Quiver Quantum Toroidal Algebra & Crystal Representations

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based on 2108.07104 & 2109.02045 (with Noshita)
See also 2101.03953 (with Harada, Matsuo, Noshita)

Introduction : q-deformation

- Deform algebras by adding parameters [Drinfeld 1987]
- In the limit $q \to 1$, the original algebra is recovered
- Hopf algebra structure & a larger symmetry.

q-Virasoro has paremeters q, t, p [Shiraishi et al. 1995]

$$T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n},$$

$$[T_n, T_m] = -\sum_{l=1}^{\infty} f_l (T_{n-l} T_{m+l} - T_{m-l} T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-p} (p^n - p^{-n}) \delta_{m+n,0}$$

- There are two invariances

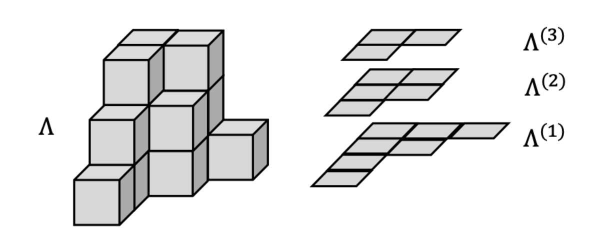
$$T_n \to T_{-n}, \quad (q, t) \to (q^{-1}, t^{-1})$$

- In the limit $q = e^h, h \to 0$, $T(z)$ becomes

$$T(z) = 2 + \beta \left(z^2 L(z) + \frac{(1-\beta)^2}{4\beta} \right) h^2 + \mathcal{O}(h^4)$$

Introduction : Affine Yangian & q-deformation

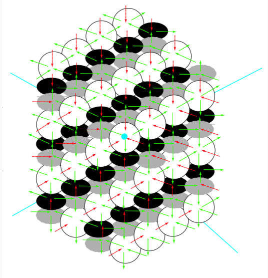
- Affine Yangian of $\mathfrak{gl}_1$ [Tsymbaliuk 2014]
- Its q-deformation is quantum toroidal $\mathfrak{gl}_1$ [Ding-Iohara 1997]
- Three representations [Feigin et al. 2012]
  - Vector representation (1d Young diagrams)
  - Fock representation (2d Young diagrams)
  - MacMahon representation (3d Young diagrams)



Introduction : Generalization of quantum toroidal $\mathfrak{gl}_1$

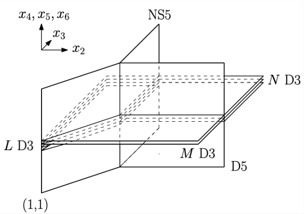
- Affine Yangian of $\mathfrak{gl}_1 \subset$ Quiver Yangian [Li-Yamazaki 2020]
- Quantum Toroidal $\mathfrak{gl}_1 \subset$ QQTA [Noshita-AW 2108.07104]

- Representations
  - 1d Young diagrams $\to$ 1d crystals
  - 2d Young diagrams $\to$ 2d crystals [Nishinaka-Yamaguchi-Yoshida 2013]
  - 3d Young diagrams $\to$ 3d crystals [Ooguri-Yamazaki 2008]
- We found the relations between 1d & 2d representations [Noshita-AW 2109.02045]



[Ooguri-Yamazaki 2008]

Introduction : q-deformation of corner VOA



[Gaiotto-Rapcak 2017]

[Gaiotto-Rapcak 2017]

$$\begin{tikzcd}
\text{Virasoro algebra} \arrow[d, "q\text{-deformation}"'] & \subset & W_N \text{ algebra} \arrow[d] & \subset & \text{corner VOA } Y_{L,M,N} \arrow[d] \\
q\text{-Virasoro algebra} & \subset & q\text{-}W_N \text{ algebra} & \subset & \boxed{q\text{-corner VOA } q\text{-}Y_{L,M,N}}
\end{tikzcd}$$

[Shiraishi et al. 1995]

[Awata et al. 1995]

We constructed it!!
[2101.03953]

Today's talk

1. Introduction $\leftarrow$ Finished

2. (q-) affine Yangian of $\mathfrak{gl}_1$ & Young diagrams representations
    - Affine Yangian of $\mathfrak{gl}_1$
    - q-deformation
    - Representations

3. (q-) quiver Yangian & crystal representations
    - Quiver diagrams and quiver Yangian
    - q-deformation
    - Representations by crystals
    - Construction of 2d crystal reps from 1d

(q-) affine Yangian of $\mathfrak{gl}_1$ & Young diagrams representations

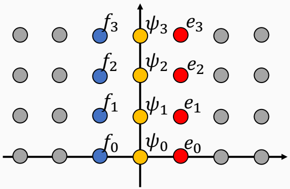
Affine Yangian of $\mathfrak{gl}_1$

- Generators are [Tsymbaliuk 2014]

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}},$$

$$f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}},$$

$$\psi(u) = 1 + \sigma \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$



- Their relations depend on h_i symmetrically

$$h_1 + h_2 + h_3 = 0, \quad e(u)e(v) \sim \varphi(u-v)e(v)e(u)$$

$$\varphi(u) = \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}$$

- Affine Yangian of $\mathfrak{gl}_1 \simeq W_{1+\infty}$

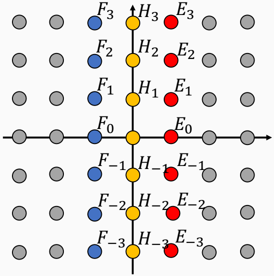
quantum toroidal $\mathfrak{gl}_1$

- q-deformation of affine Yangian of $\mathfrak{gl}_1$
- Generators are

$$E(z) = \sum_{k\in\mathbb{Z}} E_k z^{-k},$$

$$F(z) = \sum_{k\in\mathbb{Z}} F_k z^{-k},$$

$$K^{\pm}(z) = K^{\pm 1} \exp\left(\pm \sum_{r=1}^{\infty} H_{\pm r} z^{\mp}\right)$$



- Their relations depend on q_1, q_2, q_3 symmetrically

$$q_1q_2q_3 = 1, \quad E(z)E(w) \sim \varphi(z,w)E(w)E(z)$$

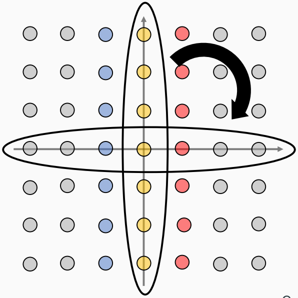
$$\varphi(z,w) = \prod_{i=1}^{3} \frac{(q_i^{1/2}z - q_i^{-1/2}w)}{(q_1^{-1/2}z - q_i^{1/2})}$$

Properties of quantum toroidal $\mathfrak{gl}_1$

- In the limit $q \to 1$, it becomes affine Yangian of $\mathfrak{gl}_1$
- The modes are $k > 0$ in the degenerate case, while $k \in \mathbb{Z}$ in q-deformed case
- Hopf algebra structure(unit, counit, product, coproduct, antipode)

- Coproduct is important here

$$\Delta E(z) = E(z) \otimes 1 + K^{-}(C_1 z) \otimes E(C_1 z)$$
$$\Delta F(z) = F(C_2 z) \otimes K^{+}(C_2 z) + 1 \otimes F(z)$$
$$\Delta K^{+}(z) = K^{+}(z) \otimes K^{+}(C_1^{-1} z)$$
$$\Delta K^{-}(z) = K^{-}(c_2^{-1} z) \otimes K^{-}(z)$$



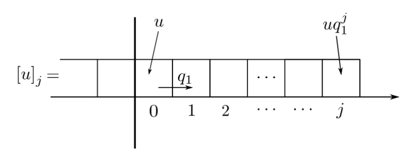
- Triality : exchange of q_1, q_2, q_3
- Miki duality : $SL(2, \mathbb{Z})$

Representations of quantum toroidal $\mathfrak{gl}_1$

- There are 3 vertical representations
  - Vector representation : 1d Young diagrams
  - Fock representation : 2d Young diagrams
  - MacMahon representation : 3d Young diagrams

- Vector representation

$$K^{\pm}(z)[u]_j = \left[\Psi_{[u]_j}(z)\right]_{\pm}[u]_j,$$
$$E(z)[u]_j = \mathcal{E}\delta(q_1^{j+1}u/z)[u]_{j+1},$$
$$F(z)[u]_{j+1} = \mathcal{F}\delta(q_1^{j+1}u/z)[u]_j$$



- $E(z)$ adds a box ,and $F(z)$ removes a box.

$$[E(z), F(w)] = \delta(w/z)K^{+}(z) - \delta(z/w)K^{-}(w)$$

Tensor product of two vector representations

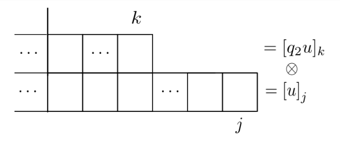
- Tensor product of two vector reps becomes Fock rep by a two-row Young diagram.

- Generators are defined by coproduct as

$$\Delta E(z) = E(z) \otimes 1 + K^{-}(z) \otimes E(z),$$

$$\Delta F(z) = F(z) \otimes K^{+}(z) + 1 \otimes F(z),$$

$$\Delta K^{\pm}(z) = K^{\pm}(z) \otimes K^{\pm}(z)$$



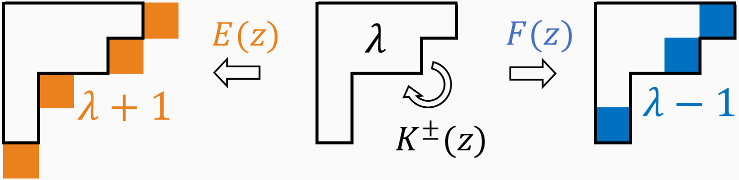
- $E(z)$ adds a box, and $F(z)$ removes a box. If the condition of Young diagram breaks, the coefficient becomes zero.

Fock representation

- Furthermore, tensor product of infinite vector reps becomes Fock rep by a general Young diagram.

$$|\lambda\rangle = \bigotimes_{j=1}^{\infty} [q_2^{j-1}u]_{\lambda_j - 1}$$

- Generators are defined by $N-1$ coproducts.
- $\Delta^{(N-1)}E(z)$ adds a box, and $\Delta^{(N-1)}F(z)$ removes a box.
- If the condition of Young diagram breaks, the coefficient becomes zero.

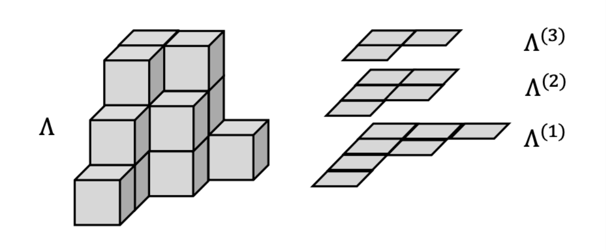


MacMahon representation

- Tensor product of vector rep (1d) $\to$ Fock rep (2d)
- Similarly,
  tensor product of Fock rep (2d) $\to$ MacMahon rep (3d)

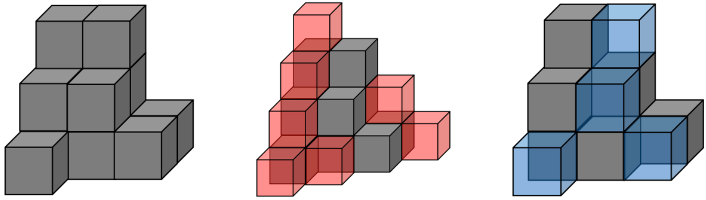
$$\Lambda = (\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \cdots) \quad \Lambda^{(i)} : \text{Young diagram}$$

- Λ is called Plane Partition, satisfying
  $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \cdots$



MacMahon representation

- Similarly to the 1d or 2d case, $K^{\pm}(z)$ acts as eigenvalue、$E(z)$ adds a box to the Plane Partition, and $F(z)$ removes a box.
- If the condition of Plane Partition breaks, the coefficient becomes zero.



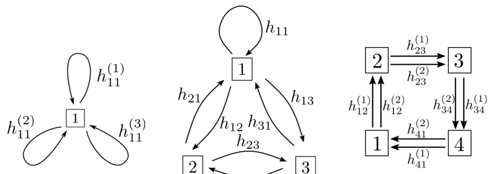
(a) Y_Λ (b) $CC(Y_\Lambda)$ (c) $CV(Y_\Lambda)$

(q-) quiver Yangian & crystal representations

Quiver data

- Quiver Yangian is defined by a quiver data and generalizes affine Yangian of $\mathfrak{gl}_1$ [Li-Yamazaki 2020]
- Quiver data = Quiver diagram (Q_0, Q_1) + Loops Q_2.
- (b) corresponds to the affine Yangian of $\mathfrak{gl}_{2|1}$

$$Q_0 = \{1,2,3\},\ Q_1 = \{1\to 1,\ 1\to 2,\ 1\to 3,\ 2\to 1,\ 2\to 3,\ 3\to 1,\ 3\to 2\},$$
$$Q_2 = \{1\to 1\to 3\to 1, 1\to 2\to 3\to 2\to 1, 1\to 1\to 2\to 1, 1\to 3\to 2\to 3\to 1\}$$



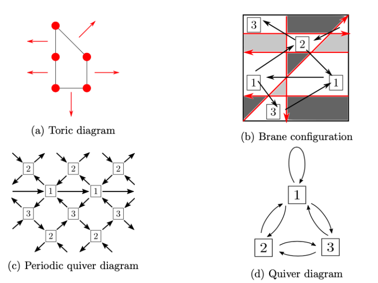
(a) One vertex case

(b) Three vertices and one self-loop case

(c) The case with $|i\to j| \neq |j\to i|$

From toric diagram to quiver diagram

- We focus on symmetric quivers constructed from toric diagram without compact 4-cycles.
- Draw the red arrows perpendicular to the toric diagram on the torus.(brane configuration)
- White region $\leftrightarrow$ vertex, their connection $\leftrightarrow$ arrows
- Loops on brane configuration $\leftrightarrow$ Q_2



(a) Toric diagram

(b) Brane configuration

(c) Periodic quiver diagram

(d) Quiver diagram

Definition of quiver Yangian

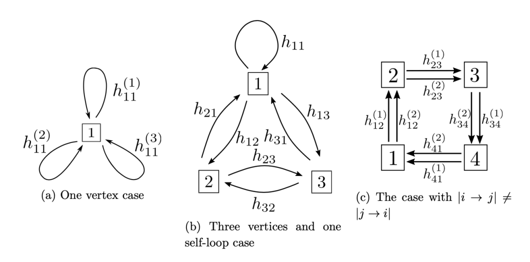
- We define a set of generators e, f, ψ for each vertex a.

$$e^{(a)}(z) = \sum_{n=0}^{\infty} \frac{e_n^{(a)}}{z^{n+1}}, \quad \psi^{(a)}(z) = \sum_{n=-\infty}^{\infty} \frac{\psi_n^{(a)}}{z^{n+1}}, \quad f^{(a)}(z) = \sum_{n=0}^{\infty} \frac{f_n^{(a)}}{z^{n+1}},$$

- Bond facters are defined by Q_1.

$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

$$e^{(a)}(z)e^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(z-w)e^{(b)}(w)e^{(a)}(z)$$



Quiver Quantum Toroidal Algebra (QQTA)

- We proposed the q-deformation of quiver Yangian called QQTA [Noshita-AW 2108.07104]
- Similar to the quiver Yangian, we use quiver data (Q_0, Q_1, Q_2) to define QQTA.
- Parameters are replaced into $h_I \to q_I = e^{\epsilon h_I}$
- We define a set of generators $E, F, K^{\pm}$ for each vertex i.

$$E_i(z) = \sum_{k\in\mathbb{Z}} E_{i,k} z^{-k}, \quad F_i(z) = \sum_{k\in\mathbb{Z}} F_{i,k} z^{-k}, \quad K_i^{\pm}(z) = K_i^{\pm 1} \exp\left(\pm \sum_{r=1}^{\infty} H_{i,\pm r} z^{\mp r}\right).$$

- Bond factors are defined by Q_1

$$\varphi^{i\Rightarrow j}(z,w) = \frac{\prod_{I\in\{j\to i\}}(q_I^{1/2} z - q_I^{-1/2} w)}{\prod_{I\in\{i\to j\}}(q_I^{-1/2} z - q_I^{1/2} w)}$$

Properties

- In the limit $\epsilon \to 0$, it reduces to quiver Yangian.

$$\frac{\prod_{I\in\{j\to i\}}(q_I^{1/2}z - q_I^{-1/2}w)}{\prod_{I\in\{i\to j\}}(q_I^{-1/2}z - q_I^{1/2}w)} \to \frac{\prod_{I\in\{j\to i\}}(x-y+h_I)}{\prod_{I\in\{i\to j\}}(x-y-h_I)},$$

- Hopf superalgebra structure, especially coproduct

$$\begin{aligned}
\Delta E_i(z) &= E_i(z)\otimes 1 + K_i^-(C_1 z)\otimes E_i(C_1 z),\\
\Delta F_i(z) &= F_i(C_2 z)\otimes K_i^+(C_2 z) + 1\otimes F_i(z),\\
\Delta K_i^+(z) &= K_i^+(z)\otimes K_i^+(C_1^{-1}z),\\
\Delta K_i^-(z) &= K_i^-(C_2^{-1}z)\otimes K_i^-(z)
\end{aligned}$$

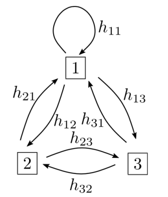
- Coproduct is important in the relation of different representations.

3d crystal

- 3d crystals are defined by the quiver data
  [Ooguri-Yamazaki 2008]

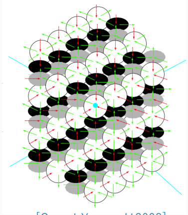
$$Q_2 = \{1 \to 1 \to 3 \to 1, 1 \to 2 \to 3 \to 2 \to 1, 1 \to 1 \to 2 \to 1, 1 \to 3 \to 2 \to 3 \to 1\}$$

- vertex in quiver diagram $\to$ atom of the crystal
  arrow between vertices $\to$ bond between atoms
- Q_2 identifies some path on crystal.



(b) Three vertices and one self-loop case

$\xrightarrow{+Q_2}$

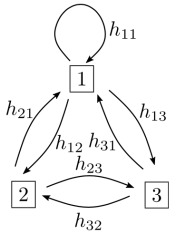

[Ooguri-Yamazaki 2008]

3d crystal

- quantum toroidal $\mathfrak{gl}_1$ acts to Plane Partitions.
- QQTA acts to general 3d crystals.
- Such crystals are defined by the quiver data

  [Ooguri-Yamazaki 2008]
- First, we obtain periodic quiver from (Q_0, Q_1, Q_2).

  $Q_2 = \{1 \to 1 \to 3 \to 1, 1 \to 2 \to 3 \to 2 \to 1, 1 \to 1 \to 2 \to 1, 1 \to 3 \to 2 \to 3 \to 1\}$



(b) Three vertices and one self-loop case

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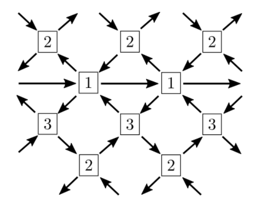
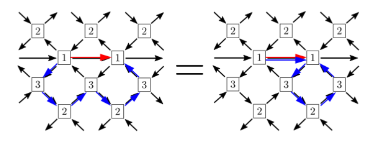
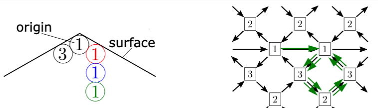


3d crystal

- a path in periodic quiver $\simeq$ an atom on 3d crystal
- However, it is not always one-to-one correspondence. Several paths may be identified to an atom (F-term relation)
- The depth of atom from the surface is the number of loops in the periodic quiver.



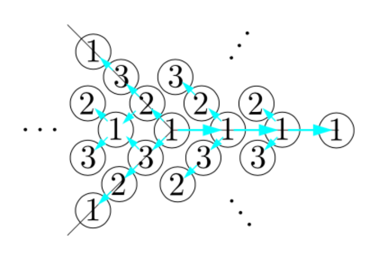


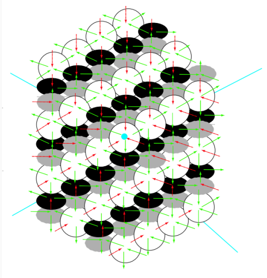
(a) Depth of some atoms

(b) A path with three loops

3d crystal

- By the identification of path and atom, we obtain 3d crystal.





[Ooguri-Yamazaki 2008]

- This is different from the Plane Partition of affine Yangian of $\mathfrak{gl}_1$.
- If we start with the quiver diagram of $\mathfrak{gl}_1$, we obtain the Plane Partition.

The action of QQTA to 3d crystal

- quantum toroidal $\mathfrak{gl}_1$ acts to Plane Partitions.
- QQTA acts to general 3d crystals.
- $E_i(z)$ adds an atom of vertex i, and $F_i(z)$ removes an atom of vertex i.

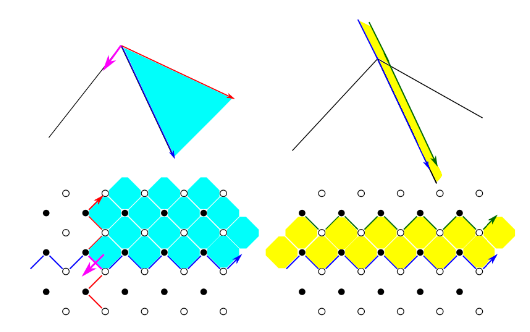
$$K_i^{\pm}(z)\,|\Lambda\rangle = [\Psi_\Lambda^{(i)}(z,u)]_{\pm}\,|\Lambda\rangle\,,$$

$$E_i(z)\,|\Lambda\rangle = \sum_{\boxed{i}\in\mathrm{Add}(\Lambda)} E^{(i)}(\Lambda\to\Lambda+\boxed{i})\delta\left(\frac{z}{uq(\boxed{i})}\right)|\Lambda+\boxed{i}\rangle\,,$$

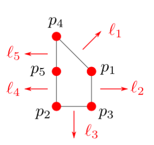
$$F_i(z)\,|\Lambda\rangle = \sum_{\boxed{i}\in\mathrm{Rem}(\Lambda)} F^{(i)}(\Lambda\to\Lambda-\boxed{i})\delta\left(\frac{z}{uq(\boxed{\mathrm{i}})}\right)|\Lambda-\boxed{i}\rangle$$

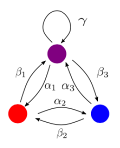
Reduction to subcrystals

- Quantum toroidal $\mathfrak{gl}_1$ has not only Plane Partition rep, but also vector rep and Young diagram rep.
- QQTA also has representations by subcrystals.
- The surface of 3d crystal is 2d crystal, and the edge of 2d crystal is 1d crystal.

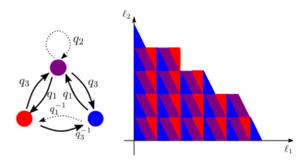


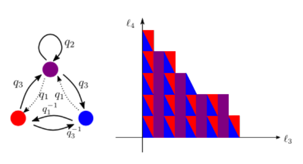
Reduction to 2d crystals

- A quiver diagrams of 3d crystal is constructed from a toric diagram. 



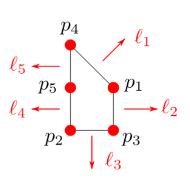
- A quiver diagrams of 2d crystal is constructed from a toric diagram with a corner divisor [Nishinaka-Yamaguchi-Yoshida2013].
- Different corner divisor creates different 2d crystals. Left figure is the case $p1$, and right is $p2$.

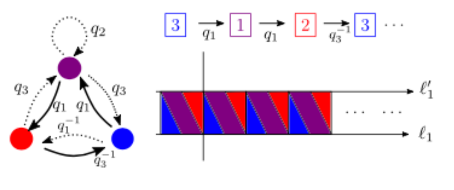




Reduction to 1d crystals

- Furthermore, removing all arrows of two neighboring corner divisors gives 1d crystal.
- The below figure is the case l_1, and arrows correponding to p_1 and p_4 are removed.





- 1d crystals and 2d crystals are called subcrystals.

1d crystal

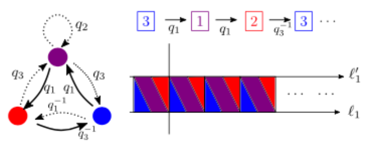
- Subcrystals are the representations of ”shifted” QQTA.
- QQTA has 4 types generators E, F, K^+, K^-, and we slightly ”shift” K^+ as $K_i^+(z) \to z^{r_i} K_i^+(z)$.
- $(r_1, \cdots, r_{Q_0})$ is parameters determined by the shape of crystal. Of course, different subcrystals give different r_i.
- This shift is essential to keep generated states on surface.

2d crystal from 1d crystal

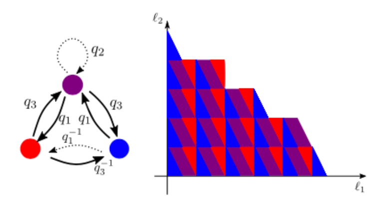
- Quantum toroidal $\mathfrak{gl}_1$ has vector rep (1d) and Fock rep (2d). [Feigin et al. 2011]
    - Fock rep is constructed from the tensor product of vector rep.
    - The action of quantum toroidal algebra to Fock rep is the coproduct of that of vector rep.
- Also in the QQTA, the representations of 2d crystals are constructed from that of 1d crystals. [Noshita-AW 2109.02045]
- For example 2d crystal correponding to p_1 is constructed from the tensor product of 1d crystal l_1.

2d crystal from 1d crystal

- 1d crystals stacked vertically become a 2d crystal.



1d crystal l_1



2d crystal p_1

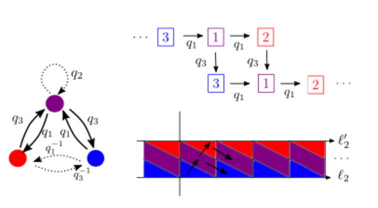
- Vacuum state of 2d crystal is defined by the tensor product of vacuum of 1d crystals $[u]_{-1}$.

$$|\emptyset\rangle = \otimes_{i=1}^{\infty}[(q_1q_3)^{i-1}u]_{-1}$$

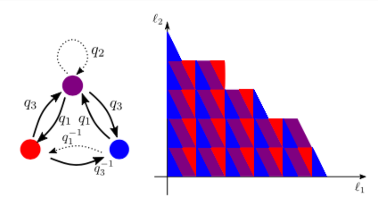
- The generators acting to 2d crystal is defined by the coproduct of these to 1d crystal. $E_i(z)$ adds an atom, and $F_i(z)$ removes an atom.

2d crystal from another 1d crystal

- In the previous slide, we construct 2d crystal from 1d crystal along with l_1.
- We can construct the same 2d crystal from 1d crystal along with l_2.
- This is the difference of slicing direction.



1d crystal l_2



2d crystal p_1

Summary

quantum toroidal $\mathfrak{gl}_1$

vector rep (1d) $\longrightarrow$ Fock rep (2d) $\longrightarrow$ MacMahon rep (3d)

$\downarrow$

(shifted) QQTA [Noshita-AW 2108.07104]

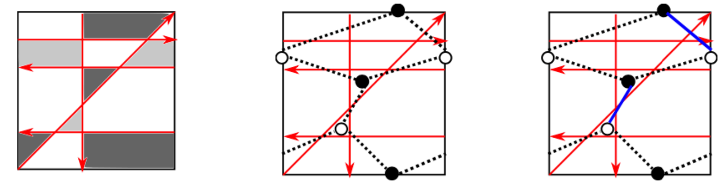
1d crystal rep $\xrightarrow{\text{tensor product}}$ 2d crystal rep   3d crystal rep

tensor product

[Noshita-AW 2109.02045]

Construction of 2d quiver

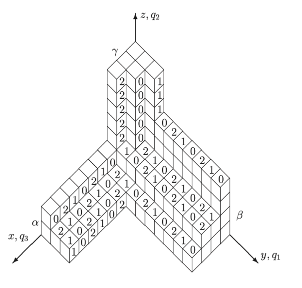
- 2d quiver is obtained from 3d quiver by remobing some arrows.
- Physically, it consists of D0-D2-D4 brane system.
  $\leftrightarrow$ 3d quiver comes from D0-D2-D6 brane system.
- We need to determine the boundary in the brane configuration by finding a perfect matching in dimer model.



(a) Brane configuration (b) Dimer model (c) Perfect matching

Related works

- Feigin-Jimbo-Miwa-Mukhin 1204.5378
  "Representations of Quantum Toroidal $\mathfrak{gl}_n$"



- Galakhov-Li-Yamazaki 2108.10286
  "Toroidal and Elliptic Quiver BPS Algebras and Beyond"

Parameters of quiver Yangian

- Parameter h_I exsits for each arrow $I \in Q_1$.
- Not all of these are independent, and the degree of freedom is 2 due to two kinds of constraints.

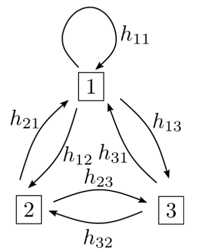
- Vertex constraints

$$\sum_{I \in Q_1(a)} \mathrm{sign}_a(I) h_I = 0$$

- Loop constraints

$$\sum_{I \in L} h_I = 0$$

L is an arbitrary loop in Q_2



(b) Three vertices and one self-loop case

$$Q_2 = \{1 \to 1 \to 3 \to 1, 1 \to 2 \to 3 \to 2 \to 1, 1 \to 1 \to 2 \to 1, 1 \to 3 \to 2 \to 3 \to 1\}$$