mathrm{End}_\mathbb{Z}(V_\mathbb{Z}))$ is the Schur algebra over $\mathbb{Z}$, and $S(n,r)_R := R \otimes_\mathbb{Z} S(n,r)_\mathbb{Z}$. Then we have canonical isomorphisms

$$R \otimes_\mathbb{Z} \mathrm{Hom}_{S(n,r)_\mathbb{Z}}(V_\mathbb{Z}^{\otimes r}, V_\mathbb{Z}^{\otimes r}) \cong \mathrm{Hom}_{S(n,r)_R}(V_R^{\otimes r}, V_R^{\otimes r}),$$
$$R \otimes_\mathbb{Z} \mathrm{Hom}_{S(n,r)_\mathbb{Z}}(D_\lambda V_\mathbb{Z}, D_\mu V_\mathbb{Z}) \cong \mathrm{Hom}_{S(n,r)_R}(D_\lambda V_R, D_\mu V_R),$$
$$R \otimes_\mathbb{Z} \mathrm{Hom}_{S(n,r)_\mathbb{Z}}(\textstyle\bigwedge_\lambda V_\mathbb{Z}, \bigwedge_\mu V_\mathbb{Z}) \cong \mathrm{Hom}_{S(n,r)_R}(\bigwedge_\lambda V_R, \bigwedge_\mu V_R).$$

The first isomorphism is easy, as

$$R \otimes_\mathbb{Z} \mathrm{Hom}_{S(n,r)_\mathbb{Z}}(V_\mathbb{Z}^{\otimes r}, V_\mathbb{Z}^{\otimes r}) \cong R \otimes_\mathbb{Z} (V_\mathbb{Z}^{\otimes r})_{\omega_r} \cong (V_R^{\otimes r})_{\omega_r} \cong \mathrm{Hom}_{S(n,r)_R}(V_R^{\otimes r}, V_R^{\otimes r}).$$

The second isomorphism also holds similarly. The third isomorphism is by the u-goodness of $\bigwedge_\mu V_\mathbb{Z}$, see [Has, Corollary III.4.1.8].

Thus we only have to prove the corresponding statement for $R = \mathbb{Z}$. However, first consider the case that $R = \mathbb{Q}$. Then for $\nu \in \Lambda(n,r)$, define

$$\mathfrak{S}_\nu = \{\sigma \in \mathfrak{S}_r \mid \forall i\ \sigma([\nu_1 + \cdots + \nu_{i-1} + 1, \nu_1 + \cdots + \nu_{i-1} + \nu_i]) \subset [\nu_1 + \cdots + \nu_{i-1} + 1, \nu_1 + \cdots + \nu_{i-1} + \nu_i]\}.$$

Also define idempotents

$$e_\nu = \frac{1}{\#\mathfrak{S}_\nu} \sum_{\sigma \in \mathfrak{S}_\nu} \sigma \in k\mathfrak{S}_r, \quad e'_\nu = \frac{1}{\#\mathfrak{S}_\nu} \sum_{\sigma \in \mathfrak{S}_\nu} (-1)^\sigma \sigma = \Psi(e_\nu) \in k\mathfrak{S}_r.$$

Then we can identify $D_\nu V_\mathbb{Q} \subset V_\mathbb{Q}^{\otimes r}$ by $e_\nu V_\mathbb{Q}^{\otimes r}$, and $\bigwedge_\nu V_\mathbb{Q} \subset V_\mathbb{Q}^{\otimes r}$ by $e'_\nu V_\mathbb{Q}^{\otimes r}$. Thus $\mathrm{Hom}_{S(n,r)_\mathbb{Q}}(D_\lambda V_\mathbb{Q}, D_\mu V_\mathbb{Q})$ and $\mathrm{Hom}_{S(n,r)_\mathbb{Q}}(\bigwedge_\lambda V_\mathbb{Q}, \bigwedge_\mu V_\mathbb{Q})$ are respectively identified with $e_\mu k\mathfrak{S}_r e_\lambda$ and $e'_\mu k\mathfrak{S}_r e'_\lambda$. So Ψ maps $\mathrm{Hom}_{S(n,r)_\mathbb{Q}}(D_\lambda V_\mathbb{Q}, D_\mu V_\mathbb{Q})$ bijectively onto $\mathrm{Hom}_{S(n,r)_\mathbb{Q}}(\bigwedge_\lambda V_\mathbb{Q}, \bigwedge_\mu V_\mathbb{Q})$, and Ψ is its inverse.

Now consider the case $R = \mathbb{Z}$. Then as $V_\mathbb{Z}^{\otimes r} \to \bigwedge_\lambda V_\mathbb{Z}$ is surjective and $\bigwedge_\mu V_\mathbb{Z} \to V_\mathbb{Z}^{\otimes r}$ is a $\mathbb{Z}$-split mono, we have

$$\mathrm{Hom}_{S(n,r)_\mathbb{Z}}(\textstyle\bigwedge_\lambda V_\mathbb{Z}, \bigwedge_\mu V_\mathbb{Z}) = (\mathrm{Hom}_{S(n,r)_\mathbb{Z}}(\bigwedge_\lambda V_\mathbb{Z}, \bigwedge_\mu V_\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Q})\cap\mathrm{Hom}_\mathbb{Z}(V_\mathbb{Z}^{\otimes r}, V_\mathbb{Z}^{\otimes r}).$$

So

$$\Psi : \mathrm{Hom}_{S(n,r)_\mathbb{Z}}(D_\lambda V_\mathbb{Z}, D_\mu V_\mathbb{Z}) \to \mathrm{Hom}_{S(n,r)_\mathbb{Z}}(\textstyle\bigwedge_\lambda V_\mathbb{Z}, \bigwedge_\mu V_\mathbb{Z})$$

is uniquely defined so that the diagram (3.17.1) is commutative.

Ψ is clearly injective, as it is injective when considered over $\mathbb{Q}$. The surjectivity is difficult, and we omit the proof. See [AB]. □

(3.18) Now set $T = \bigoplus_{\lambda\in\Lambda(n,r)} \bigwedge_\lambda V$. Then T could be called a characteristic module of $S(n,r)$. Note that

$$S(n,r) = (\mathrm{End}_{S(n,r)} S(n,r))^{\mathrm{op}} = (\mathrm{End}_{S(n,r)}(\bigoplus_{\lambda\in\Lambda(n,r)} D_\lambda V))^{\mathrm{op}} \xrightarrow{\Psi} (\mathrm{End}_{S(n,r)} T)^{\mathrm{op}}$$

is an algebra isomorphism, as can be seen easily from the fact that $\Psi : k\mathfrak{S}_r \to k\mathfrak{S}_r$ is an algebra isomorphism. As $\mathrm{Hom}_{S(n,r)}(T, ?)$ is a functor from $S(n,r)\,\mathrm{mod}$ to $(\mathrm{End}_{S(n,r)} T)^{\mathrm{op}}\,\mathrm{mod}$, we have that it is also considered as a functor from $S(n,r)\,\mathrm{mod}$ to itself.

Now we invoke the following Ringel's theorem.

3.19 Theorem (Ringel [Rin, Theorem 6]). Let (A, Λ, L) be a quasi-hereditary algebra, and T its characteristic module. Set $A' = (\mathrm{End}_A T)^{\mathrm{op}}$, $\Lambda' = \Lambda^{\mathrm{op}}$, $F := \mathrm{Hom}_A(T, ?) : A\,\mathrm{mod} \to A'\,\mathrm{mod}$, Then (A', λ', L') is a quasi-hereditary algebra, and $F(\nabla_A(\lambda)) \cong \Delta_{A'}(\lambda)$, where $L'(\lambda) = \mathrm{top}(F(\nabla_A(\lambda)))$.

(3.20) Set $\nabla_A = \{\nabla_A(\lambda) \mid \lambda \in \Lambda\}$, $\Delta_A = \{\Delta_A(\lambda) \mid \lambda \in \Lambda\}$, $\nabla_{A'} = \{\nabla_{A'}(\lambda) \mid \lambda \in \Lambda'\}$, and $\Delta_{A'} = \{\Delta_{A'}(\lambda) \mid \lambda \in \Lambda'\}$. As T is a tilting module (in the sense of [Miy]), $F : \mathcal{F}(\nabla_A) \to \mathcal{F}(\Delta_{A'})$ is an exact equivalence, whose quasi-inverse $G : \mathcal{F}(\Delta_{A'}) \to \mathcal{F}(\nabla_A)$ is given by $G = T\otimes_{A'}?$ [Miy]. This equivalence induces an equivalence $\omega_A \to \mathrm{proj}\, A'$.

(3.21) Through the isomorphism $S(n,r) \cong (\mathrm{End}_{S(n,r)} T)^{\mathrm{op}}$, we get a functor $F = \mathrm{Hom}_{S(n,r)}(T, ?) : S(n,r)\,\mathrm{mod} \to S(n,r)\,\mathrm{mod}$. Note that $F(\bigwedge_\lambda V) = D_\lambda V$ almost by the definition of F. Let $T(\lambda)$ be the indecomposable tilting module of highest weight λ. Then $F(T(\tilde{\lambda})) = P(\lambda)$ by Lemma 2.28 and

Lemma 3.14. From this, the simple $L(\lambda)$ corresponds to the simple $L(\tilde{\lambda})$ by F, as $\tilde{\tilde{\lambda}} = \lambda$ for $\Lambda^+(n,r)$. As $\tilde{?}$ is order-reversing, the map $(S(n,r), \Lambda^+(n,r), L) \to (S(n,r)', \Lambda^+(n,r)', L')$ is an isomorphism of quasi-hereditary algebra, which is appropriately defined, where $S(n,r) \to S(n,r)' = (\mathrm{End}_{S(n,r)} T)^{\mathrm{op}}$ is given above, and $\Lambda^+(n,r) \to \Lambda^+(n,r)'$ is the order reversing map $\tilde{?}$.

Thus we have,

3.22 Theorem. Let $n \geq r$. Then $T = \bigoplus_{\lambda \in \Lambda(n,r)} \bigwedge_\lambda V$ is a characteristic module (which may not be basic), and $S(n,r) \cong (\mathrm{End}_{S(n,r)} T)^{\mathrm{op}}$. The tilting $F : \mathrm{Hom}_{S(n,r)}(T, ?)$ gives an exact equivalence $F : \mathcal{F}(\nabla) \to \mathcal{F}(\Delta)$. We have $F(\nabla(\lambda)) = \Delta(\tilde{\lambda})$.

3.23 Corollary (Akin–Buchsbaum [AB]). Let $n \geq r$. Then

$$\mathrm{Ext}^i_{S(n,r)}(\nabla(\lambda), \nabla(\mu)) \cong \mathrm{Ext}^i_{S(n,r)}(\Delta(\tilde{\lambda}), \Delta(\tilde{\mu}))$$

for $\lambda, \mu \in \Lambda^+(n,r)$ and $i \geq 0$.

3.24 Corollary. Let $n \geq r$. Then $c'(\lambda, \mu)$ in Lemma 3.14 agrees with $c(\tilde{\lambda}, \tilde{\mu})$ (in Lemma 2.28).

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