Origin of Pure Spinor Superstring

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“Origin of PSS” in a nutshell

Berkovits’ Pure Spinor Formalism

$\uparrow$Canonical (Dirac) quantization

Derived from a classical action (essentially GS)
with reparam, kappa, plus a hidden sym.

Motivation

- Modern string theory rounds about D-branes; But strings in (D-brane sourced) Ramond-Ramond BG's: Poorly understood
  ➜ Super-Poincaré covariant formulation should help:

- Standard RNS & GS:

  Either SUSY or Lorentz is opaque

  – RNS:
    - Worldsheet SUSY $\rightarrow$ powerful scheme of super CFT,
      but with spin structures, GSO, supermoduli, bosonization, picture etc.
    - No built-in spacetime spinor $\rightarrow$ Conceptually hard to handle RR fields

  – GS:
    - Fully covariant classically
    - Less understood worldsheet sym (kappa)
      $\rightarrow$ Hard to quantize except in (semi-)LC gauge

- PS formalism Berkovits (2000–)
  – A new covariant worldsheet formalism with some promising results
  – Like GS, has built-in spacetime SUSY ➜ no conceptual difficulty with RR

With PS formalism, one hopes to understand:

- Strings in various BG's
  - esp. AdS+RR

- String side of the AdS/CFT conjecture
  - String spectrum in $AdS_5 \times S^5$
  - *Quantum* integrability of string side

- Higher genus computation

- Supermembrane / M(atrix) theory

Plan:

1. Introduction
   - Motivation $\surd$
   - Review of GS; treatment of kappa & its difficulty
   - Siegel's approach to GS
2. Review of PS
   - Basics, Achievments & Challenges
3. Origin of PS
   - Classical action for PS
   - Derivation of PS via canonical quantization
4. Summary and Outlook

Classical GS action

- type IIB: $(x^m, \theta^{A\alpha}) = (x^m, \theta^\alpha, \widehat{\theta}^\alpha)$ $\quad i = 0, 1;\ m = 0 \sim 9;\ \alpha = 1 \sim 16;\ A = 1, 2$

$$\mathcal{L}_{\mathsf{GS}} = \mathcal{L}_{\mathsf{K}} + \mathcal{L}_{\mathsf{WZ}}$$

$$\begin{cases} \mathcal{L}_{\mathsf{K}} = -\frac{1}{2}\sqrt{-g}g^{ij}\Pi_i^m \Pi_{jm} \\ \mathcal{L}_{\mathsf{WZ}} = \epsilon^{ij}\Pi_i^m (W_{jm} - \widehat{W}_{jm}) - \epsilon^{ij} W_i^m \widehat{W}_{jm} \end{cases}$$

where

$$\begin{aligned} \Pi_i^m &= \partial_i x^m - W_i^m - \widehat{W}_i^m, \\ W_i^m &= i\theta\gamma^m\partial_i\theta, \quad \widehat{W}_i^m = i\widehat{\theta}\gamma^m\partial_i\widehat{\theta}, \end{aligned} \qquad \Gamma^m = \begin{pmatrix} 0 & \gamma^{m\alpha\beta} \\ \gamma^m_{\alpha\beta} & 0 \end{pmatrix} \quad \begin{matrix} SO(9,1) \\ 32 \times 32 \end{matrix}$$

- Wess-Zumino term:
  - Topological (no dep on g_{ij})
  - Gives rise to κ-sym (local fermionic sym; halves on-shell DOF's of $\theta^{A\alpha}$)

- Due to kappa,
  - (Semi-)LC quant: easy
  - Covariant quant: hard

➜ Shall demonstrate for superparticle

Difficulty of Covariant Quant of GS

- $\mathcal{N} = 1$ Brink-Schwarz Superparticle (generalisation to $\mathcal{N} = 2$ is trivial)

$$L_{\text{BS}} = \frac{1}{2e}\Pi^2, \quad \Pi^m = \dot{x}^m - i\theta\gamma^m\dot{\theta}$$

  - Zero-mode part of string
  - ∃Kappa without WZ

- Symmetries:
  - Super-Poincaré
  - Reparam: $\tau \to \tau'$
  - Kappa: for a fermionic param $\kappa_\alpha(\tau)$

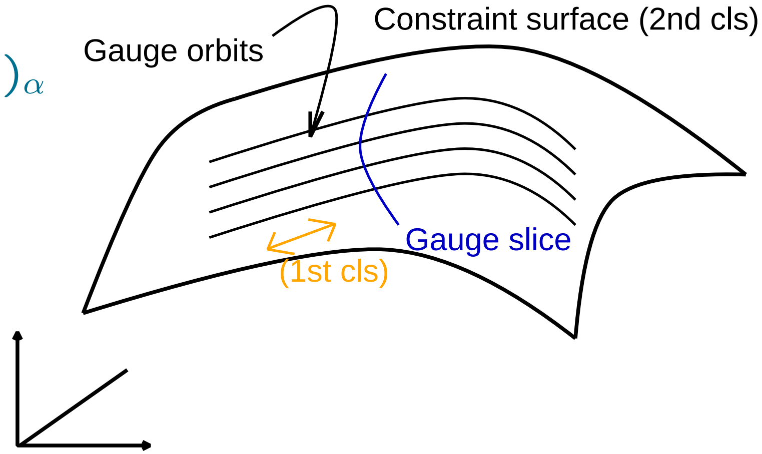
$$\delta\theta^\alpha = ip_m(\gamma^m\kappa)^\alpha, \quad \delta x^m = i\theta\gamma^m\delta\theta, \quad \delta e = 4e\dot{\theta}\kappa$$

Canonical Treatment of BS I

- Fix reparam to "conformal gauge" $e = 1$
- Canonical momenta & Poisson bracket:

$$p_m = \frac{\partial L}{\partial \dot{x}^m} = \Pi_m, \qquad p_\alpha = \frac{\partial L}{\partial \dot{\theta}^\alpha} = i p_m (\gamma^m \theta)_\alpha$$

$$\{x^m, p_n\}_{\mathrm{P}} = \delta^m{}_n, \qquad \{\theta^\alpha, p_\beta\}_{\mathrm{P}} = -\delta^\alpha{}_\beta$$



- Phase space constraints:

$$D_\alpha = p_\alpha - i p_m (\gamma^m \theta)_\alpha \approx 0$$

$$T = \frac{1}{2} p^m p_m \approx 0 \quad \text{(Virasoro)}$$

- Constraint algebra:

$$\{T, T\}_{\mathrm{P}} = \{T, D_\alpha\}_{\mathrm{P}} = 0, \qquad \{D_\alpha, D_\beta\}_{\mathrm{P}} = 2 i p_m \gamma^m_{\alpha\beta} \leftarrow \text{ rank 8 } (p^2 = 0)$$

➜ $1 + 8$ first class (gauge syms)
8 second class (rels between p_α and θ^α) $\rightarrow$ constraint surface

For (Dirac) quantization, must separate explicitly; hard to maintain covariance

Canonical treatment of BS II

- Explicit $8+8$ splitting of D_α (assume $p^+ \neq 0$):

$$D_{\dot a} = p_{\dot a} - ip^- \theta_{\dot a} - ip^i \gamma^i_{\dot a b} \theta_b \;\longrightarrow\; K_{\dot a} = D_{\dot a} - \frac{1}{p^+} p^i \gamma^i_{\dot a b} D_b \;:\; \text{1st cls (kappa)}$$

$$D_a = p_a - ip^+ \theta_a - ip^i \gamma^i_{a \dot b} \theta_{\dot b} \;:\; \text{2nd cls}$$

Indeed from $\{D_\alpha, D_\beta\}_{\mathsf{P}} = 2ip_m \gamma^m_{\alpha\beta}$, easy to see $\qquad \phi_I = (T, K_{\dot a}, D_a)$

$$\begin{array}{l} \{D_a, D_b\}_{\mathsf{P}} = 2ip^+ \delta_{ab}, \quad \{D_a, K_{\dot b}\}_{\mathsf{P}} = 0 \\ \{K_{\dot a}, K_{\dot b}\}_{\mathsf{P}} = \frac{2i}{p^+} p^2 \delta_{\dot a \dot b} \approx 0 \quad (\propto \text{Virasoro}) \end{array} \quad \Rightarrow \quad \{\phi_I, \phi_J\}_{\mathsf{P}} \approx \begin{pmatrix} 0 & 0 \\ 0 & 2p^+ \delta_{\dot a \dot b} \end{pmatrix}$$

- A way to see relation between kappa and conformal syms:
  (kappa)^2 = (Virasoro)

—

Vectors: $V^\pm = V^0 \pm V^9$, $\quad \gamma^\pm = \gamma^0 \pm \gamma^9$

Spinors: $\theta^\alpha = (\theta_a, \theta_{\dot a})$

$$\gamma^0 = \begin{pmatrix} 1_8 & 0 \\ 0 & 1_8 \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \gamma^i_{a \dot b} \\ \gamma^i_{\dot a b} & 0 \end{pmatrix}, \; \gamma^9 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix} \quad \Rightarrow \quad \gamma^\pm : \text{SO(8) chirality projector}$$

Canonical treatment of BS III

- Quantization is most easily done by fixing kappa

$$\text{Semi lightcone gauge: } \gamma^{+}\theta = 0 \Leftrightarrow \theta_{\dot{a}} = 0$$
$$\Rightarrow \quad (K_{\dot{a}}, \theta_{\dot{a}}) \ : \text{ 2nd class}$$

- Dirac bracket on $\phi_I \equiv (D_a, K_{\dot{a}}, \theta_{\dot{a}}) = 0$ (Symplectic structure on $\phi_I = 0$)

$$\begin{aligned}\{A,\, B\}_{\mathsf{D}^*} &= \{A,\, B\}_{\mathsf{P}} - \{A,\, \phi_I\}_{\mathsf{P}}\{\phi_I,\, \phi_J\}_{\mathsf{P}}^{-1}\{\phi_J,\, B\}_{\mathsf{P}} \\ &= \{A,\, B\}_{\mathsf{P}} - \frac{1}{2ip^+}\{A,\, D_a\}_{\mathsf{P}}\{D_a,\, B\}_{\mathsf{P}} \\ &\quad + \{A,\, K_{\dot{a}}\}_{\mathsf{P}}\{\theta_{\dot{a}},\, B\}_{\mathsf{P}} + \{A,\, \theta_{\dot{a}}\}_{\mathsf{P}}\{K_{\dot{a}},\, B\}_{\mathsf{P}}\end{aligned}$$

- Indep variables $(x_m, p_n; S_a = \sqrt{2p^+}\theta_a)$ has canonical brackets:
  $(D_a = p_a - ip^+\theta_a = 0 \rightarrow (\theta_a, p_a)$ become self-conjugate)

$$\{x_m,\, p_n\}_{\mathsf{D}^*} = \eta_{mn}, \quad \{S_a,\, S_b\}_{\mathsf{D}^*} = i\delta_{ab}$$

  with single 1st cls constraint $T = \frac{1}{2}p^2 = 0$

Canonical quantization of BS

- Quantization is now easy:
  - Replace $\{*, *\}_{\mathrm{D}^*}$ by quantum commutators:

$$\{x_m,\, p_n\}_{\mathrm{D}^*} = \eta_{mn} \to [x_m, p_n] = i\eta_{mn}$$
$$\{S_a,\, S_b\}_{\mathrm{D}^*} = i\delta_{ab} \to \{S_a, S_b\} = \delta_{ab}$$

  - On quantum states, impose $T = \frac{1}{2}p^2 = 0$ (or invoke BRST)

- Remarks:
  - Usually, we further fix T imposing $x^+ = \tau$ etc. (full LC gauge)
  - $(x_m, p_n; S_a)$ with canonical brackets are sometimes called "free"

- To sum up,

  (1) In a non-covariant gauge ($\theta_{\dot{a}} = 0$) BS can be systematically quantized:

  $L_{\mathrm{BS}} \to$ free fields + constraint T

  (2) Covariant quantization is bound to be difficult:

  $D_\alpha = 0$ contains $8 + 8$ first and second cls constraint

Siegel's approach to GS (1986)

- Gave up classical action. Instead, asked

> "Can one find a set of 1st cls constraints ('BRST')
> s.t. Free fields + 'BRST' = superstring?"

$$\mathcal{L} = -\frac{1}{2}\partial x \cdot \bar{\partial} x - p_\alpha \partial\theta^\alpha - \widehat{p}_\alpha \bar{\partial}\widehat{\theta}^\alpha$$
$$= -\frac{1}{2}\Pi \cdot \bar{\Pi} - d_\alpha \partial\theta^\alpha - \widehat{d}_\alpha \bar{\partial}\widehat{\theta}^\alpha \quad \text{(SUSY inv)}$$

At the time
1. Appropriate 'BRST' was not found.
2. Also, $c_{\text{tot}} \neq 0$

- Using pure spinor λ^α, one can. — Berkovits (2000)

❤ Review of PS Superstring

Berkovits (2000-)

Pure Spinor (PS)

Cartan (1913)

- Bosonic chiral spinor λ^α with non-linear PS constraint:

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$$

- Not all 10 conditions are indep because of a Fierz

$$(\lambda\gamma^m\lambda)(\lambda\gamma_m\lambda) = 0$$

- In fact, only 5 constraints are indep.
  ➜ PS: $16 - 5 = 11$ indep components.

  Under $U(5) \subset SO(9,1)$: $16 = 1 + 10 + \bar{5}$

$$\text{PS cond} \Leftrightarrow \lambda_I = \frac{1}{8}\lambda_+^{-1}\epsilon_{IJKLM}\lambda_{JK}\lambda_{LM}$$

PS formalism for superstring

(Shall concentrate on the holomorphic sector)

- Worldsheet "free" CFT with vanishing center
  - GS sector: x^m, $(p_\alpha, \theta^\alpha)$
  - PS sector: $(\omega_\alpha, \lambda^\alpha) \leftarrow$ weight $(1,0)$

$$T(z) = -\frac{1}{2}\partial x^m \partial x_m|_{(c=10)} - p_\alpha \partial\theta^\alpha|_{(-32)} - \omega_\alpha \partial\lambda^\alpha|_{(22)}$$

  - "Free": not genuinely;
    But physical quantities can be computed with covariant rules

- "BRST" op: bosonic "ghost" against

$$d_\alpha = p_\alpha + i(\gamma^m\theta)_\alpha p_m + \frac{1}{2}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta), \quad d_\alpha(z)d_\beta(w) = 2\gamma^m\Pi_m/(z-w)$$

$$Q_B = \int \lambda^\alpha d_\alpha \quad \Rightarrow \quad Q_B^2 = 2\int \lambda\gamma^m\lambda\Pi_m = 0$$

Cohomology of Q_B = Spacetime spectrum

Passed various checks, and more!

Achievements of PS: Basics

- Explicit forms of Q_{B}-inv massless vertex ops:

$$U(z) = \lambda^{\alpha} A_{\alpha}(x, \theta) \quad \text{(unintegrated)}$$

- Superfield form. of Super-Maxwell:
  – $Q_{\text{B}} U = 0 \rightarrow$ Correct constraint:

$$\gamma^{\alpha\beta}_{m_1 m_2 \ldots m_5} D_{\alpha} A_{\beta} = 0, \quad (D_{\alpha} = \partial_{\alpha} - i(\gamma^m \theta)\partial_m)$$

  – $\delta U = Q_{\text{B}} \Omega \rightarrow$ Gauge transf.
  – $A_{\alpha}(x, \theta) \ni a_m(x), \psi^{\alpha}(x)$: photon, photino wave func; for plane waves:

$$U^{\text{B}}_m \sim (\lambda \gamma_m \theta) \mathrm{e}^{ik \cdot x}, \quad U^{F}_{\alpha} \sim (\lambda \gamma_m \theta)(\gamma^m \theta)_{\alpha} \mathrm{e}^{ik \cdot x}$$
$$k^2 = 0 \quad (\text{weight} = 0)$$

  Also, integrated vertex ops are available

- Similar for 1st-massive modes (alas, much much more complicated)

- Super-Poincaré covariant 'rule' to compute N-pt tree amplitudes

$$\mathcal{A} = \langle U_1(z_1)U_2(z_2)U_3(z_3) \int [dz_4]V_4(z_4) \cdots \int [dz_N]V_N(z_N)\rangle$$

  – $\langle \ldots \rangle$: CFT correlation func. of vertex ops.
  – Zero-mode prescription for θ and λ: Read off the coeff. of

$$(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^r\theta)(\theta\gamma_{mnr}\theta)$$

➜ Coincides with RNS results.

- Similar for loops ➜ finiteness proof (2004)

- Construction of boundary states

Achievements of PS: Applications

- PS analogue of Metsaev-Tseytlin GS action for $AdS_5 \times S^5$
  - One-loop conformal invariant
  - Construction of infinite non-local conserved charges

- Supermembrane (difficult)

Extended formulations with PS cond removed

Genuine free $(\omega_\alpha, \lambda^\alpha)$ with ghosts which effectively impose PS cond

- SB: covariant
- Komaba: minimal extension
  - Add BRST quartet $(\omega_I, \lambda_I, b_I, c_I)$ $(SO(9,1) \to U(5))$
  - Quantum operator mapping to RNS / GS: $Q_{\text{PS}} \sim Q_{\text{RNS}} \sim Q_{\text{GS}}$

- Relation to doubly supersymmetric formulation (aka superembedding)

Challenges of PS

- All rules postulated by hand
  - What is Q_{B} the BRST of?
  - Is λ^α a BRST ghost?; Origin of $\lambda\gamma^m\lambda = 0$?
  - Where is Virasoro? ($Q = cT$ vs λd?)

  Nature of worldsheet symmetry is obscure

  (Drawback of the powerful free field postulate)

➜ Desirable to have the underlying classical action for PS
  - Geometrical understanding of Berkovits' measure
  - Application to 11D supermembranes

Can we find one? ...Yes!

❤ Origin of PS Superstring

[hep-th/0502208] (with Y. Kazama)

Classical PS Action I

- Recall the classical GS action:

$$\mathcal{L}_{\mathrm{GS}} = \mathcal{L}_{\mathrm{K}} + \mathcal{L}_{\mathrm{WZ}}$$

$$\begin{cases} \mathcal{L}_{\mathrm{K}} = -\frac{1}{2}\sqrt{-g}g^{ij}\Pi_i^m \Pi_{jm} \\ \mathcal{L}_{\mathrm{WZ}} = -\epsilon^{ij}\Pi_i^m (W_{jm} - \widehat{W}_{jm}) - \epsilon^{ij} W_i^m \widehat{W}_{jm} \end{cases}$$

$$\Pi_i^m = \partial_i x^m - W_i^m - \widehat{W}_i^m, \qquad W_i^m = i\tilde{\theta}\gamma^m \partial_i \tilde{\theta}$$

- Classical PS can be obtained simply by
(1) gauging the origin of $\tilde{\theta}^A$:

$$\tilde{\theta}^A \to \Theta^A = \tilde{\theta}^A - \theta^A$$

(2) shifting (← to have SUSY in $(x^m, \theta^{A\alpha})$ sector)

$$x^m \to y^m = x^m - i(\theta\gamma^m\tilde{\theta}) - i(\hat{\theta}\gamma^m\tilde{\hat{\theta}})$$

Classical PS Action II

- Then, GS action turns to

$$\mathcal{L}_{\mathsf{PS}} = \mathcal{L}_{\mathsf{K}} + \mathcal{L}_{\mathsf{WZ}}$$

$$\begin{cases} \mathcal{L}_{\mathsf{K}} = -\frac{1}{2}\sqrt{-g}g^{ij}\Pi_i^m\Pi_{jm} \\ \mathcal{L}_{\mathsf{WZ}} = \epsilon^{ij}\Pi_i^m(W_{jm} - \widehat{W}_{jm}) - \epsilon^{ij}W_i^m\widehat{W}_{jm} \end{cases}$$

$$\Pi_i^m = \partial_i y^m - W_i^m - \widehat{W}_i^m, \quad W_i^m = i\Theta\gamma^m\partial_i\Theta$$

$$\Theta = \tilde{\theta} - \theta, \quad y^m = x^m - i(\theta\gamma^m\tilde{\theta}) - i(\hat{\theta}\gamma^m\hat{\tilde{\theta}})$$

- By construction, has 'hidden local symmetry' $\times$ 2:

$$\delta\tilde{\theta} = \delta\theta = \chi, \; \delta x^m = i\chi\gamma^m\Theta, \quad (\delta\Theta = \delta y^m = 0)$$

  – can be used to kill θ (equivalence to GS is manifest)
  – *or*, kept along quantization
    ➜ Berkovits' 'BRST' sym: $Q_{\mathsf{B}} = \lambda^\alpha d_\alpha$

Symmetries of Classical PS

- Common with GS:
  - Reparam. + Weyl
  - Kappa among $(y^m, \Theta^\alpha, g_{ij})$
  - Super-Poincaré

$$\delta\tilde{\theta} = 0,\ \delta\theta = -\epsilon,\ \delta x^m = -i\epsilon\gamma^m\theta, \quad (\delta\Theta = \epsilon,\ \delta y^m = i\epsilon\gamma^m\Theta)$$

- New to PS:
  - Hidden local symmetry

We now canonically quantize $\mathcal{L}_{\mathrm{PS}}$
  - $\tilde{\theta}^{A\alpha}$ (GS): fix to semi-LC
  - Physical sector $(x^m, \theta^{A\alpha})$: can maintain covariance

Canonical Quantization of PS

- Canonical mom.

$$k_m = \frac{\partial\mathcal{L}}{\partial\dot{x}^m}, \quad \tilde{k}_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\tilde{\theta}}^\alpha}, \quad k_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\theta}^\alpha}$$

- Poisson bracket:

$$\{x^m(\sigma), k^n(\sigma')\}_{\mathsf{P}} = \eta^{mn}\delta(\sigma-\sigma'), \quad \{\tilde{\theta}^\alpha(\sigma), \tilde{p}_\beta(\sigma')\}_{\mathsf{P}} = -\delta^\alpha{}_\beta\delta(\sigma-\sigma')$$
$$\{\theta^\alpha(\sigma), p_\beta(\sigma')\}_{\mathsf{P}} = -\delta^\alpha{}_\beta\delta(\sigma-\sigma'), \quad (\text{rest}) = 0$$

- Constraints (left/right(hatted): particle $\times 2$):

$$T = \frac{1}{4}\Pi^m\Pi_m + \partial_1\Theta^\alpha\tilde{D}_\alpha = (\text{Virasoro}) \approx 0, \qquad \hat{T} = \cdots$$
$$(\Pi^m \equiv k^m + \partial_1 y^m - 2W_1^m \sim \Pi_0^m + \Pi_1^m) \qquad (\hat{\Pi}^m \sim \hat{\Pi}_0^m - \hat{\Pi}_1^m)$$
$$\tilde{D}_\alpha = \tilde{k}_\alpha - i(\gamma^m\theta)_\alpha k_m - i(\gamma^m\Theta)_\alpha(k_m + \Pi_1^m - W_{1m}) \approx 0, \qquad \hat{\tilde{D}}_\alpha = \cdots$$
$$D_\alpha = k_\alpha + i(\gamma^m\tilde{\theta})_\alpha k_m + i(\gamma^m\Theta)_\alpha(k_m + \Pi_1^m - W_{1m}) \approx 0, \qquad \hat{D}_\alpha = \cdots$$

Expect (1st cls: 1+8+16, 2nd cls: 8)$\times$2 constraints.

- Constraint algebra (omit δ-funcs):

$$\begin{array}{l} \{T,\, *\}_{\mathrm{P}} \approx 0, \quad \{\tilde{D}_\alpha,\, \tilde{D}_\beta\}_{\mathrm{P}} = 2i\gamma^m_{\alpha\beta}\Pi_m \\ \{D_\alpha,\, D_\beta\}_{\mathrm{P}} = -\{\tilde{D}_\alpha,\, D_\beta\}_{\mathrm{P}} = 2i\gamma^m_{\alpha\beta}\Pi_m \end{array} \quad \Rightarrow \quad \{\Delta_\alpha,\, *\}_{\mathrm{P}} = 0$$

– T (Virasoro): just measures weight $\rightarrow$ 1st cls
– $\tilde{D}_\alpha$: just as GS $\rightarrow$ 8×1st cls (κ) and 8×2nd cls

– $\Delta_\alpha = D_\alpha + \tilde{D}_\alpha = p_\alpha + \tilde{p}_\alpha - ip_m(\Theta\gamma^m)$:
generates the 'hidden sym' $\rightarrow$ 16×1st cls

- Below,
(1) we fix κ by imposing semi-LC gauge: $\gamma^+\tilde{\theta} = 0 \Leftrightarrow \tilde{\theta}_{\dot{a}} = 0$
(2) Reduce the phase space to

$$\Sigma^* : \quad \tilde{\theta}_{\dot{a}} = \tilde{D}_\alpha = 0 \quad \Leftrightarrow \quad (x^m, p_n, S_a; p_\alpha, \theta^\alpha)$$

(3) On Σ^*, we will be left with $1 + 16$ 1st cls constraints (T, D_α)
➜ Can invoke BRST: introduce ghosts (b, c) and $(\omega_\alpha, \lambda^\alpha)$ to define

$$Q = \lambda^\alpha \Delta_\alpha + cT + \cdots, \quad Q^2 = 0$$

((b, c) and S_a kill a part of λ^α ➜ PS constraint)

A digression on left/right splitting

- Similar for $(\hat{T}, \hat{\tilde{\Delta}}_\alpha, \hat{D}_\alpha)$; but at this stage,

Holomorphicity $\neq$ Spinor species ($A = 1, 2$)

  – Constraint alg splits into holomorphic/anti-hol part:

  $$\{D_\alpha,\, \hat{D}_\beta\}_{\mathsf{P}} = 0, \quad \text{etc.}$$

  – But unlike GS, "left/right" variables are still entangling:

  $$\Pi^m = k^m + \underbrace{\partial_1 x^m - i\partial_1(\theta\gamma^m\tilde{\theta} + \hat{\theta}\gamma^m\hat{\tilde{\theta}})}_{\partial_1 y^m} - 2W_1^m$$

- More on this later

Explicit separation of κ

- Assuming $\Pi^+ \neq 0$, able to separate 1st and 2nd cls. in $\tilde{D}_\alpha$:

$$\tilde{D}_\alpha \to \begin{cases} \tilde{D}_a & \to \{\tilde{D}_a, \tilde{D}_b\}_{\mathrm{P}} = 2i\delta_{ab}\Pi^+ \\ \tilde{K}_{\dot{a}} = \tilde{D}_{\dot{a}} - \frac{1}{\Pi^+}\Pi^i \gamma^i_{\dot{a}b}\tilde{D}_b & \to \text{1st cls.} = \text{Kappa sym.} \end{cases}$$

Indeed

$$\{\tilde{K}_{\dot{a}}, \tilde{K}_{\dot{b}}\}_{\mathrm{P}} = -8i\delta_{\dot{a}\dot{b}}\mathcal{T} + f^\alpha \tilde{D}_\alpha \approx 0, \qquad \mathcal{T} \equiv T/\Pi^+$$

- $\mathcal{T} = T/\Pi^+$ is a good combination:

$$\{\mathcal{T}, \mathcal{T}\}_{\mathrm{P}} = 0 \quad (\leftarrow \Pi^+ \text{ weight } 1)$$

  – $\mathcal{T} \approx 0 \Leftrightarrow T \approx 0$ ($\Pi^+ \neq 0$) ➜ Can use $\mathcal{T}$ instead of T

1st cls constraints on reduced phase space

- On the constrained surface Σ^*: $\phi_I \equiv (D_a, \tilde{\theta}_{\dot{a}}, \tilde{K}_{\dot{a}}) \approx 0$
  with Dirac bracket

$$\{A,\, B\}_{\mathsf{D}^*} = \{A,\, B\}_{\mathsf{P}} - \{A,\, \phi_I\}_{\mathsf{P}}\{\phi_I,\, \phi_J\}_{\mathsf{P}}^{-1}\{\phi_J,\, B\}_{\mathsf{P}}$$

  $\mathcal{T}$ and $D_\alpha = (D_a, D_{\dot{a}})$ yield $(1+16)$ 1st cls constraints
  with simple algebra:

$$\{D_{\dot{a}}(\sigma),\, D_{\dot{b}}(\sigma')\}_{\mathsf{D}^*} = -8i\delta_{\dot{a}\dot{b}}\mathcal{T}\delta(\sigma-\sigma')$$
$$\{\mathcal{T}(\sigma),\, *(\sigma')\}_{\mathsf{D}^*} = \{D_a(\sigma),\, *(\sigma')\}_{\mathsf{D}^*} = 0$$

  – Now $D_{\dot{a}}$ carries the info of Kappa $\tilde{K}_{\dot{a}}$:

$$(\text{Kappa})^2 = \text{Virasoro}$$

So far, so good.

Non-Canonical Dirac Brackets

- However, the phase space variables are no more canonical wrt the induced Symplectic structure (Dirac bracket) on

$$\Sigma^* : \quad \tilde{D}_a = \tilde{K}_{\dot{a}} = \tilde{\theta}_{\dot{a}} = 0$$

eg.

$$\{x^m(\sigma), k^n(\sigma')\}_{\mathrm{D}^*} = \eta^{mn}\delta(\sigma - \sigma') + (i/2\Pi^+)(\gamma^m\tilde{\theta})_c(\gamma^n\Theta)_c\partial_\sigma\delta(\sigma - \sigma')$$
$$\{k^m(\sigma), k^n(\sigma')\}_{\mathrm{D}^*} = -(i/2)\partial_\sigma[(1/\Pi^+)(\gamma^m\Theta)_c(\gamma^n\Theta)_c\partial_\sigma\delta(\sigma - \sigma')]$$

- Is not surprising; but problematic for quantization.

Can one find a canonical basis?

Yes, and with local combination of original fields.

Canonical Basis

$$\{x^m(\sigma),\, p^n(\sigma')\}_{\mathsf{D}^*} = \eta^{mn}\delta(\sigma-\sigma'), \qquad \{\theta^\alpha(\sigma),\, p_\beta(\sigma')\}_{\mathsf{D}^*} = -\delta^\alpha_\beta\delta(\sigma-\sigma')$$
$$\{S_a(\sigma),\, S_b(\sigma')\}_{\mathsf{D}^*} = i\delta_{ab}\delta(\sigma-\sigma'), \qquad \text{(rest)} = 0$$

for

$$\begin{aligned}
p^m &= k^m + i\partial_1(\theta\gamma^m\tilde{\theta}) - i\partial_1(\hat{\theta}\gamma^m\hat{\tilde{\theta}}) \\
p_a &= k_a - i(\partial_1 x^+ - i\theta\gamma^+\partial_1\theta)\tilde{\theta}_a + [2(\gamma^i\partial_1\theta)_a\tilde{\theta}\gamma^i\theta + (\gamma^i\theta)_a\partial_1(\tilde{\theta}\gamma^i\theta)] \\
p_{\dot{a}} &= k_{\dot{a}} + i(\gamma^m\theta)_{\dot{a}}\big[-2i\tilde{\theta}\gamma_m\partial_1\theta + i\tilde{\theta}\gamma_m\partial_1\tilde{\theta} - i\partial_1(\tilde{\theta}\gamma_m\theta)\big] \\
&\qquad - i(\gamma^i\tilde{\theta})_{\dot{a}}\big[\partial_1 x^i - 3i\theta\gamma^i\partial_1\theta + 2i\theta\gamma^i\partial_1\tilde{\theta} + i\partial_1(\hat{\tilde{\theta}}\gamma^i\hat{\theta})\big] \\
S_a &= \sqrt{2\Pi^+}\tilde{\theta}_a
\end{aligned}$$

& similar for $\widehat{S}_a$, $\widehat{p}_a$ and $\widehat{p}_{\dot{a}}$

- Also, complete left/right separation takes place:

$$\text{eg. } \Pi^m = k^m + \partial_1 x^m + i\partial_1(\theta\gamma^m\tilde{\theta} + \hat{\theta}\gamma^m\hat{\tilde{\theta}}) - 2W_1^m$$
$$\Rightarrow \quad \Pi^m = p^m + \partial_1 x^m + 2i\partial_1(\theta\gamma^m\tilde{\theta}) - 2W_1^m$$

Quantization

- Quantization is trivial: just replace $\{A,\, B\}_{\mathsf{D}^*} \to -i[A, B\}$:

$$[p^m, x^n] = -i\eta^{mn}, \quad \{p_\alpha, \theta^\beta\} = -i\delta_\alpha^\beta, \quad \{S_a, S_b\} = -\delta_{ab}$$

- Spectrum dictated by $(1+16)\times 2$ 1st cls constraints:

$$\mathcal{T}, \quad (D_a, D_{\dot{a}}) \approx 0$$
$$\{D_{\dot{a}}(\sigma),\, D_{\dot{b}}(\sigma')\}_{\mathsf{D}^*} = -8i\delta_{\dot{a}\dot{b}}\mathcal{T}(w)\delta(\sigma-\sigma')$$
$$\{\mathcal{T}(\sigma),\, *(\sigma')\}_{\mathsf{D}^*} = \{D_a(\sigma),\, *(\sigma')\}_{\mathsf{D}^*} = 0$$

Obtained system as a free CFT

- Or in CFT language, we get conformal fields with free field OPE's:

$$x^m(z)x^n(w) = -\eta^{mn}\log(z-w), \quad S_a(z)S_b(w) = \delta_{ab}/(z-w)$$
$$p_\alpha(z)\theta^\beta(w) = \delta_\alpha{}^\beta/(z-w)$$

- Constraints are

$$\mathcal{T} = \frac{1}{4}\frac{\Pi\cdot\Pi}{\Pi^+}, \quad D_a = d_a + i\sqrt{2\Pi^+}S_a$$
$$D_{\dot a} = d_{\dot a} + i\sqrt{\frac{2}{\Pi^+}}\Pi^i(\gamma^i S)_{\dot a} + \frac{2}{\Pi^+}(\gamma^i S)_{\dot a}(S\gamma^i\partial\theta)$$

where

$$\Pi^m = \pi^m - \frac{1}{2\pi^+}(S\gamma^m\partial S) - i\sqrt{\frac{2}{\pi^+}}(S\gamma\partial\theta), \quad (\Pi^+ = \pi^+)$$
$$\pi^m = i\partial x^m + \theta\gamma^m\partial\theta$$
$$d_\alpha = p_\alpha + i(\gamma_m\theta)_\alpha\partial x^m + \frac{1}{2}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)$$

- Separating S_a from $(x^m, p_\alpha, \theta^\alpha)$, we get

$$\mathcal{T} = \frac{1}{2}\frac{\pi^m\pi_m}{\pi^+} - \frac{1}{2\pi^+}S_c\partial S_c + i\sqrt{\frac{2}{\pi^+}}S_c\partial\theta_c + i\sqrt{\frac{2}{(\pi^+)^3}}\pi^i(S\gamma^i\partial\theta) - \frac{1}{(\pi^+)^2}(S\gamma^i\partial\theta)^2 + \frac{4\partial^2\theta_{\dot{c}}\partial\theta_{\dot{c}}}{(\pi^+)^2} - \frac{\partial^2\log\pi^+}{2\pi^+}$$

$$D_a = d_a + i\sqrt{2\pi^+}S_a$$

$$D_{\dot{a}} = d_{\dot{a}} + i\sqrt{\frac{2}{\pi^+}}\pi^i(\gamma^i S)_{\dot{a}} - \frac{1}{\pi^+}(\gamma^i S)_{\dot{a}}(S\gamma^i\partial\theta) + \frac{4\partial^2\theta_{\dot{a}}}{\pi^+} - \frac{2\partial\pi^+\partial\theta_{\dot{a}}}{(\pi^+)^2}$$

- ∃ some ordering ambiguities;
  Can fix it demanding the constraint alg close quantum mechanically:

$$D_{\dot{a}}(z)D_{\dot{b}}(w) = \frac{-4\delta_{\dot{a}\dot{b}}\mathcal{T}(w)}{z-w}, \quad (\text{rest}) = 0$$

- Those were already engineered by Berkovits-Marchioro [hep-th/0412198] as a 1st cls algebra ‘containing’ d_α.

Derivation of PS

- Now, we can BRST quantize the system in a standard way: Introduce fermionic (b, c), bosonic $(\tilde{\omega}_\alpha, \tilde{\lambda}^\alpha)$ ghosts and define

$$\tilde{Q} = \int (\tilde{\lambda}^\alpha D_\alpha + \mathcal{T} c - \tilde{\lambda}_{\dot{a}} \tilde{\lambda}_{\dot{a}} b)$$

  ➜ $\tilde{Q}^2 = 0$; Cohom of $\tilde{Q}$ = spectrum

This completes the quantization.
$\tilde{Q}$ must have correct cohom; a tricky way to quantize GS

- Furthermore, can show the equivalence with PS:

$$\text{cohom of } \tilde{Q} = \text{cohom of } Q_{\mathrm{B}} \quad (Q_{\mathrm{B}} = \int \lambda^\alpha d_\alpha)$$

  where λ^α is constrained $\lambda\gamma^m\lambda = 0$ (PS)

- Basic idea of the proof:
  (1) Split $\tilde{\lambda}$ to PS direction and the rest: $\tilde{\lambda} = (\lambda, \lambda^\perp) = 11 + 5$
  (2) $(\lambda^\perp, \omega^\perp; c, b, S_a)$ forms 5 KO quartet and decouples cohomologically

Derivation of Q_{B} via Similarity Transf Berkovits-Marchioro

- Bring a constant spinor $r_{\dot{a}}$ satisfying

$$r_{\dot{a}}\tilde{\lambda}_{\dot{a}} = 1, \quad r_{\dot{a}}r_{\dot{a}} = 0$$

➜ Able to define a projector to PS space:

$$\mathcal{P}_{\dot{a}\dot{b}} = \tilde{\lambda}_{\dot{a}}r_{\dot{b}} - \frac{1}{2}r_{\dot{a}}\tilde{\lambda}_{\dot{b}}, \quad \mathcal{P}^{\perp}_{\dot{a}\dot{b}} = \delta_{\dot{a}\dot{b}} - \mathcal{P}_{\dot{a}\dot{b}}$$

$$\mathcal{P}_{ab} = \frac{1}{2}\gamma^{i}_{a\dot{b}}\mathcal{P}_{\dot{b}\dot{c}}\tilde{\lambda}_{\dot{c}}(\gamma^{i}r)_{b}, \quad \mathcal{P}^{\perp}_{ab} = \delta_{ab} - \mathcal{P}_{ab}$$

$$\Rightarrow \quad \lambda^{\alpha} \equiv (\mathcal{P}\tilde{\lambda})^{\alpha}, \ \lambda\gamma^{m}\lambda = 0, \quad \lambda^{\perp\alpha} \equiv (\mathcal{P}^{\perp}\tilde{\lambda})^{\alpha}$$

- Then, one can show

$$\mathrm{e}^{Z}\mathrm{e}^{Y}\mathrm{e}^{X}\tilde{Q}\mathrm{e}^{-X}\mathrm{e}^{-Y}\mathrm{e}^{-Z} = \delta_{b} + \delta_{\perp} + Q_{\mathsf{B}}$$

$$X = c(r_{\dot{c}}D_{\dot{c}}), \quad Y = -\frac{1}{2}(\mathcal{P}^{\perp}S)_{c}(\mathcal{P}S)_{c}, \quad Z = -\frac{d_{c}(\mathcal{P}S)_{c}}{\sqrt{2}i} + \frac{4(\partial\theta_{\dot{a}}\lambda_{\dot{a}})(\partial\theta_{\dot{b}}r_{\dot{b}})}{\pi^{+}}$$

where

$$\delta_{b} = 2b(\tilde{\lambda}\gamma^{+}\tilde{\lambda}), \quad \delta_{\perp} = \sqrt{2}i\lambda^{\perp}_{c}(\mathcal{P}S)_{c}, \quad Q_{\mathsf{B}} = \lambda^{\alpha}d_{\alpha}$$

$$\{\delta_{b}, \delta_{\perp}\} = \{\delta_{b}, Q_{\mathsf{B}}\} = \{\delta_{\perp}, Q_{\mathsf{B}}\} = 0$$

❤ Summary

- Derived Berkovits' Q_{B} from first principles

- Origin of PS
  - Classical GS action with a hidden local sym. (completely covariant)
  - Hidden sym $\rightarrow Q_{\mathsf{B}} = \lambda^\alpha d_\alpha$ (λ^α: BRST ghost)
  - Reparam (b, c)-ghost + GS $S_a \rightarrow$ 5 PS constraint $\lambda\gamma^m\lambda = 0$

❤ Outlooks

- Should be able to derive Berkovits' measure (Tree & Loop)
  - Improve the last step; more clever gauge fixing?

- Supermembrane (or super p-branes)
  - Just like Siegel/Berkovits, ask
    "Can one find a clever set of 1st cls constraint
      s.t. Free fields + 'BRST' = Supermembrane"
  - Appropriate constraints are not known;
    classical PS action might provide some hints.

- Application to strings in various BG's
  - Derivation of PS in curved BG. esp. $AdS_5 \times S^5$