Approaches To the Quantization of Nambu Mechanics and Nambu Brackets:

An overview and generalized Hamilton-Jacobi theory

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Generalized Hamiltonian Dynamics*

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Taking the Liouville theorem as a guiding principle, we propose a possible generalization of classical Hamiltonian dynamics to a three-dimensional phase space. The equation of motion involves two Hamiltonians and three canonical variables. The fact that the Euler equations for a rotator can be cast into this form suggests the potential usefulness of the formalism. In this article we study its general properties and the problem of quantization.

Nambu's attempt of quantization was unfortunately not successful.

"One is repeatedly led to discover that the quantized version is essentially equivalent to the ordinary quantum theory. This may be an indication that quantum theory is pretty much unique, although its classical analog may not be."

Nambu's Original motivation:
an extension of
Liouville theorem

Hamiltonian dynamics :

area-preserving flow in phase space $(\xi^1, \xi^2) = (q, p)$

$$\frac{dX}{dt} = \{H, X\} = D^i(H)\partial_i X \qquad D^i(H) \equiv \epsilon^{ij}\frac{\partial H}{\partial \xi^j}$$

$$\partial_i D^i(H) = 0$$

Nambu dynamics :

volume-preserving flow in an extended phase space (ξ^1, ξ^2, ξ^3)

$$\frac{dX}{dt} = \underbrace{\{H, G, X\}}_{\text{Nambu bracket}} = D^i(H, G)\partial_i X$$

$$\partial_i D^i(H, G) = 0$$

$$\{K, L, M\} = \frac{\partial(K, L, M)}{\partial(\xi^1, \xi^2, \xi^3)} = \epsilon^{ijk}\partial_i K \partial_j L \partial_k M$$

Remark: There is a "gauge freedom" with respect to the choice of Hamiltonians as emphasized by Nambu himself.

"N"-gauge symmetry:

$$(H,G) \to (H',G')$$

$$H\delta G - H'\delta G' = \delta\Lambda$$

$$\frac{\partial\Lambda}{\partial G} = H, \quad \frac{\partial\Lambda}{\partial G'} = -H' \qquad \Lambda = \Lambda(G,G')$$

Since $\frac{\partial(H',G')}{\partial(H,G)} = 1$, the Nambu equations of motion are invariant.

$$\frac{d\xi^1}{dt} = \frac{\partial(H,G)}{\partial(\xi^2,\xi^3)}, \quad \frac{d\xi^2}{dt} = \frac{\partial(H,G)}{\partial(\xi^3,\xi^1)}, \quad \frac{d\xi^3}{dt} = \frac{\partial(H,G)}{\partial(\xi^1,\xi^2)}$$

Note also that Nambu equations of motion is equivalent to the choice of vector potential for D_i in the so-called Clebsch representation (also mentioned by Nambu):

$$D_i = \frac{1}{2}\epsilon_{ijk}F_{jk} \quad F_{jk} = \partial_j A_k - \partial_k A_j \quad A_k = H\partial_k G + \partial_k\psi \quad \text{or} \quad -G\partial_k H + \partial_k\psi$$

This suggests a more general form $H\partial_k G \to \sum_a H_a\partial_k G_a$ (no conserved Hamiltonian)

Basic properties of Nambu bracket:

1. skew symmetry

$$\{A_1, A_2, A_3\} = (-1)^{\epsilon(p)}\{A_{p(1)}, A_{p(2)}, A_{p(3)}\}$$

2. Leipniz rule

$$\{A_1A_2, A_3, A_4\} = A_1\{A_2, A_3, A_4\} + \{A_1, A_3, A_4\}A_2$$

3. Jacobi-like ("fundamental") identity (FI)

$$\{A_1, A_2, \{A_3, A_4, A_5\}\} =$$

$$\{\{A_1, A_2, A_3\}, A_4, A_5\}+\{A_3, \{A_1, A_2, A_4\}, A_5\}+\{A_3, A_4, \{A_1, A_2, A_5\}\}$$

Takhtajan, and others, ~1993

The FI is the most crucial consistency condition for interpreting the Nambu equations of motion as

infinitesimal canonical transformations,

though Nambu himself was unaware of this fact at the time of his proposal.

Why Nambu mechanics and its quantization ?

- interesting by itself: adds *flexibility to our thinking*, with respect to the methodology of dynamical descriptions of any physical and mathematical phenomena.

- a challenge: quantization seems to be *as* (more) *difficult as* (than) the *direct quantization of general relativity has been.*

  Why is it so difficult?

- *possible relevance to* (covariant) formulations of Matrix theories, M-theory membranes, and string/M theory, in general.

$\cup$

! quantum gravity !

In my opinion, there is no completely satisfactory quantum Nambu mechanics to this day.

Nambu-type symmetry and Nambu bracket naturally appear in classical theory of relativistic membrane:

Classical world-volume action (Dirac, 1962) can be rewritten by introducing an auxiliary variable *e*.

$$A_{\rm mem} = -\frac{1}{\ell_{11}^3}\int d^3\xi\Big(\frac{1}{e}\{X^\mu, X^\nu, X^\sigma\}_{\rm N}\{X_\mu, X_\nu, X_\sigma\}_{\rm N} - e\Big),$$

$$\{X^\mu, X^\nu, X^\sigma\}_{\rm N} \equiv \sum_{a,b,c}\epsilon^{abc}\partial_a X^\mu \partial_b X^\nu \partial_c X^\sigma,$$

symmetry transformation :3D diffeo. transformation $\delta X^\mu = \{F, G, X^\mu\}_{\rm N}$

If we replace the Nambu bracket by an appropriate discretized version (or *quantized* version), we would obtain regularized (super) membrane theory, which could be manifestly covariant and would hopefully be a covariantized version of so-called M(atrix) theory!

This was how I first became aware of Nambu's work around 1997.
Similar idea for a regularization of relativistic string had been briefly discussed by Nambu far back in 1976.

From a more general viewpoint, we can possibly consider two ***basic*** approaches to the quantization of Nambu mechanics.

1. Matrix-type Mechanics

Nambu himself studied only this direction.

However, he had not taken FI into account, unfortunately.
For example, such a dynamics cannot be laid properly into the framework of canonical transformations for the interpretation of Nambu equation of motion.
(The triple canonical structure would not be preserved by the time evolution.)
Actually, this seems to be one of the reasons why he met with little success in his attempt at quantization.

For more details, see, e.g. my lecture note (especially, part II),
T.Y. arXiv 1612:08513, published in "Non-Commutative Geometry and Physics 4" (eds.Y. Maeda et al., World Scientific, 2017) ;
also a general account in JPS Bulletin, Butsuri 72, 231(2017).

One among early attempts of matrix-type quantization, *FI being taken into account,* was our work in 1999, Awata, Li, Minic, T.Y. (hep-th/9906248, JHEP 02(2001)013).

In this work, we have proposed various different possibilities of realizing FI in terms of **square** matrices and **cubic (or higher)** matrices.

More recently, I have slightly extended this method in the case of square matrices, and applied it to construct a possible ***covariantized*** version of the M(atrix) theory.

$$X^\mu(\tau) = (X^\mu_{\mathrm{M}}(\tau), \boldsymbol{X}^\mu(\tau))$$

T.Y., JHEP06(2016)058 [arXiv: 1603.06402]

"M"-variables (auxiliary but dynamical)

$N \times N$ hermitian matrix variables

- **3-bracket**

$$[X, Y, Z] \equiv (0, X_{\mathrm{M}}[\boldsymbol{Y}, \boldsymbol{Z}] + Y_{\mathrm{M}}[\boldsymbol{Z}, \boldsymbol{X}] + Z_{\mathrm{M}}[\boldsymbol{X}, \boldsymbol{Y}])$$

◇ total skew symmetry

$$[X, Y, Z] = -[Y, X, Z] = -[X, Z, Y] = -[Z, Y, X]$$

◇ fundamental identity

$$[F, G, [X, Y, Z]] = [[F, G, X], Y, Z] + [X, [F, G, Y], Z] + [X, Y, [F, G, Z]]$$

These variables are treated as ordinary canonical coordinates for D-particles in the sense of 11 dimensions of M-theory.

The Nambu transformations appear as the characteristic symmetry for this system, ***not as the equations of motion.***

Remarks:

1 Nambu's Leibnitz rule (basic property 2.) is not satisfied, and does not play any important role for this construction: its absence is not necessarily a defect but is actually useful in constraining the form of possible actions and observables of the theory.

2 Nambu's original suggestion about a candidate for the bracket for three square matrices A,B,C was *different, but was close to* this form :

$$\begin{aligned}[A, B, C]_{\mathrm{N}} &\equiv ABC + BCA + CAB - BAC - ACB - CBA \\ &= A[B, C] + B[C, A] + C[A, B] \\ &= [A, B]C + [B, C]A + [C, A]B \quad , \quad \textit{FI not satisfied !}\end{aligned}$$

He then studied how the derivation law could be satisfied with this ansatz.
His conclusion was that the equation of motion reduced to the usual Heisenberg eq. :

typically as

$$i\frac{dF}{dt} = [H, G, F]_{\mathrm{N}} \xrightarrow[H = \alpha G + \beta]{[H,G] = 0} \beta[G, F]$$

essentially usual form!

However,

3 If we assume the Nambu-type 3 bracket for cubic (and higher) matrices, FI can be satisfied.

For an example,

$$[A, B, C] \equiv (ABC) + (BCA) + (CAB) - (CBA) - (ACB) - (BAC)$$

$$(ABC)_{ijk} \equiv \sum_{p} A_{ijp} \langle B \rangle C_{pjk} = \sum_{pqm} A_{ijp} B_{qmq} C_{pjk} \qquad \langle A \rangle \equiv \sum_{pm} A_{pmp}$$

In this talk, I will not pursue this and other similar (or related) possibilities further.

Hopefully, it is reserved as future works for everyone.

4 Distinction between Nambu *dynamics* and Nambu *symmetry*

If we consider relativistic extended objects, we can naturally introduce higher symplectic forms

$$d\omega^{(n)} = p_{\mu_1\mu_2\cdots\mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}$$

and the world-volume action as

$$A_n = \int d^n\xi \left[p_{\mu_1\mu_2\cdots\mu_n} \{x^{\mu_1}, x^{\mu_2}, \ldots, x^{\mu_n}\} + \underline{\frac{e}{2} p_{\mu_1\mu_2\cdots\mu_n} p^{\mu_1\mu_2\cdots\mu_n}} \right]$$

Nambu (n=2, 1980) , Hoppe (n=3, 1982), Sugamoto (n>2, 1983)
….. with various different gauge choices.

However, the dynamical evolution of this system is governed by a **single Hamiltonian.** The appearance of this higher symplectic structure only reflects the existence of *n*-dimensional volume-preserving diffeomorphism ("Nambu symmetry").

Indeed, this system can directly be converted to the standard quadratic action with world-volume metric. As such, dynamics can be formulated naturally **in the usual Hamiltonian form.**

For more details (n=2 case), see T.Y. PTP 97,941(1997).

2. Wave-type Mechanics

As a preliminary step toward a wave-mechanical quantization,
one natural approach would be to try to construct
Hamilton-Jacobi-type formalisms for Nambu Mechanics.

Apparently, however, no one has ever pursued such a formalism seriously.

"It's difficult!" (from a conversation with Takhtajan, 2017, in ICTS, Bangarole, India)

Once the wave-mechanical approach is established, it should also be possible to construct path-integral formulation.

Remark: There has been also another possibility, such as an attempt by
Dito, Flato, Steinheimer, Takhtajan , 1996 (CMP 183,1,1997; hep-th/9602016),
pursuing a possibility of deformation-type (called ***"Zariski"*** quantization)
with both Leibnitz and FI being taken into account.
(NB: naive deformations in terms of the usual Moyal-type do not satisfy FI)
Unfortunately, to my knowledge, application of this method to concrete physical systems has been extremely scarce.

I have nothing to say about such approaches in this talk.

Obstacles, at least *naively*, against HJ formalism:

- canonical triplet does not lend, at least apparently, natural decomposition of the phase space into pairs of generalized coordinates and momenta.

$$\{\xi^i, \xi^j, \xi^k\} = \epsilon^{ijk}$$

- no explicit formulation of finite canonical transformations is known, as opposed to the infinitesimal ones.

- there is no action integral defined for each one-dimensional trajectory in the phase space, as opposed to a known (*due to Takhtajan*) action functional for a ***continuous family, forming a surface, of the one-dimensional trajectories.***

But this method introduces a huge unphysical redundancy; it does not seem appropriate for our purpose of constructing Hamilton-Jacobi like formalism and also toward quantization.

Indeed, to my knowledge, there has been no serious attempt toward Hamilton-Jacobi formalism until my own attempt in 2016-2017.

I will spend the rest of my talk to present a possible approach to such a formalism and discuss its implication for the quantization problem of Nambu mechanics on the basis of this work.

Our vision (or prospect) will be quite different from Nambu's conclusion.

T.Y., PTEP(2017)023A01 [arXiv: 1612.08509]

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Generalized Hamilton–Jacobi theory of Nambu mechanics

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We develop a Hamilton–Jacobi-like formulation of Nambu mechanics. The Nambu mechanics, originally proposed by Nambu more than four decades ago, provides a remarkable extension of the standard Hamilton equations of motion in even-dimensional phase space with a single Hamiltonian to a phase space of three (and more generally, arbitrary) dimensions with two Hamiltonians (n Hamiltonians in the case of $(n+1)$-dimensional phase space) from the viewpoint of the Liouville theorem. However, it has not been formulated seriously in the spirit of Hamilton–Jacobi theory. The present study is motivated to suggest a possible direction towards quantization from a new perspective.

Subject Index A00, A13, B23, B25

Historical remark 1:

An interesting reformulation of the HJ formalism *that does not presuppose any knowledge of an action functional nor of canonical transformations* was emphasized by Einstein ("free of surprising tricks of trades",1917) , at least in its physical essence, in his attempt at generalizing the Sommerfeld-Epstein quantization to non-separable cases and giving a coordinate-independent formulation of semi-classical quantum theory.

$$\oint p_i dq^i$$

:the same value for all closed curves that can be continuously transformed to each other if

$$\frac{\partial p_j}{\partial q^i} - \frac{\partial p_i}{\partial q^j} = 0$$

It seems that both de Broglie and Schrödinger had been greatly influenced by Einstein's work in their inception of wave mechanics.

The spirit of my attempt toward HJ-like formalism for Nambu mechanics is along this line of thought.

Historical remark 2:

Attempt to generalize the Hamilton-Jacobi theory to systems with higher symplectic forms has a long history: there have been works by Caratheodory (1929), De Donder (1935), Weyl (1935), Lepage (1942), ….., Kastrup (1977), Rinke (1980), Nambu (1980) and others.

For a review, see, e.g., Kastrup, Phys. Rep. 101, 1 (1983)

Any field theory, including world-volume theories of extended objects, can formally have a higher symplectic form whose rank coincides with the dimensions of the base space.

However, to my knowledge, all those works have dealt only with the various case of a single Hamiltonian, including Nambu's 1980 paper on string dynamics which is essentially equivalent with De Donder-Weyl formulation.

My attempt has little relation to these works.

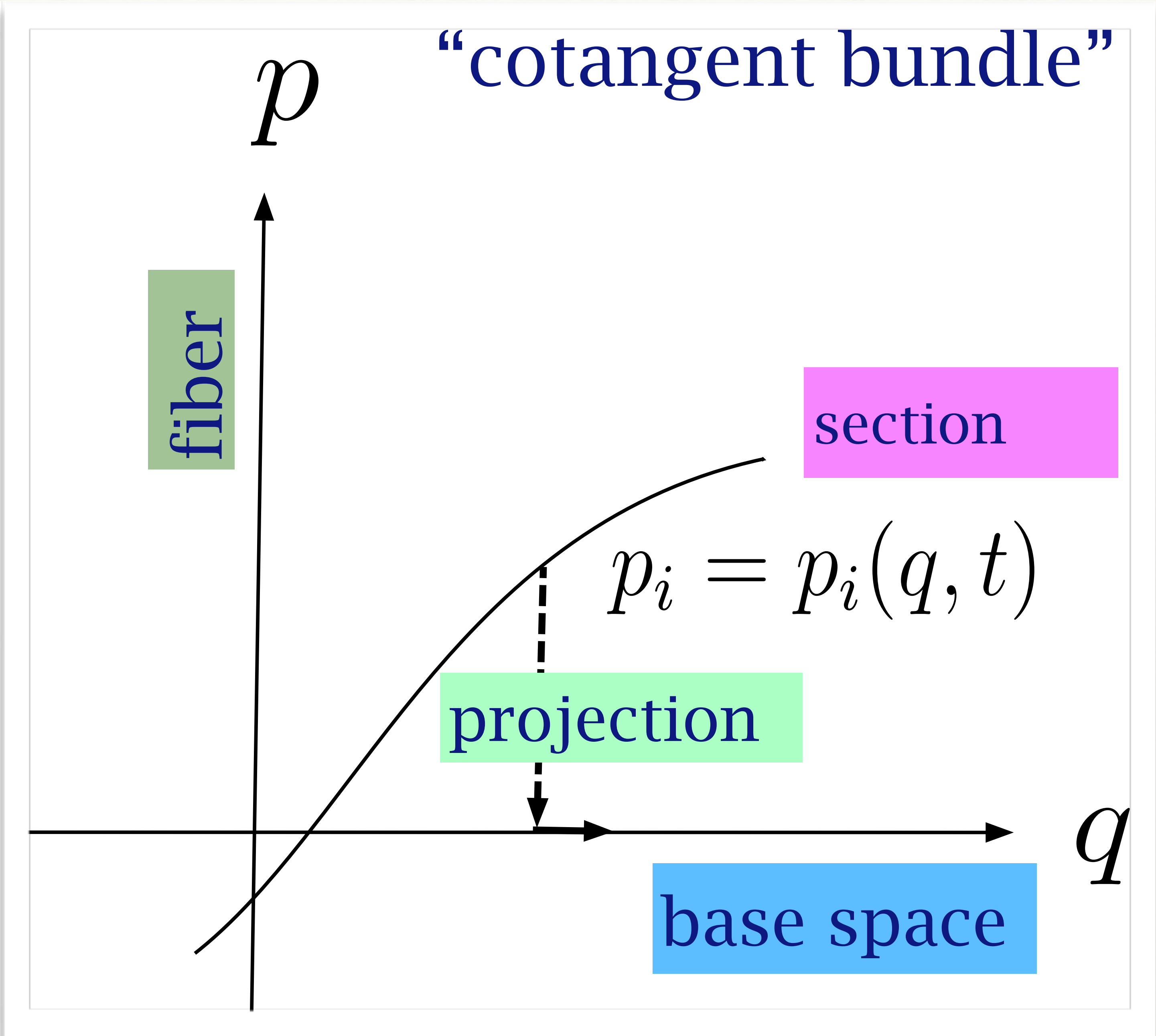
Conventional HJ formalism and geometrical interpretation

$S(q,t;Q)$: action as a field on the base (configuration) space (q^i, t)

$$p_i(q,t) = \frac{\partial S}{\partial q^i}$$: generalized momenta

$$\frac{\partial S}{\partial t} + \bar{H} = 0 \qquad \bar{H} = \bar{H}(q,t) = H(p(q,t),q)$$

Hamilton-Jacobi equation as governing equation



projection

$$\bar{\omega}^{(2)} \equiv \omega^{(2)}|_{(q,t)} = \frac{1}{2}\left(\frac{\partial p_j}{\partial q^i} - \frac{\partial p_i}{\partial q^j}\right) dq^i \wedge dq^j - \left(\frac{\partial p_i}{\partial t} + \frac{\partial \bar{H}}{\partial q^i}\right) dq^i \wedge dt = 0$$

$$\bar{\omega}^{(1)} \equiv \omega^{(1)}|_{(q,t)} = dS = \frac{\partial S}{\partial q^i} dq^i + \frac{\partial S}{\partial t} dt$$

$$\omega^{(2)} = dp_i \wedge dq^i - dH \wedge dt = d\omega^{(1)} \qquad \omega^{(1)} = p_i dq^i - H dt \qquad H = H(p,q)$$

Poincaré-Cartan

closed and exact 2-form in phase space (p_i, q^i, t)

In the geometrical picture, the action (variational) principle is replaced by the requirement

$$p_i = p_i(q, t)$$: **gradient flow**

$$\bar{\omega}^{(1)} \equiv \omega^{(1)}|_{(q,t)} = dS = \frac{\partial S}{\partial q^i} dq^i + \frac{\partial S}{\partial t} dt \longleftarrow \omega^{(1)} = p_i dq^i - H dt$$

Rationale: Starting with the Hamilton eps. of motion,
(emphasized by Einstein in 1917)

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \qquad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}$$

we obtain the p. d. stream equations ("Euler-Einstein" equation) for the **momentum fields**

$$\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_j}\frac{\partial p_i}{\partial q^j} = 0 \Rightarrow \frac{\partial}{\partial q^i}\underbrace{\left(\frac{\partial J}{\partial t} + \bar{H}\right)}_{f} = 0$$

and define the action function s.t.

$$\partial J/\partial t - f = \partial S/\partial t$$

Together with the requirement

$$\boxed{p_i = \frac{\partial J}{\partial q^i} \longleftrightarrow \frac{\partial p_j}{\partial q^i} - \frac{\partial p_i}{\partial q^j} = 0}$$

this is essentially the HJ equation:

$$\boxed{p_i(q, t) = \frac{\partial S}{\partial q^i} \qquad \frac{\partial S}{\partial t} + \bar{H} = 0 \qquad \bar{H} = \bar{H}(q, t) = H(p(q, t), q)}$$

Jacobi Theorem

Conversely, the Hamilton equations of motion are reproduced by imposing the **Jacobi conditions** on the “complete solutions” of HJ equation: $S(q,t;Q)$

$$\frac{\partial S}{\partial Q^i} = -P_i$$

$2n$ integration constants

- **connection with the Schrödinger quantization in the semiclassical approximation.**

$$S \to \tilde{S} = S + P_i Q^i$$
$$\partial \tilde{S}/\partial Q_i = 0$$

$$\psi \sim \int d^n Q \, e^{i\tilde{S}/\hbar}$$

superposition principle

Let us try to investigate whether and how this avenue of quantization can be extended to Nambu mechanics!

Three steps in our generalization of the HJ formalism to the Nambu mechanics

- Step I : decompose the phase space into the base space and the fiber, and derive the **EE equations for sections**.
- Step II : demand the **vanishing of appropriate forms** in the phase space under the **projection** to the base space.
- Step III: find appropriate **Jacobi-like conditions**, enabling us to reproduce the Nambu equations of motion in the phase space with the **necessary and sufficient number of integration constants.**

The last step will be most difficult, but be crucial to quantization.

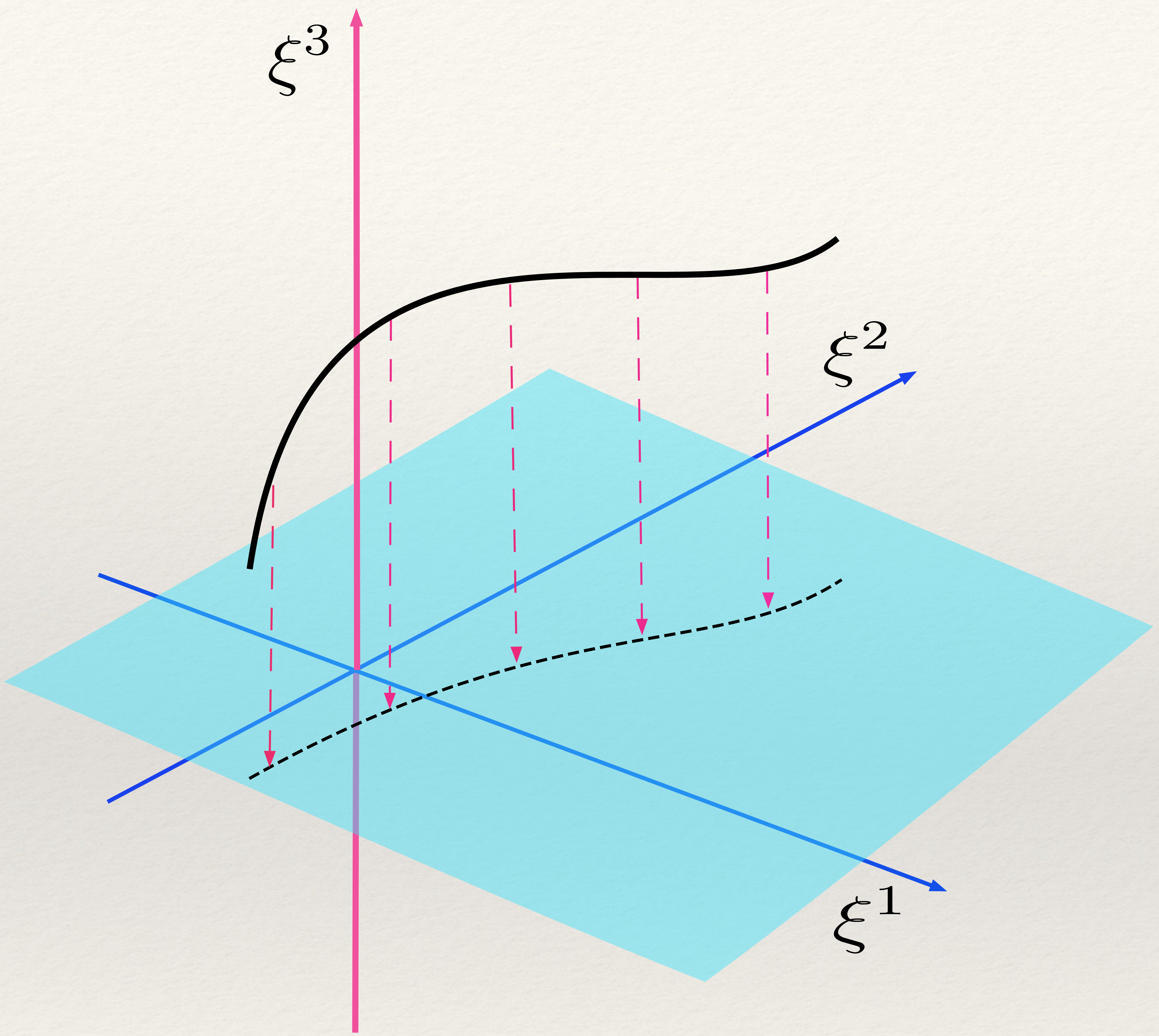
In the case of 3-dimensional phase space, there are basically two possibilities for the step I .

$$(1/2)\ \text{decomposition} : (\xi^1, \xi^2, \xi^3) \to \big(\underline{\xi^1, \xi^2}, \underline{\xi^3(\xi_1, \xi_2, t)}\big)$$

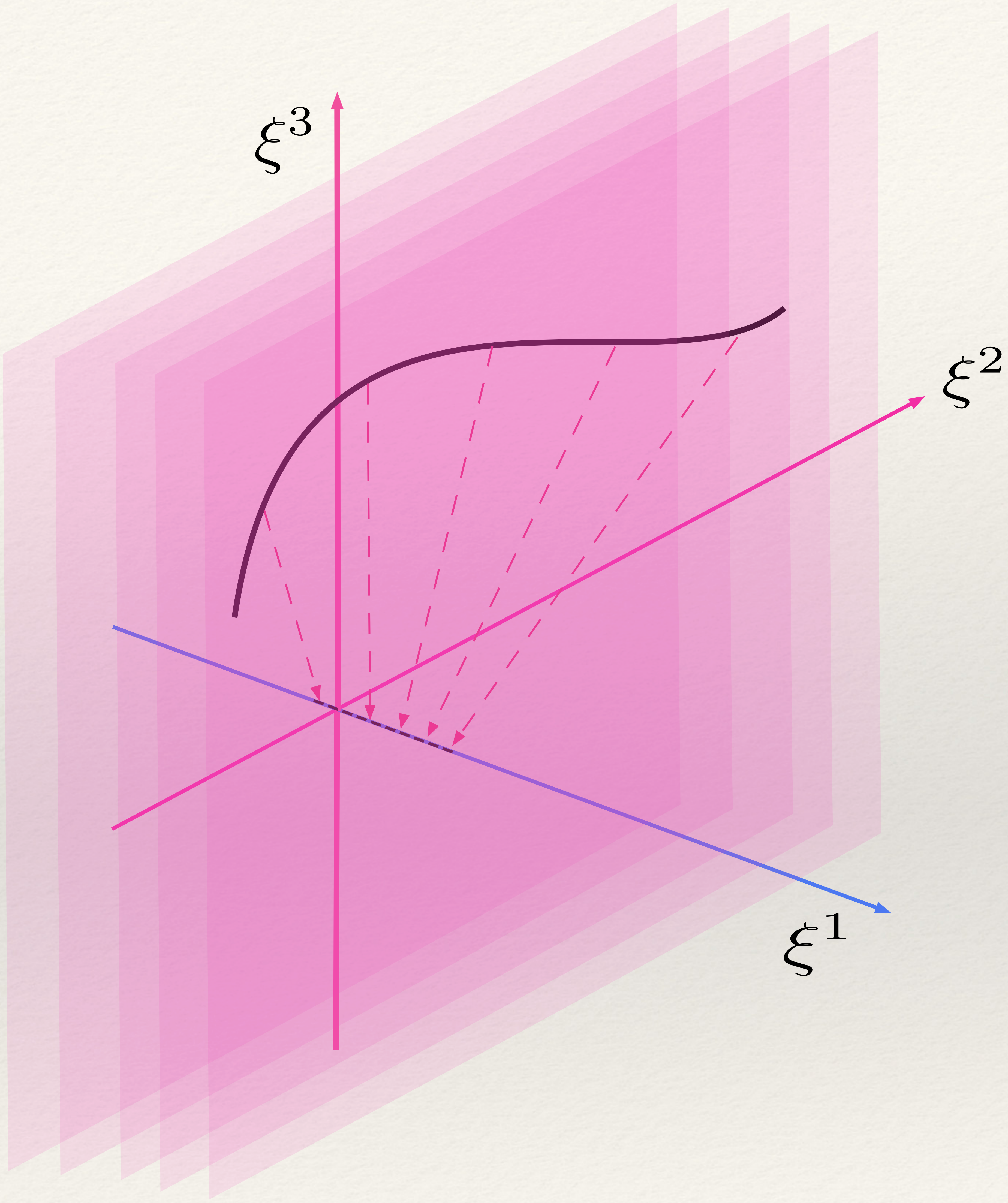
$$(2/1)\ \text{decomposition} : (\xi^1, \xi^2, \xi^3) \to \big(\underline{\xi^1}, \underline{\xi^2(\xi^1, t), \xi^3(\xi_1, t)}\big)$$

$(\underline{\text{fiber}}/\underline{\text{base}})$

$(1/2)$ decomposition : $(\xi^1, \xi^2, \xi^3) \to \big(\xi^1, \xi^2, \xi^3(\xi_1, \xi_2, t)\big)$



$(2/1)$ decomposition : $(\xi^1, \xi^2, \xi^3) \to \big(\xi^1, \xi^2(\xi^1, t), \xi^3(\xi_1, t)\big)$



We will briefly outline our results in each case.

(1/2) formalism : $\xi^3 = \xi^3(\xi^1, \xi^2, t)$

$$\bar{H} = \bar{H}(\xi^1, \xi^2, t) = H(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2, t)), \quad \bar{G} = \bar{G}(\xi^1, \xi^2, t) = G(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2, t))$$

- Step I : derive the EE stream equation starting with the Nambu eq. of motion

$$\partial_t \xi^3 = \partial_1(\bar{H}\partial_2\bar{G}) - \partial_2(\bar{H}\partial_1\bar{G})$$

- Step II : This result shows that we can naturally define a (2+1)-vector field

$S_\mu \quad (\mu = 1, 2, 0,\ \partial_t = \partial_0)$ such that

$$\boxed{\xi^3 \equiv \epsilon^{3ij}\partial_i S_j = \partial_1 S_2 - \partial_2 S_1 \qquad \partial_t S_i = \bar{H}\partial_i\bar{G} + \partial_i S_0}$$

the first version of generalized HJ equations

with a characteristic "S"-gauge symmetry $\quad S_\mu \to S_\mu + \partial_\mu \lambda$

(independent of the N gauge symmetry $\frac{\partial \Lambda}{\partial G} = H, \quad \frac{\partial \Lambda}{\partial G'} = -H' \quad S_0' = S_0 + \Lambda$)

In terms of differential forms, the system of these equations is equivalent to

$$\bar{\Omega}^{(1)} \equiv S_i d\xi^i + S_0 dt \equiv S_\mu d\xi^\mu$$

$$\bar{\Omega}^{(2)} \equiv d\bar{\Omega}^{(1)} = (\partial_1 S_2 - \partial_2 S_1) d\xi^1 \wedge d\xi^2 + (\partial_i S_0 - \partial_0 S_i) d\xi^i \wedge dt$$

$$= \xi^3 d\xi^1 \wedge d\xi^2 - \bar{H}(\partial_1 \bar{G} d\xi^1 + \partial_2 \bar{G} d\xi^2) \wedge dt.$$

$$0 = \bar{\Omega}^{(3)} = \underline{\partial_t \xi^3} d\xi^1 \wedge d\xi^2 \wedge dt - \underline{(\partial_1 \bar{H} \partial_2 \bar{G} - \partial_2 \bar{H} \partial_1 \bar{G})} d\xi^1 \wedge d\xi^2 \wedge dt$$

$\bar{\Omega}^{(2)}$ coincides with the **(1/2) projection** of the following 2-form and 3-form in 3D Nambu phase space :

$$\boxed{\begin{aligned} \Omega^{(2)} &\equiv \xi^3 d\xi^1 \wedge d\xi^2 - H dG \wedge dt \\ \Omega^{(3)} &\equiv d\Omega^{(2)} = d\xi^1 \wedge d\xi^2 \wedge d\xi^3 - dH \wedge dG \wedge dt \end{aligned}}$$

(studied previously by Estabrook (1973) , Takhtajan (1994) independently of HJ formalism.)

Comparing to the conventional HJ formalism:

$$(\bar{\omega}^{(1)}, S) \to (\bar{\Omega}^{(2)}, \bar{\Omega}^{(1)})$$

$$(\omega^{(2)}, \omega^{(1)}) \to (\Omega^{(3)}, \Omega^{(2)})$$

explains why there is no action integral defined along each single 1-dimensional trajectory in the phase space.

Remark:

Geometrical meaning of $\Omega^{(3)} \equiv d\Omega^{(2)} = d\xi^1 \wedge d\xi^2 \wedge d\xi^3 - dH \wedge dG \wedge dt$

It satisfies the null condition with vector fields $\tilde{X} = \sum_{i=1}^{3} X^i \partial_i + \frac{\partial}{\partial t}$

$$i_{\tilde{X}}(\Omega^{(3)}) = 0 \qquad X^i = \epsilon^{ijk} \partial_j H \partial_k G$$

Or more explicitly,

$$\Omega^{(3)} = (d\xi^1 - X^1 dt) \wedge (d\xi^2 - X^2 dt) \wedge (d\xi^3 - X^3 dt)$$

This property replaces the role of the action principle as a variational characterization of the equations of motion (=integral curve of $\tilde{X}$) for a continuous family of its trajectories.

Recall that in the ordinary Hamiltonian mechanics $\omega^{(1)} = p_i\, dq^i - H\, dt$

$$\omega^{(2)} = dp_i \wedge dq^i - dH \wedge dt = d\omega^{(1)}$$

$$= (dp_j + \partial_j H dt) \wedge \left(dq^j - \frac{\partial H}{\partial p_j} dt\right) \qquad i_{\tilde{V}}(\omega^{(2)}) = 0$$

$$\tilde{V} = V + \frac{\partial}{\partial t}$$

$$V \equiv \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}$$

❖ Step III: **what is the analogue of Jacobi condition ?**

A fundamental difficulty:

The usual procedure of finding an appropriate finite canonical transformation such that we can unfold the dynamics to vanishing Hamiltonian cannot be straightforwardly extended to Nambu mechanics.

$$p_i\, dq^i - \bar{H}\, dt = dS + P_i\, dQ^i \quad \Longleftrightarrow \quad \frac{\partial S}{\partial q^i} = p_i, \qquad \frac{\partial S}{\partial Q^i} = -P_i$$

generating function

Jacobi condition

It would suggest us defining a "generating" 1-form $\Sigma' = \Sigma'_\mu\, d\xi^\mu + \Sigma'_{Q_1} dQ_1$ such that

$$\xi^3\, d\xi^1 \wedge d\xi^2 - \bar{H}\, d\bar{G} \wedge dt = d\Sigma' + Q_3\, dQ_1 \wedge dQ_2 \qquad (Q_1, Q_2, Q_3)$$

initial conditions at t=0

But this is not in general allowed since it would require the existence of 4 independent canonical variables (apart from t), instead of 3 that is the dimension of the phase space ?!

We found two ways for circumventing this difficulty.

Approach (i): assume special canonical coordinates (and/or N-gauge choices) such that $\partial_3 G = 0$ ("axial gauge")

$$\bar{G} = \bar{G}(\xi^1, \xi^2, t) = G(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2, t)) = G(\xi^1, \xi^2) \longleftrightarrow \partial_t \bar{G} = 0$$

and choose $S_0 = 0$ gauge with respect to the S-gauge symmetry.

Then, it is easy to see that the system is equivalent to

$$S_i = -S\partial_i G$$

$$\bar{H} = \bar{H}(\xi^1, \xi^2, t) = H(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2, t))$$

$$\partial_t S = -\bar{H}$$

$$\underline{\xi^3 = d_G S}$$

with a scalar *S*,

$$d_G \equiv \partial_1 G \partial_2 - \partial_2 G \partial_1$$ *: a sort of angular (or vortical) momentum!*

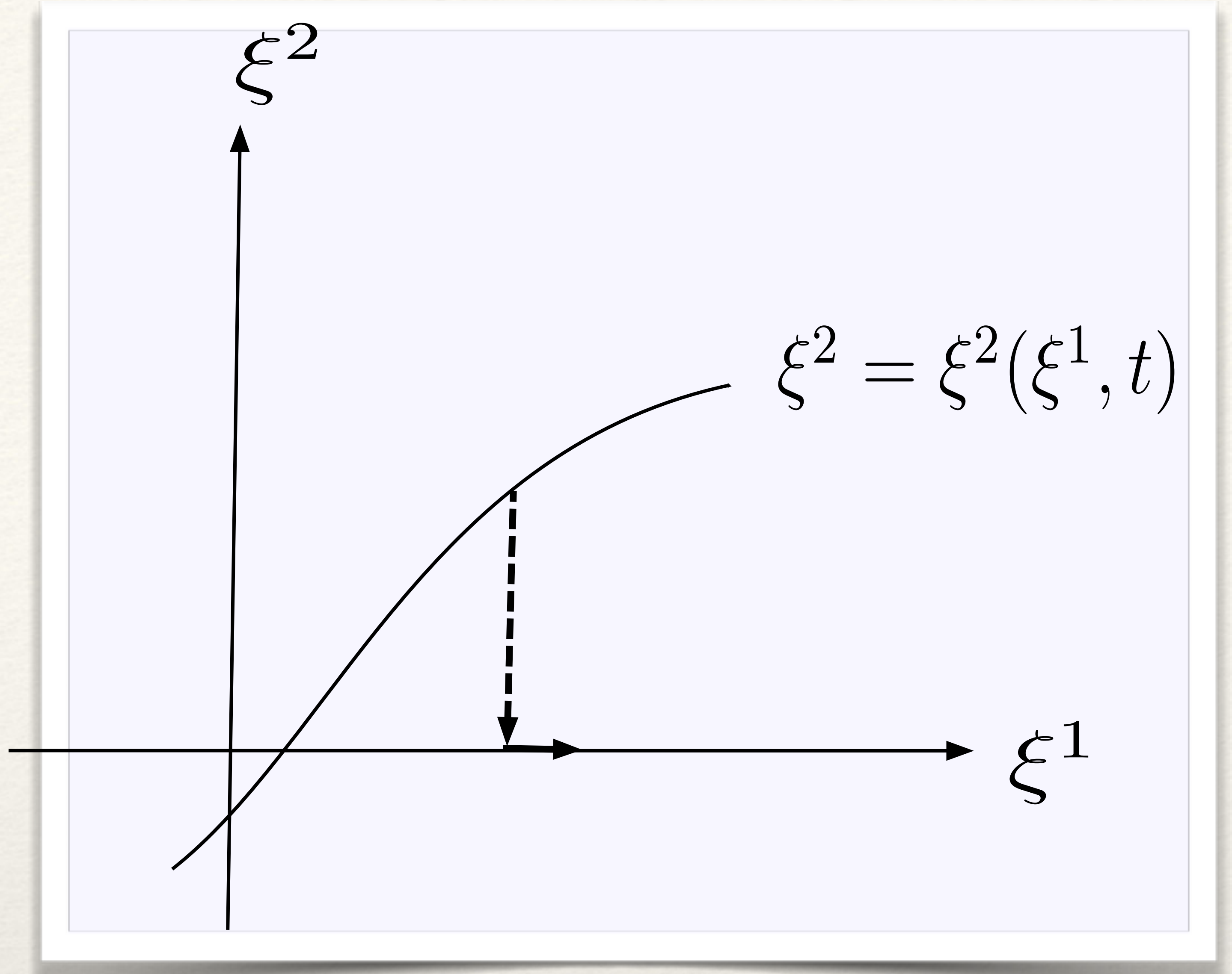
and the original Nambu equations of motion are reproduced by demanding, for the complete solution $S = S(\xi^1, \xi^2, t; Q_1)$,

$$\frac{\partial S}{\partial Q_1} = Q_3 \qquad G = \bar{G} = Q_2$$

$\Rightarrow$ $\xi^i = \xi^i(t; Q_1, Q_2, Q_3)$: general solution with three integration constants

Approach (ii): (1/1/1) formalism, namely, *further reduction of the base space* (ξ^1, ξ^2) *into (1/1) by setting* $\xi^2 = \xi^2(\xi^1, t)$ *under the constraint*

$$\bar{H} = H\big(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2; E)\big) = E$$



EE equation for $\xi^2(\xi^1, t)$:

$$\partial_t \xi^2 = \partial_3 H \partial_1 \bar{\bar{G}}$$

$$\bar{\bar{G}}(\xi^1, t) = \bar{G}\big(\xi^1, \xi^2(\xi^1, t)\big) = G\Big(\xi^1, \xi^2(\xi^1, t), \xi^3\big(\xi^1, \xi^2(\xi^1, t)\big)\Big)$$

The **second version** of our generalized HJ equation is obtained from the following 1-form on the phase space (ξ^1, ξ^2, t)

$$\Omega'^{(1)} = p_1 d\xi^1 - \bar{G} dt$$

$$p_1 = p_1(\xi^1, \xi^2) = -\int^{\xi^2} \frac{dx}{\partial_3 H(\xi^1, x)}$$

under the projection $\xi^2 = \xi^2(\xi^1, t)$, by requiring $dT = \Omega'^{(1)}$

$$\boxed{\begin{aligned} &\partial_t T + \bar{\bar{G}} = 0, \\ &p_1 = \partial_1 T \end{aligned} \quad \text{with **Jacobi condition**} \quad \frac{\partial T}{\partial Q_1} = P \;: \quad \xi^i = \xi^i(t; Q_1, P, E)}$$

(2/1) formalism : section=2d vector field $(\xi^2(\xi^1,t),\xi^3(\xi^1,t))$ on 1d base space ξ^1

$$\hat{H}(\xi^1,t)\equiv H\left(\xi^1,\xi^2(\xi^1,t),\xi^3(\xi^1,t)\right) \quad \textit{etc.}$$

- Step I : EE equation

$$\partial_t\xi^2=\partial_3 H\partial_1\hat{G}-\partial_3 G\partial_1\hat{H}$$
$$\partial_t\xi^3=\partial_1\hat{H}\partial_2 G-\partial_1\hat{G}\partial_2 H$$

coincides with the vanishing condition of following **two** 2-forms, under the projection

$$\Omega_2^{(2)}\equiv d\xi^2\wedge d\xi^1+(\partial_3 H dG-\partial_3 G dH)\wedge dt \quad\rightarrow\quad \hat{\Omega}_2^{(2)}\equiv-\partial_t\xi^2 d\xi^1\wedge dt+(\partial_3 H\partial_1\hat{G}-\partial_3 G\partial_1\hat{H})d\xi^1\wedge dt\;=0$$
$$\Omega_3^{(2)}\equiv d\xi^3\wedge d\xi^1-(\partial_2 H dG-\partial_2 G dH)\wedge dt \qquad \hat{\Omega}_3^{(2)}\equiv-\partial_t\xi^3 d\xi^1\wedge dt-(\partial_2 H\partial_1\hat{G}-\partial_2 G\partial_1\hat{H})d\xi^2\wedge dt=0$$

invariant under the rotation in the fibre 23-space

Geometrically,

$$i_{\tilde{X}}(\Omega_2^{(2)})=i_{\tilde{X}}(\Omega_3^{(2)})=0$$

or more explicitly

$$\Omega_2^{(2)}=(d\xi^1-X^1dt)\wedge(d\xi^2-X^2dt)$$
$$\Omega_3^{(2)}=(d\xi^3-X^3dt)\wedge(d\xi^1-X^1dt)$$

Connection with the 3-form of the (1/2) formalism: $\Omega^{(3)}=\Omega_2^{(2)}\wedge(d\xi^3-X^3dt)=\Omega_3^{(2)}\wedge(d\xi^2-X^2dt)$

- Step II and III: possible to show that this system is equivalent to that of the previous (1/1/1) formalism under the constraint, $\hat{H} = H\big(\xi^1, \xi^2(\xi^1, t), \xi^3(\xi^1, t)\big) = E$

since the EE equations reduce to $\partial_t \xi^2 = \partial_3 H \partial_1 \hat{G}, \qquad \partial_t \xi^3 = -\partial_2 H \partial_1 \hat{G}$

identical to those of (1/1/1) with $\bar{\bar{G}} = \hat{G}$

Hence, we have essentially the identical results for the generalized Hamilton-Jacobi equations as those of the (1/1/1) formalism.

Example: Euler Top

canonical triplet

Euler equation $\frac{dL_i}{dt} = -\sum_{j,k=1}^{3} \epsilon_{ijk}\left(\frac{1}{I_j} - \frac{1}{I_k}\right) L_j L_k$ $\qquad \xi^i = L_i \qquad \{\xi^i, \xi^j, \xi^k\} = \epsilon^{ijk}$

Hamiltonians $H = \frac{1}{2}\left((\xi^2)^2 + (\xi^2)^2 + (\xi^3)^2\right) \qquad G = \frac{1}{2}\left(\frac{(\xi^1)^2}{I_1} + \frac{(\xi^2)^2}{I_2} + \frac{(\xi^3)^2}{I_3}\right)$

- **(1/2) formalism :** $\xi^3 = \xi^3(\xi^1, \xi^2, t)$

By N-gauge transformation with $\Lambda = H^2/2I_3$, G is transformed to

$$\boxed{G = \frac{\alpha}{2}(\xi^1)^2 + \frac{\beta}{2}(\xi^2)^2 \qquad \alpha = \frac{I_3 - I_1}{I_3 I_1}, \quad \beta = \frac{I_3 - I_2}{I_3 I_2}}$$

GHJ equation then takes the form

$$\frac{\partial S}{\partial t} = -\bar{H} = -\frac{1}{2}\left((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2\right) \qquad \xi^3 = \partial_1 G \partial_2 S - \partial_2 G \partial_1 S = \alpha \xi^1 \partial_2 S - \beta \xi^2 \partial_1 S$$

The complete solution (two integration constants E , G) is

$$S = -E(t - t_0) + \frac{A}{\alpha\beta} \int_0^{\xi^1/\sqrt{(\xi^1)^2 + \frac{\beta}{\alpha}(\xi^2)^2}} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}}\, dx$$

$$A = \sqrt{2\alpha\beta(E - G/\beta)} \qquad k^2 = \sqrt{\frac{G(\beta - \alpha)}{\alpha\beta(E - G/\beta)}}$$

additional constant

fundamental elliptic integral of the 2nd kind

Jacobi condition: $\frac{\partial S}{\partial E} = 0$

(1/1/1) or (2/1) formalism :

GHJ equation takes the form $\partial_t T = -\bar{\bar{G}} = -\frac{1}{2}(\alpha(\xi^1)^2 + \beta(\xi^2)^2)$ $\quad \xi^2 = -\sqrt{2E - (\xi^1)^2}\sin(\partial_1 T)$

The complete solution with one integration constant F with an additional energy constraint

$$\hat{H} = H\big(\xi^1, \xi^2(\xi^1, t), \xi^3(\xi^1, t)\big) = E$$

$$T = \int^{\xi^1} \arcsin\sqrt{\frac{2F - \alpha x^2}{\beta(2E - x^2)}}\, dx$$

Jacobi condition: $-t_0 = \dfrac{\partial T}{\partial F}$

In both cases, by imposing the Jacobi conditions, we obtain the well known general solution of the Euler equations in terms of elliptic functions of modulus k :

$$\xi^1 = \sqrt{\frac{2G}{\alpha}}\,\mathrm{sn}\, u \qquad \xi^2 = \sqrt{\frac{2G}{\beta}}\,\mathrm{cn}\, u \qquad \xi^3 = -\sqrt{2(E - G/\beta)}\,\mathrm{dn}\, u \qquad u = A(t - t_0)$$

3 integration constants

in a ***much more direct way*** than the standard canonical HJ method that uses Euler angles as the generalized coordinates.

Our GHJ formalism is also useful from a practical point of view!

To summarize what we have discussed so far :

In both (1/2) and (2/1) formalisms, we arrived at the generalized HJ equations, in which the momentum variable in the conventional HJ formalism are replaced by somewhat more complex structures. The dynamics is not a simple gradient flow, but some sort of rotating flow.

Note: *Extension to general n-dimensional Nambu mechanics in which Nambu brackets are n-dimensional Jacobian is also straightforward.*

It was then shown that the Nambu equations of motion are obtained by appropriate generalized Jacobi conditions.

This suggests that there exist natural routes toward the quantization of Nambu mechanics, à la Schrödinger's wave mechanics.

There is in fact a reasonable interpretation of our results, also from the (algebraic) viewpoint of **Nambu bracket satisfying FI**, which is in harmony with this expectation.

- For any realization of Nambu bracket **satisfying the fundamental identity**

$$\{A,G,\{B,F,C\}\} = \{\{A,G,B\},F,C\} + \{B,\{A,G,F\},C\} + \{B,F,\{A,G,C\}\}$$

we can define a subordinate Poisson bracket, satisfying the Jacobi identity (Takhtajan)

$$\{A,B\}_G \equiv \{A,G,B\}$$

Then, if we choose such that $\partial_3 G = 0$, we have

$$\underline{\{\xi^1,\xi^2\}_G = 0}, \quad \underline{\{\xi^3,\xi^1\}_G = -\partial_2 G, \quad \{\xi^3,\xi^2\}_G = \partial_1 G} \qquad \text{and} \qquad \boxed{\frac{d\xi^i}{dt} = \{H,\xi^i\}_G}$$

This is consistent with the following quantization of (1/2) formalism:

wave function $\langle \underline{\xi^1,\xi^2}|1(t)\rangle \quad \sim \; e^{iS(\xi^1,\xi^2,t)/\hbar}$

"momentum operator"
(or "rotation operator") $\underline{\xi^3 \to -i\hbar(\partial_1 G\partial_2 - \partial_2 G\partial_1) = -i\hbar d_G}$

Schrödinger eq. $\boxed{i\hbar\partial_t\langle\xi^1,\xi^2|1(t)\rangle = H(\xi^1,\xi^2,-i\hbar d_G)\langle\xi^1,\xi^2|1(t)\rangle}$

quantization (1/2)

Similarly, (1/1/1) formalism corresponds ($\{A,B\}_H \equiv \{H,A,B\}$, $\partial_3 H \neq 0$) to

$$\{\xi^1,\xi^2\}_H = \partial_3 H, \quad \{\xi^3,\xi^1\}_H = \partial_2 H, \quad \{\xi^3,\xi^2\}_H = -\partial_1 H$$

$$\boxed{\frac{d\xi^i}{dt} = \{G,\xi^i\}_H}$$

under the constraint $H = E$: $\quad \xi^3 = \xi^3(\xi^1,\xi^2;E)$

$$\boxed{0 = \partial_2 H(\xi^1,\xi^2,\xi^3(\xi^1,\xi^2;E)) = \partial_2 H + \partial_3 H \partial_2 \xi^3}$$

$$\boxed{0 = \partial_1 H(\xi^1,\xi^2,\xi^3(\xi^1,\xi^2;E)) = \partial_1 H + \partial_3 H \partial_1 \xi^3}$$

in harmony with the following quantization

$\xi^2 \longrightarrow \hat{\xi}^2$ momentum operator formally defined by $\displaystyle -i\hbar\partial_1 = -\int^{\hat{\xi}^2} \frac{dx}{\partial_3 H(\xi^1,x)}$

Schrödinger eq. $\quad \boxed{i\hbar\partial_t \langle \xi^1|2(t)\rangle = \bar{G}(\xi^1,\hat{\xi}^2)\langle \xi^1|2(t)\rangle}$

wave function $\quad \langle \xi^1|2(t)\rangle \sim \quad e^{iT(\xi^1,t)/\hbar}$

quantization (2/1)

The generalized Jacobi condition naturally fits the saddle point condition.

Apparently, these two formulations are **entirely different quantum mechanics, using different Hilbert spaces and different Schrödinger (or Heisenberg) equations.**

In principle, there could be infinitely many different quantizations, **depending upon the choices of the subordinated Poisson brackets.**

However, they give one and the same ***classical*** Nambu equations of motion, being connected to each other by N-gauge (and/or canonical coordinate) transformations.

Our conclusion has both similarity and dissimilarity
to Nambu's original conclusion:

"One is repeatedly led to discover that the quantized version is essentially equivalent to the ordinary quantum theory. This may be an indication that quantum theory is pretty much unique, although its classical analog may not be."

Is quantum theory unique?

Key issues to quantization of Nambu mechanics ?

(1) **quantize the subordinated Poisson brackets** defined by the usual commutator algebras with variable choices of two Hamiltonians (H, G) with respect to N-gauge transformations;

(2) **enlarge the usual framework of quantum mechanics to an extended scheme,** allowing (possibly infinitely) many different Hilbert spaces corresponding to different choices of Poisson brackets and different Hamiltonians;

(3) construct a **transformation theory** by which we can transform systems among the sets of Hilbert spaces and corresponding Hamiltonians in some covariant fashion, such that it gives the N-gauge and canonical coordinate transformations in the classical limit.

(4) find **probabilistic interpretations** of the formalism, in such a manner that different and allowed choices of Hilbert spaces and Hamiltonians in the sense of (1) to (3) in the framework of transformation theory give **physically unique** results.

Basic questions

Should the framework of quantum mechanics itself be enlarged ?

Remember that the Wheeler-DeWittt equation
in canonical quantum gravity already requires some extension
(albeit in a sense quite different from the present case)
of the framework of quantum mechanics.

After all, however, why should we be so serious
about Nambu quantum mechanics ?

Is there any compelling theoretical reason for us to pursue this problem ?

Implications for string/M theory, in general ?

Hopefully, these questions might become relevant in the near future !

And perhaps
we should learn more from Nambu-san's passion and imagination for physics !

Thanks for listening.

His talk at Osaka Univ.in 2011
(second last in his life, 1921-2015)

A particle physicist's view of fluid dynamics
--An old sake in a new cup--

*1. Motivation
*2. Literature
Lamb, Landau-Lifshitz, Tatsumi,..
*3. Two pictures----Euler and Laplace

"Running stream never ceases, yet it is not the same water. Kamo-no-Chomei 1155-1216
"Worn out at journey's end, dreams still wander along ergodic paths." Basho 1644-1694

「行く川の流れは絶えずして、しかも、もとの水にあらず」 鴨長明
「旅に病で夢は枯野をかけ廻る」 松尾芭蕉

Background *2-Hamiltonian dynamics*
Y. Nambu, Phys. Rev.D7(1973) 2405

3D phase space triplet *(p,q,r)*
triple Poisson bracket *[p,q,r] = 1*
dO/dt = [O,H,G] = (∂O,∂H,∂G)/(∂p,∂q,∂r)

Example: **rigid rotator**
triplet L_i = angular momenta