Open-closed homotopy algebra in superstring field theory

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2022/06/17 online talk for “Tagen-seminar”

based on arXiv:2204.01249 [hep-th]

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Introduction and summary

Introduction

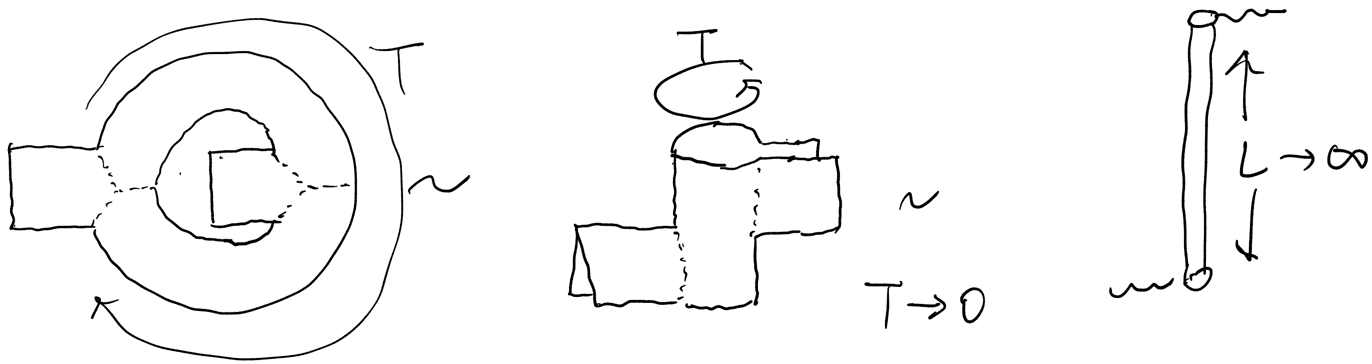
Why open-closed string field theory?

• From open string theory perspective:

∘ To enable the manifest factorization of closed strings appeared in the quantum open-string field theory.

• We known the pure open string theory cannot be unitary! [Kaku-Kikkawa]

• The cubic string field theory contains only an open string field but develops closed string poles at one-loop level. [Giddings-Martinec-Witten]



• We need to incorporate a closed string field to calculate the S-matrix at loop order by the conventional LSZ formula.

- From closed string theory perspective:
  - ◦ To incorporate the nontrivial D-brane backgrounds.

• The open-closed interactions describe the interactions between closed strings and D-branes. [Zwiebach]

• Some nonperturbative effects are also incorporated via D-instanton. [Sen]

These can be realized by the quantum open-closed string field theory, but, in this talk, we consider the classical open-closed string field theory as the first step.

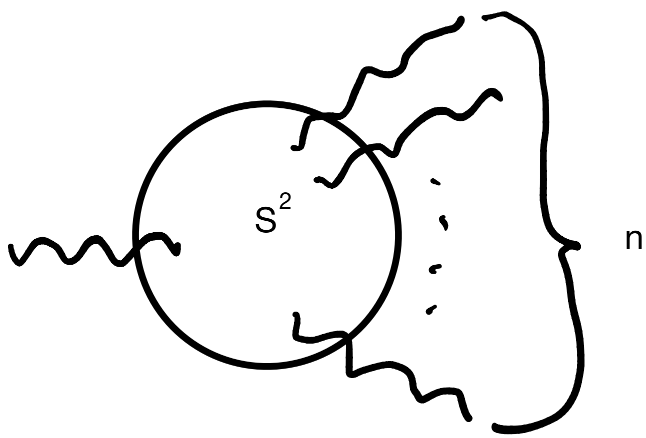
Summary

(Open, Closed) string fields: (Ψ, Φ)

Symplectic form $\quad \Omega = \Omega^o + \Omega^c$

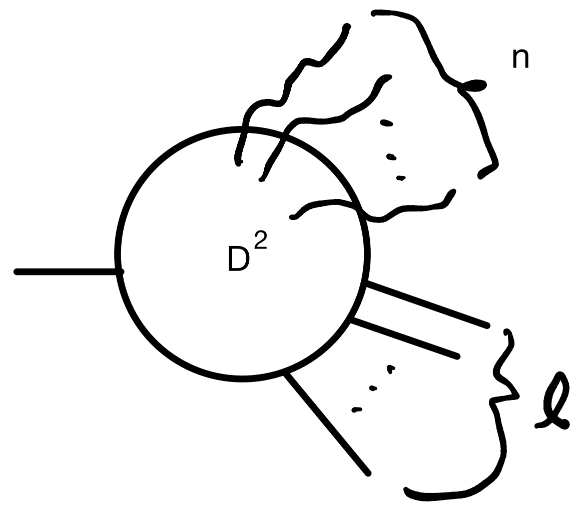
Interactions

(Closed)$^{n+1}$



L_n

(Closed)^n-(Open)$^{l+1}$



$N_{n,l}$

Action

$$I = \sum_{\substack{l,n=0\\ l+n>0}}^{\infty} \frac{1}{(l+1)n!}\Omega^{o}\Big(\Psi, N_{n,l}\big((\Phi_{b.g.})^{n};\Psi^{l}\big)\Big)\,,$$

is invariant under

$$\delta\Psi = \sum_{l,n=0}^{\infty}\sum_{m=0}^{l}\frac{1}{n!}N_{n,l+1}\Big(\big(\Phi_{b.g.}\big)^{n};\Psi^{l-m},\Lambda,\Psi^{m}\Big)\,.$$

and represents the open superstring field theory on general closed superstring backgrounds:

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)!}L_{n+1}\Big(\big(\Phi_{b.g.}\big)^{n+1}\Big) = 0\,,$$

if $(L_n, N_{n,l})$ satisfies the cyclic open-closed homotopy algebra.

Cyclicity

$$\Omega^o(\Psi_1, N_{n,l}(\Phi_1, \cdots, \Phi_n; \Psi_2, \cdots, \Psi_{l+1}))$$
$$= -(-1)^{\deg(\Psi_1)(|\Phi_1|+\cdots+|\Phi_l|+1)}\Omega^o(N_{n,l}(\Phi_1, \cdots, \Phi_n; \Psi_1, \cdots, \Psi_l), \Psi_{l+1}),$$

$$\left(\Omega^c(\Phi_1, L_n(\Phi_2, \cdots, \Phi_{n+1})) = -(-1)^{|\Phi_1|}\Omega^c(L_n(\Phi_1, \cdots, \Phi_n), \Phi_{n+1}).\right)$$

Open-closed homotopy algebra (OCHA)

$$0 = \sum_\sigma \sum_{m=1}^{n} (-1)^{\epsilon(\sigma)} \frac{1}{m!(n-m)!} L_{n-m+1}(Lm(\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(m)}), \Phi_{\sigma(m+1)}, \cdots, \Phi_{\sigma(n)}),$$

$$0 = \sum_\sigma \sum_{m=1}^{n} (-1)^{\epsilon(\sigma)} \frac{1}{m!(n-m)!} N_{n-m+1,l}(Lm(\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(m)}), \Phi_{\sigma(m+1)}, \cdots, \Phi_{\sigma(n)}; \Psi_1, \cdots, \Psi_l)$$
$$+ \sum_\sigma \sum_{m=0}^{n} \sum_{j=0}^{l} \sum_{i=0}^{l-j} (-1)^{\mu m,i(\sigma)} \frac{1}{m!(n-m)!} N_{m,l-j+1}(\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(m)};$$
$$\Psi_1, \cdots, \Psi_i, N_{n-m,j}(\Phi_{\sigma(m+1)}, \cdots, \Phi_{\sigma(n)}; \Psi_{i+1}, \cdots, \Psi_{i+j}), \Psi_{i+j+1}, \cdots, \Psi_l).$$

For superstring field theory, Ψ and Φ have several components with the picture numbers and satisfy some constraints:

$$\Psi = \Psi_{NS} + \Psi_{R} \in \mathcal{H}_o^{res}$$

$$\Phi = \Phi_{NS\text{-}NS} + \Phi_{R\text{-}NS} + \Phi_{NS\text{-}R} + \Phi_{R\text{-}R} \in \mathcal{H}_c^{res}$$

We must distinguish these sectors and give L_{n+1} and $N_{n,l}$ their own picture numbers so that the output is consistent with the constraints and have correct picture numbers:

$$(\star) \qquad L_{n+1}^{(n-r,n-\bar{r})}\,\big|_{(2r,2\bar{r})}\,, \qquad N_{n,l}^{(2n+l-r-1)}\,\big|_{2r}$$

Assuming "bosonic" OCHA $\left(L_{n+1}^{(0,0)}\,\big|_{(2r2\bar{r})}, N_{n,l}^{(0)}\,\big|_{2r}\right)$ is known, we will provide a prescription to give $(\star)$ by appropriately inserting X_0's and/or ξ_0's.

Some basic matters

Open string field:

$$\Psi = \Psi_{NS} + \Psi_R \in \mathcal{H}_o^{res}$$

Closed string field:

$$\Phi = \Phi_{NS\text{-}NS} + \Phi_{R\text{-}NS} + \Phi_{NS\text{-}R} + \Phi_{R\text{-}R} \ \in \mathcal{H}_c^{res}$$

with $\mathrm{gh}(\Psi) = 1, \ \mathrm{gh}(\Phi) = 2$, where $\mathcal{H}_o^{res}$ and $\mathcal{H}_c^{res}$ are the subspace of

$$\begin{aligned} \mathcal{H}_o &= \mathcal{H}^{NS}_{(-1)} + \mathcal{H}^{R}_{(-1/2)}, \\ \mathcal{H}_c &= \mathcal{H}^{NS\text{-}NS}_{(-1,-1)} + \mathcal{H}^{R\text{-}NS}_{(-1/2,-1)} + \mathcal{H}^{NS\text{-}R}_{(-1,-1/2)} + \mathcal{H}^{R\text{-}R}_{(-1/2,-1/2)}, \end{aligned}$$

restricted by the closed-string constraints $b_0^- \Phi = L_0^- \Phi = 0$ and

$$\mathcal{P}^o_{XY}\Psi \ = \ \Psi, \qquad \mathcal{P}^c_{XY}\Phi \ = \ \Phi\,.$$

Here, $\mathcal{P}^o_{XY} = \mathcal{G}^o(\mathcal{G}^o)^{-1}$ and $\mathcal{P}^c_{XY} = \mathcal{G}^c(\mathcal{G}^c)^{-1}$ are the projection operators defined by

$$\begin{aligned} \mathcal{G}^o &= \pi^0_o + X^o \pi^1_o, & (\mathcal{G}^o)^{-1} &= \pi^0_o + Y^o \pi^1_o, \\ X^o &= -\delta(\beta_0) G_0 + \delta'(\beta_0) b_0, & Y^o &= -c_0 \delta'(\gamma_0), \end{aligned}$$

with the projectors $\pi^0_o \Psi = \Psi_{NS}$ and $\pi^1_o \Psi = \Psi_R$ and

$$\begin{aligned} \mathcal{G}^c &= \pi^{(0,0)}_c + X^c \pi^{(1,0)}_c + \bar{X}^c \pi^{(0,1)}_c + X^c \bar{X}^c \pi^{(1,1)}_c, \\ X^c &= -\delta(\beta_0) G_0 + \frac{1}{2}\delta'(\beta_0) b^+_0, \qquad \bar{X}^c = -\delta(\bar{\beta}_0)\bar{G}_0 + \frac{1}{2}\delta'(\bar{\beta}_0) b^+_0, \\ (\mathcal{G}^c)^{-1} &= \pi^{(0,0)}_c + Y^c \pi^{(1,0)}_c + \bar{Y}^c \pi^{(0,1)}_c + Y^c \bar{Y}^c \pi^{(1,1)}_c, \\ Y^c &= -2\frac{G}{L^+}\delta(\gamma_0), \qquad \bar{Y}^c = -2\frac{\bar{G}}{L^+}\delta(\bar{\gamma}_0), \end{aligned}$$

with the projectors $\pi^{(0,0)}\Phi = \Phi_{NS\text{-}NS}$, $\pi^{(1,0)}\Phi = \Phi_{R\text{-}NS}$, $\pi^{(0,1)}\Phi = \Phi_{NS\text{-}R}$, and $\pi^{(1,1)}\Phi = \Phi_{R\text{-}R}$. Both $(\mathcal{G}^c, (\mathcal{G}^c)^{-1})$ and $(\mathcal{G}^o, (\mathcal{G}^o)^{-1})$ satisfy

$$\mathcal{P}_{XY}\mathcal{G} = \mathcal{G}, \qquad \mathcal{G}^{-1}\mathcal{P}_{XY} = \mathcal{G}^{-1}, \qquad [Q, \mathcal{G}] = 0.$$

Under these constraints, the string fields expanding in the ghost zero-modes are restricted to only two components as

$$\begin{aligned}
\Psi_{NS} &= \psi_{NS} - c_0\tilde{\psi}_{NS}, \qquad \Psi_{R} \;=\; \psi_{R} - (\gamma_0 + c_0 G)\tilde{\psi}_{R},\\
\Phi_{NS\text{-}NS} &= \phi_{NS\text{-}NS} - c_0^{+}\tilde{\phi}_{NS\text{-}NS}\,,\\
\Phi_{R\text{-}NS} &= \phi_{R\text{-}NS} - \frac{1}{2}(\gamma_0 + 2c_0^{+}G)\tilde{\phi}_{R\text{-}NS}\,,\\
\Phi_{NS\text{-}R} &= \phi_{NS\text{-}R} - \frac{1}{2}(\bar{\gamma}_0 + 2c_0^{+}\bar{G})\tilde{\phi}_{NS\text{-}R}\\
\Phi_{R\text{-}R} &= \phi_{R\text{-}R} - \frac{1}{2}(\gamma_0\bar{G} - \bar{\gamma}_0 G + 2c_0^{+}G\bar{G})\tilde{\phi}_{R\text{-}R}\,.
\end{aligned}$$

(After relaxing the ghost number restriction,) the first and the second components become the fields and the antifields, respectively, via the BV quantization on a gauge-fixed basis.

(See also the structure of the symplectic forms Ω below.)

Symplectic forms:

Symplectic forms in $\mathcal{H}_o$ and $\mathcal{H}_c$:

$$\omega_s^o(\Psi_1\,,\Psi_2) \;=\; (-1)^{\deg\Psi_1}\langle\Phi_1|\Phi_2\rangle\,, \qquad \omega_s^c(\Phi_1\,,\Phi_2) \;=\; \langle\Phi_1|c_0^-|\Phi_2\rangle,$$

Natural symplectic forms in $\mathcal{H}_o^{res}$ and $\mathcal{H}_c^{res}$:

$$\Omega^o(\Psi_1\,,\Psi_2) \;=\omega_s^o(\Psi_1\,,(\mathcal{G}^o)^{-1}\Psi_2) = \sum_i \Big(\langle\psi_i|\tilde{\psi}_i\rangle + \langle\tilde{\psi}_i|\psi_i\rangle\Big)\,,$$

$$\Omega^c(\Phi_1\,,\Phi_2) \;=\omega_s^c(\Phi_1\,,(\mathcal{G}^c)^{-1}\Phi_2) \;= \sum_i \Big(\langle\phi_i|\tilde{\phi}_i\rangle + \langle\tilde{\phi}_i|\phi_i\rangle\Big)\,.$$

We also use the natural symplectic forms ω_l^o and ω_l^c in the large Hilbert spaces satisfying

$$\omega_s^o(\Psi_1, \Psi_2) \;= \omega_l^o(\xi_0\Psi_1, \Psi_2), \qquad \text{if } \Psi_1, \Psi_2 \in \mathcal{H}_o,$$

$$\omega_s^c(\Phi_1, \Phi_2) \;= \omega_l^c(\xi_0\bar{\xi}_0\Phi_1, \Phi_2), \qquad \text{if } \Phi_1, \Phi_2 \in \mathcal{H}_c.$$

Interactions with OCHA structure

(1) closed string interaction $(\sim \ S^2)$

$$
\begin{array}{cccccc}
L_n \ : & (\mathcal{H}_{res})^{\wedge n} & \longrightarrow & \mathcal{H}_{res}, & (n \geq 1), \\
 & \Cup & & \Cup & \\
 & \Phi_1 \wedge \cdots \wedge \Phi_n & \longmapsto & L_n(\Phi_1, \cdots, \Phi_n), &
\end{array}
$$

with $L_1 = Q_c$, where $\Phi_1 \wedge \cdots \wedge \Phi_n = \sum_\sigma (-1)^{\epsilon(\sigma)} \Phi_{\sigma(1)} \otimes \cdots \otimes \Phi_{\sigma(n)}$.

(2) open and closed string interaction $(\sim \ D^2)$

$$
\begin{array}{cccc}
N_{n,l} \ : \ (\mathcal{H}_c^{res})^{\wedge n} \otimes (\mathcal{H}_o^{res})^{\oplus l} & \longrightarrow & \mathcal{H}_o^{res}, & (n, l \geq 0, \ n + l > 0), \\
\Cup & & \Cup & \\
(\Phi_1 \wedge \cdots \wedge \Phi_n) \otimes (\Psi_1 \otimes \cdots \otimes \Psi_l) & \longmapsto & N_{n,l}(\Phi_1, \ \cdots, \Phi_n; \Psi_1, \cdots, \Psi_l). &
\end{array}
$$

with $N_{0,1} = Q_o$.

Coalgebra representation

These interactions are represented by degree odd coderivation acting on (Symmetrized) tensor algebra $\mathcal{SH}_c^{res} \otimes \mathcal{TH}_o^{res}$,

$$\mathcal{SH}_c^{res} \otimes \mathcal{TH}_o^{res} \;=\; \sum_{n,l=0}^{\infty} (\mathcal{H}_c^{res})^{\wedge n} \otimes (\mathcal{H}_o^{res})^{\otimes l},$$

as

$$\boldsymbol{L} \;=\; \sum_{n=1}^{\infty} \boldsymbol{L}_n \;=\; \sum_{n=1}^{\infty} \sum_{m,l=0}^{\infty} \Big(\big(L_n \wedge \mathbb{I}_m \big) \otimes \mathbb{I}^{\otimes l} \Big) \pi_{m+n,l},$$

$$\boldsymbol{N} \;=\; \sum_{\substack{n,l=0 \\ n+l>0}} \boldsymbol{N}_{n,l} \;=\; \sum_{\substack{n,l=0 \\ n+l>0}} \sum_{m,j,k=0}^{\infty} \Big(\mathbb{I}_m \otimes \big(\mathbb{I}^{\otimes j} \otimes N_{n,l} \otimes \mathbb{I}^{\otimes k} \big) \Big) \pi_{m+n,j+k+l}.$$

If $\boldsymbol{L}$ and $\boldsymbol{N}$ satisfy

$$[\boldsymbol{L}+\boldsymbol{N},\boldsymbol{L}+\boldsymbol{N}] \;=\; 0 \quad \text{or} \quad [\boldsymbol{L},\boldsymbol{L}] \;=\; [\boldsymbol{L},\boldsymbol{N}]+\frac{1}{2}[\boldsymbol{N},\boldsymbol{N}] \;=\; 0\,,$$

$(\mathcal{H}=\mathcal{H}_c\oplus\mathcal{H}_o,\boldsymbol{L},\boldsymbol{N})$ is called OCHA.

Substructures

$(\mathcal{H}_c,\boldsymbol{L})$: L_∞ algebra, $(\mathcal{H}_o,\{N_{0,m}\}\equiv\boldsymbol{M})$: A_∞ algebra

If it additionally satisfies

Cyclicity

$$\begin{aligned}
&\Omega^o(N_{n,m}(\Phi_1,\cdots,\Phi_n;\Psi_1,\cdots,\Psi_m),\Psi_{m+1})\\
&\qquad = \;(-1)^{|\Psi_1|(|\Psi_2|+\cdots+|\Psi_{m+1}|)}\Omega^o(N_{n,m}(\Phi_1,\cdots,\Phi_n;\Psi_2,\cdots,\Psi_{m+1}),\Psi_1)\,,\\
&\Omega^c(L_n(\Phi_1,\cdots,\Phi_n),\Phi_{n+1}) \;=\; (-1)^{\epsilon(n)}\Omega^c(L_n(\Phi_{\sigma(1)},\cdots,\Phi_{\sigma(n)}),\Phi_{\sigma(n+1)})\,,
\end{aligned}$$

$(\mathcal{H}=\mathcal{H}_c\oplus\mathcal{H}_o,\Omega=\Omega^c+\Omega^o,\boldsymbol{L},\boldsymbol{N})$ is called cyclic OCHA.

OCHA in superstring field theory

We need a cyclic OCHA $(\mathcal{H}^{res}, \Omega, \boldsymbol{L}, \boldsymbol{N})$ with additional properties

(1) Distinguish the type of fields by assigning any of (a), (b) or both:

(a) R(amond) number $|_{2r} = \#$ of R inputs $-$ $\#$ of R output

(b) cyclic R number $|^{2r} = \#$ of R inputs $+$ $\#$ of R output

Note: (a) is additive in commutator and (b) is compatible with cyclicity

(2) Have appropriate picture numbers

$$\boldsymbol{L} \ = \sum_{n=0}^{\infty} \sum_{\substack{p,\bar{p},r,\bar{r}=0 \\ p+r=\bar{p}+\bar{r}=n}}^{\infty} \boldsymbol{L}_{n+1}^{(p,\bar{p})} \,|_{(2r,2\bar{r})} \,, \qquad \boldsymbol{N} \ = \ \sum_{\substack{n,l,r=0 \\ n+l>0}}^{\infty} \boldsymbol{N}_{n,l}^{(2n+l-r-1)} \,|_{2r} \,,$$

where $r = r_o + r_c + \bar{r}_c$ in $\boldsymbol{N}$.

(3) Be consistent with the constraints

$$\mathcal{P}^{c}_{XY}\pi_1\boldsymbol{L} \ = \ \pi_1\boldsymbol{L} \,, \qquad \mathcal{P}^{o}_{XY}\pi_1\boldsymbol{N} \ = \ \pi_1\boldsymbol{N} \,.$$

Construction of $(\mathcal{H}^{res}, \Omega, \boldsymbol{L}, \boldsymbol{N})$

Assumption

$(\boldsymbol{L}, \boldsymbol{N})$ without picture numbers

$$\boldsymbol{Q}_c + \boldsymbol{L}_B \;=\; \sum_{n=0}^{\infty} \sum_{\substack{m,\bar{m},r,\bar{r}=0 \\ p+m=\bar{p}+\bar{m}=n}}^{\infty} \boldsymbol{L}_{n+1}^{(0,0)}\, |_{(2r,2\bar{r})}\,, \qquad \boldsymbol{Q}_o + \boldsymbol{N}_B \;=\; \sum_{\substack{n,l,r=0 \\ n+l>0}}^{\infty} \boldsymbol{N}_{n,l}^{(0)}\, |_{2r}\,,$$

is already known.

This can be constructed if we know the triangulation of the (bosonic) moduli space suitable for the SFT.

Step 1

Construct cyclic OCHA $(\mathcal{H}_l = \mathcal{H}_l^c + \mathcal{H}_l^o, \omega_l = \omega_l^c + \omega_l^o, \boldsymbol{\mathcal{O}}_1, \boldsymbol{\mathcal{O}}_2)$ with

$$\begin{aligned}
\pi_1 \boldsymbol{\mathcal{O}}_1 &= \pi_1(\boldsymbol{Q}_c - \boldsymbol{\eta}_c - \bar{\boldsymbol{\eta}}_c + \boldsymbol{B}) - \left(1 - \frac{1}{2}(X - \bar{X})\right) \pi_1^{(1,1)} \boldsymbol{B}\,, \\
\pi_1 \boldsymbol{\mathcal{O}}_2 &= \pi_1(\boldsymbol{Q}_o - \boldsymbol{\eta}_o + \boldsymbol{C})\,,
\end{aligned}$$

where

$$\boldsymbol{B} = \sum_{n=0}^{\infty} \sum_{\substack{p,\bar{p},r,\bar{r}=0 \\ p+r=\bar{p}+\bar{r}=n}}^{\infty} \boldsymbol{B}_{n+1}^{(p,\bar{p})} \,|^{(2r,2\bar{r})}\,, \qquad \boldsymbol{C} = \sum_{\substack{n,l,r=0 \\ n+l>0}}^{\infty} \boldsymbol{C}_{n,l}^{(2n+l-r-1)} \,|^{2r}\,,$$

have picture number deficit:

$$\begin{aligned}
\pi^{(0,0)}\boldsymbol{B} &: \quad (0,0) & \qquad \pi^{0}\boldsymbol{C} &: \quad 0 \\
\pi^{(1,0)}\boldsymbol{B} &: \quad (1,0) & \qquad \pi^{1}\boldsymbol{C} &: \quad 1 \\
\pi^{(0,1)}\boldsymbol{B} &: \quad (0,1) & & \\
\pi^{(1,1)}\boldsymbol{B} &: \quad (1,1) & &
\end{aligned}$$

OCHA relations are rewritten as

$$[\boldsymbol{\mathcal{O}}_1, \boldsymbol{\mathcal{O}}_1] \;=\; 0\,, \;\left\{\begin{array}{lcl} [\boldsymbol{Q}, \boldsymbol{B}] + \frac{1}{2}[\boldsymbol{B}, \boldsymbol{B}]^{11} & & \\ [\boldsymbol{\eta}, \boldsymbol{B}] - \frac{1}{2}[\boldsymbol{B}, \boldsymbol{B}]^{21} - \frac{1}{2}[\boldsymbol{B}, \boldsymbol{B}]^{11}_{\bar{X}^c} & = & 0\,, \\ [\bar{\boldsymbol{\eta}}, \boldsymbol{B}] - \frac{1}{2}[\boldsymbol{B}, \boldsymbol{B}]^{12} - \frac{1}{2}[\boldsymbol{B}, \boldsymbol{B}]^{22}_{X^c} & = & 0\,, \end{array}\right. \;,$$

$$[\boldsymbol{\mathcal{O}}_1, \boldsymbol{\mathcal{O}}_2] + \frac{1}{2}[\boldsymbol{\mathcal{O}}_2, \boldsymbol{\mathcal{O}}_2] \;=\; 0\,, \;\left\{\begin{array}{lcl} [\boldsymbol{Q}, \boldsymbol{C}] + [\boldsymbol{B}, \boldsymbol{C}]^{11} + \frac{1}{2}[\boldsymbol{C}, \boldsymbol{C}]^{1} & = & 0\,, \\ [\boldsymbol{\eta}, \boldsymbol{C}] - \frac{1}{2}[\boldsymbol{C}, \boldsymbol{C}]^{2} - [\boldsymbol{B}, \boldsymbol{C}]^{21+12} & & \\ \qquad\qquad - [\boldsymbol{B}, \boldsymbol{C}]^{22}_{X^c + \bar{X}^c} & = & 0\,, \end{array}\right.$$

where commutators with superscript ij or k (and also subscript X) are defined by inserting the projection operator $\pi_c^{(i-1,j-1)}$ or π_o^{k-1} (and the operator X) in the middle. The coderivations $(\boldsymbol{Q}, \boldsymbol{\eta})$ act as $(\boldsymbol{Q}_c, \boldsymbol{\eta}_c + \bar{\boldsymbol{\eta}}_c)$ on $\mathcal{H}_c$ and $(\boldsymbol{Q}_o, \boldsymbol{\eta}_o)$ on $\mathcal{H}_o$.

The former is the L_∞-relation of $\boldsymbol{\mathcal{O}}_1$, which already appeared when we constructed the type II superstring field theory.

Step 2

If we find $\boldsymbol{B}$ and $\boldsymbol{C}$ then transform them as

$$\hat{\boldsymbol{F}}^{-1}\Big(\boldsymbol{Q}+\pi_c^{(0,0)}\boldsymbol{B}+\pi_o^0\boldsymbol{C}\Big)\hat{\boldsymbol{F}} \;=\; \boldsymbol{Q}+\mathcal{G}^c\boldsymbol{B}\hat{\boldsymbol{F}}+\mathcal{G}^o\boldsymbol{C}\hat{\boldsymbol{F}} \;\equiv\; \boldsymbol{L}+\boldsymbol{N}\,,$$

$$\hat{\boldsymbol{F}}^{-1}\bigg(\boldsymbol{\eta}-\Big((\pi_c^{(1,0)}+\pi_c^{(0,1)})\boldsymbol{B}+\pi_c^{(1,1)}(X^c+\bar{X}^c)\boldsymbol{B}\Big)-\pi_o^1\boldsymbol{C}\bigg)\hat{\boldsymbol{F}} \;=\; \boldsymbol{\eta}\,.$$

by the cohomomorphism

$$\pi_1\hat{\boldsymbol{F}}^{-1} \;=\; \pi_1\mathbb{I}-\Big(\Xi^c\pi_1^{(1,0)}+\bar{\Xi}^c\pi_1^{(0,1)}+\frac{1}{2}(\bar{\Xi}^cX^c+\Xi^c\bar{X}^c)\pi_1^{(1,1)}\Big)\boldsymbol{B}-\Xi^o\pi_1^1\boldsymbol{C}\,.$$

Then, $\boldsymbol{L}$ and $\boldsymbol{N}$ have correct picture numbers and satisfy

$$[\boldsymbol{L}+\boldsymbol{N},\boldsymbol{L}+\boldsymbol{N}] \;=\; 0\,,\qquad \mathcal{P}^c_{XY}\boldsymbol{L} \;=\; \boldsymbol{L}\,,\qquad \mathcal{P}^o_{XY}\boldsymbol{N} \;=\; \boldsymbol{N}\,.$$

Recall that $\mathcal{P}_{XY}\mathcal{G}=\mathcal{G}$ and

$$\mathcal{G}^o \;=\; \pi_o^0+X^o\pi_o^1,\qquad \mathcal{G}^c \;=\; \pi_c^{(0,0)}+X^c\pi_c^{(1,0)}+\bar{X}^c\pi_c^{(0,1)}+X^c\bar{X}^c\pi_c^{(1,1)}\,.$$

Construction of $(\mathbf{B},\boldsymbol{C})$

The above construction of $\boldsymbol{B}$ is that in the symmetric construction proposed previously. We consider generating functions

$$\begin{aligned}
\boldsymbol{B}(s,\bar{s},t) &= \sum_{\substack{n,p,\bar{p},r,\bar{r}=0\\ n\geq p+r,\, n\geq \bar{p}+\bar{r}}}^{\infty} s^{n-p-r}\bar{s}^{n-\bar{p}-\bar{r}}t^{p+\bar{p}}\boldsymbol{B}_{n+1}^{(p,\bar{p})}\mid^{(2r,2\bar{r})} \equiv \sum_{p,\bar{p}=0}^{\infty} t^{p+\bar{p}}\boldsymbol{B}^{(p,\bar{p})}(s,\bar{s})\,,\\
\boldsymbol{C}(s,t) &= \sum_{\substack{n,l,p,r=0\\ 2n+l\geq p+r+1}} s^{2n+l-p-r-1}t^{p}\boldsymbol{C}_{n,l}^{(p)}\mid^{2r} \equiv \sum_{p=0}^{\infty} t^{p}\boldsymbol{C}^{(p)}(s)\,,
\end{aligned}$$

and extend the OCHA relations as follows.

$$\boldsymbol{I}_B(s,\bar{s},t) \equiv [\boldsymbol{Q},\boldsymbol{B}(s,\bar{s},t)] + \frac{1}{2}[\boldsymbol{B}(s,\bar{s},t),\boldsymbol{B}(s,\bar{s},t)]_{\mathfrak{c}_1(s,\bar{s},t)} = 0\,,$$

$$\boldsymbol{J}_B(s,\bar{s},t) \equiv [\boldsymbol{\eta},\boldsymbol{B}(s,\bar{s},t)] - \frac{1}{2}[\boldsymbol{B}(s,\bar{s},t),\boldsymbol{B}(s,\bar{s},t)]_{\mathfrak{c}_2(t)} = 0\,,$$

$$\bar{\boldsymbol{J}}_B(s,\bar{s},t) \equiv [\bar{\boldsymbol{\eta}},\boldsymbol{B}(s,\bar{s},t)] - \frac{1}{2}[\boldsymbol{B}(s,\bar{s},t),\boldsymbol{B}(s,\bar{s},t)]_{\mathfrak{c}_2(t)} = 0\,,$$

$$\boldsymbol{I}_C(s,t) \equiv [\boldsymbol{Q},\boldsymbol{C}(s,t)] + \frac{1}{2}[\boldsymbol{C}(s,t),\boldsymbol{C}(s,t)]_{\mathfrak{o}_1(s)} + [\boldsymbol{B}(s,s,t),\boldsymbol{C}(s,t)]_{\mathfrak{c}_1(s,s,t)} = 0\,,$$

$$\boldsymbol{J}_C(s,t) \equiv [\boldsymbol{\eta},\boldsymbol{C}(s,t)] - \frac{1}{2}[\boldsymbol{C}(s,t),\boldsymbol{C}(s,t)]_{\mathfrak{o}_2(t)} - [\boldsymbol{B}(s,s,t),\boldsymbol{C}(s,t)]_{\mathfrak{c}_2(s)+\bar{\mathfrak{c}}_2(s)} = 0\,,$$

with

$$\mathfrak{c}_1(s,\bar{s},t) \;=\; \pi^{(0,0)} + s\pi^{(1,0)} + \bar{s}\pi^{(0,1)} + (s\bar{s} + t(s\bar{X} + \bar{s}X))\pi^{(1,1)}\,,$$

$$\mathfrak{c}_2(t) \;=\; t\pi^{(1,0)} + \frac{t^2}{2}\bar{X}\pi^{(1,1)}\,, \qquad \bar{\mathfrak{c}}_2(t) \;=\; t\pi^{(0,1)} + \frac{t^2}{2}X\pi^{(1,1)}\,,$$

$$\mathfrak{o}_1(s) \;=\; \pi^0 + s\pi^1\,, \qquad \mathfrak{o}_2(t) \;=\; t\pi^1\,.$$

These interpolate the superstring's OCHA relations at $(s,\bar{s},t) = (0,0,1)$ and the bosonic-string's OCHA relations at $(s,\bar{s},t) = (1,1,0)$.

We can show that if $\boldsymbol{B}(s,\bar{s},t)$ satisfies the differential equations

$$\begin{aligned}
\partial_t \boldsymbol{B}(s,\bar{s},t) &= [\boldsymbol{Q},(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}})(s,\bar{s},t)] + [\boldsymbol{B}(s,\bar{s},t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}})(s,\bar{s},t)]_{\mathfrak{c}_1(s,\bar{s},t)} \\
&\quad + \frac{1}{2}[\boldsymbol{B}(s,\bar{s},t),\boldsymbol{B}(s,\bar{s},t)]_{\mathfrak{d}(s,\bar{s})}\;, \\
\partial_s \boldsymbol{B}(s,\bar{s},t) &= [\boldsymbol{\eta},\boldsymbol{\lambda}(s,\bar{s},t)] - [\boldsymbol{B}(s,\bar{s},t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}})(s,\bar{s},t)]_{\mathfrak{c}_2(t)} \\
\partial_{\bar{s}} \boldsymbol{B}(s,\bar{s},t) &= [\bar{\boldsymbol{\eta}},\bar{\boldsymbol{\lambda}}(s,\bar{s},t)] - [\boldsymbol{B}(s,\bar{s},t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}})(s,\bar{s},t)]_{\bar{\mathfrak{c}}_2(t)}\;,
\end{aligned}$$

with $\mathfrak{d}(s,\bar{s}) = (s\bar{\Xi}+\bar{s}\Xi)\pi^{(1,1)}$ by introducing the degree odd coderivations (gauge products)

$$\begin{aligned}
\boldsymbol{\lambda}(s,\bar{s},t) &= \sum_{n=0}^{\infty} \sum_{\substack{m,\bar{m},p,\bar{p},r,\bar{r}=0 \\ m+p+r+1=\bar{m}+\bar{p}+\bar{r}=n}}^{\infty} s^m \bar{s}^{\bar{m}} t^{p+\bar{p}} \boldsymbol{\lambda}_{n+1}^{(p+1,\bar{p})}\,\big|^{(2r,2\bar{r})} \equiv \sum_{p,\bar{p}=0}^{\infty} t^{p+\bar{p}} \boldsymbol{\lambda}^{(p+1,\bar{p})}(s,\bar{s})\;, \\
\bar{\boldsymbol{\lambda}}(s,\bar{s},t) &= \sum_{n=0}^{\infty} \sum_{\substack{m,\bar{m},p,\bar{p},r,\bar{r}=0 \\ m+p+r=\bar{m}+\bar{p}+\bar{r}+1=n}}^{\infty} s^m \bar{s}^{\bar{m}} t^{p+\bar{p}} \bar{\boldsymbol{\lambda}}_{n+1}^{(p,\bar{p}+1)}\,\big|^{(2r,2\bar{r})} \equiv \sum_{p,\bar{p}=0}^{\infty} t^{p+\bar{p}} \bar{\boldsymbol{\lambda}}^{(p,\bar{p}+1)}(s,\bar{s})\;,
\end{aligned}$$

and $\boldsymbol{C}(s,t)$ satisfies the differential equations

$$\begin{aligned}
\partial_t \boldsymbol{C}(s,t) &= [\boldsymbol{Q},\boldsymbol{\nu}(s,t)] + [\boldsymbol{C}(s,t),\boldsymbol{\nu}(s,t)]_{\mathfrak{o}_1(s)} \\
&\quad + [\boldsymbol{B}(s,s,t),\boldsymbol{\nu}(s,t)]_{\mathfrak{c}_1(s,s,t)} + [\boldsymbol{C}(s,t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}})(s,s,t)]_{\mathfrak{c}_1(s,s,t)} \\
&\quad + [\boldsymbol{B}(s,s,t),\boldsymbol{C}(s,t)]_{\mathfrak{d}(s,s)} \,, \\
\partial_s \boldsymbol{C}(s,t) &= [\boldsymbol{\eta},\boldsymbol{\nu}(s,t)] - [\boldsymbol{C}(s,t),\boldsymbol{\nu}(s,t)]_{\mathfrak{o}_2(t)} \\
&\quad - [\boldsymbol{B}(s,s,t),\boldsymbol{\nu}(s,t)]_{\mathfrak{c}_2(t)+\bar{\mathfrak{c}}_2(t)} - [\boldsymbol{C}(s,t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}})(s,s,t)]_{\mathfrak{c}_2(t)+\bar{\mathfrak{c}}_2(t)} \,,
\end{aligned}$$

with the gauge product

$$\boldsymbol{\nu}(s,t) = \sum_{\substack{n,l,p,r=0\\ 2n+l\geq p+r+2}} s^{2n+l-p-r-2} t^p \boldsymbol{\nu}^{(p+1)}_{n,l}\,\Big|^{2r} \equiv \sum_{p=0}^{\infty} t^p \boldsymbol{\nu}^{(p+1)}(s)\,,$$

then

$$\partial_t \boldsymbol{I}_B(s,\bar{s},t)\,,\quad \partial_t \boldsymbol{J}_B(s,\bar{s},t)\,,\quad \partial_t \bar{\boldsymbol{J}}_B(s,\bar{s},t)\,,\quad \partial_t \boldsymbol{I}_C(s,t)\,,\quad \text{and}\quad \partial_t \boldsymbol{J}_C(s,t)$$

are proportional to $\partial_s \boldsymbol{I}_B(s,\bar{s},t)$, $\partial_{\bar{s}} \boldsymbol{I}_B(s,\bar{s},t)$, $\partial_s \boldsymbol{I}_C(s,t)$,

$$\boldsymbol{I}_B(s,\bar{s},t)\,,\quad \boldsymbol{J}_B(s,\bar{s},t)\,,\quad \bar{\boldsymbol{J}}_B(s,\bar{s},t)\,,\quad \boldsymbol{I}_C(s,t)\,,\quad \text{or}\quad \boldsymbol{J}_C(s,t)\ :$$

$$
\begin{aligned}
\partial_t \boldsymbol{I}_B(s,\bar{s},t) &= [\boldsymbol{I}_B(s,\bar{s},t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}}(s,\bar{s},t))]_{\mathfrak{c}_1(s,\bar{s},t)} + [\boldsymbol{I}_B(s,\bar{s},t),\boldsymbol{B}(s,\bar{s},t)]_{\mathfrak{d}(s,\bar{s})}\,,\\
\partial_t \boldsymbol{J}_B(s,\bar{s},t) &= [\boldsymbol{J}_B(s,\bar{s},t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}}(s,\bar{s},t))]_{\mathfrak{c}_1(s,\bar{s},t)} + [\boldsymbol{J}_B(s,\bar{s},t),\boldsymbol{B}(s,\bar{s},t)]_{\mathfrak{d}(s,\bar{s})}\\
&\quad - \partial_s \boldsymbol{I}_B(s,\bar{s},t) - [\boldsymbol{I}_B(s,\bar{s},t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}}(s,\bar{s},t))]_{\mathfrak{c}_2(t)}\,,\\
\partial_t \bar{\boldsymbol{J}}_B(s,\bar{s},t) &= [\bar{\boldsymbol{J}}_B(s,\bar{s},t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}}(s,\bar{s},t))]_{\mathfrak{c}_1(s,\bar{s},t)} + [\bar{\boldsymbol{J}}_B(s,\bar{s},t),\boldsymbol{B}(s,\bar{s},t)]_{\mathfrak{d}(s,\bar{s})}\\
&\quad - \partial_{\bar{s}} \boldsymbol{I}_B(s,\bar{s},t) - [\boldsymbol{I}_B(s,\bar{s},t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}}(s,\bar{s},t))]_{\bar{\mathfrak{c}}_2(t)}\,,\\
\partial_t \boldsymbol{I}_C(s,t) &= [\boldsymbol{I}_C(s,t),\boldsymbol{\nu}(s,t)]_{\mathfrak{o}_1(s)} + [\boldsymbol{I}_C(s,t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}}(s,\bar{s},t))]_{\mathfrak{c}_1(s,s,t)}\\
&\quad + [\boldsymbol{I}_C(s,t),\boldsymbol{B}(s,s,t)]_{\mathfrak{d}(s,s)}\,,\\
\partial_t \boldsymbol{J}_C(s,t) &= [\boldsymbol{J}_C(s,t),\boldsymbol{\nu}(s,t)]_{\mathfrak{o}_1(s)} + [\boldsymbol{J}_C(s,t),(\boldsymbol{\lambda}+\bar{\boldsymbol{\lambda}}(s,\bar{s},t))]_{\mathfrak{c}_1(s,s,t)}\\
&\quad + [\boldsymbol{J}_C(s,t),\boldsymbol{B}(s,s,t)]_{\mathfrak{d}(s,s)} - \partial_s \boldsymbol{I}_C(s,t) - [\boldsymbol{I}_C(s,t),\boldsymbol{\nu}(s,t)]_{\mathfrak{o}_2(t)}\,.
\end{aligned}
$$

These imply if

$$\boldsymbol{I}_B(s,\bar{s},0)=\boldsymbol{J}_B(s,\bar{s},0)=\bar{\boldsymbol{J}}_B(s,\bar{s},0)=\boldsymbol{I}_C(s,0)=\boldsymbol{J}_C(s,0)=0\,,$$

then for ${}^{\forall}t \geq 0$,

$$\boldsymbol{I}_B(s,\bar{s},t)=\boldsymbol{J}_B(s,\bar{s},t)=\bar{\boldsymbol{J}}_B(s,\bar{s},t)=\boldsymbol{I}_C(s,t)=\boldsymbol{J}_C(s,t)=0\,.$$

On the other hand, the equations at $t = 0$,

$$\boldsymbol{I}_B(s, \bar{s}, 0) = \boldsymbol{J}_B(s, \bar{s}, 0) = \bar{\boldsymbol{J}}_B(s, \bar{s}, 0) = \boldsymbol{I}_C(s, 0) = \boldsymbol{J}_C(s, 0) = 0\,,$$

are satisfied if we set

$$(*) \qquad \boldsymbol{B}(1, 1, 0) = \boldsymbol{L}_B\,, \qquad \boldsymbol{C}(1, 0) = \boldsymbol{N}_B\,,$$

which are known by the assumption.

Therefore, we can obtain $\boldsymbol{B}(s, \bar{s}, t)$ and $\boldsymbol{C}(s, t)$ by solving the differential equations with the initial conditions $(*)$.

To solve the equations explicitly, we first suppose that those for the L_∞ algebra are already solved and $\boldsymbol{B}$, $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$ are given.

Then, rewrite the equations for $\boldsymbol{C}$ as

$$\begin{aligned}
\partial_s \boldsymbol{C}^{(p)}(s) \ &= [\boldsymbol{\eta}, \boldsymbol{\nu}^{(p+1)}(s)] - \sum_{p'=0}^{p-1} [\boldsymbol{C}^{(p')}(s), \boldsymbol{\nu}^{(p-p')}(s)]^2 \\
&\quad - \sum_{p',\bar{p}'=0}^{p'+\bar{p}'\leq p-1} [\boldsymbol{B}^{(p',\bar{p}')}(s,s), \boldsymbol{\nu}^{(p-p'-\bar{p}')}(s)]^{21+12} \\
&\quad - \sum_{p',\bar{p}'=0}^{p'+\bar{p}'\leq p-2} [\boldsymbol{B}^{(p',\bar{p}')}(s,s), \boldsymbol{\nu}^{(p-p'-\bar{p}'-1)}(s)]^{22}_{\frac{1}{2}(X^c+\bar{X}^c)} \\
&\quad - \sum_{p',\bar{p}'=0}^{p'+\bar{p}'\leq p-1} [\boldsymbol{C}^{(p-p'-\bar{p}'-1)}(s), (\boldsymbol{\lambda}^{(p'+1,\bar{p}')} + \bar{\boldsymbol{\lambda}}^{(p',\bar{p}'+1)})(s,s)]^{21+12} \\
&\quad - \sum_{p',\bar{p}'=0}^{p'+\bar{p}'\leq p-2} [\boldsymbol{C}^{(p-p'-\bar{p}'-2)}(s), (\boldsymbol{\lambda}^{(p'+1,\bar{p}')} + \bar{\boldsymbol{\lambda}}^{(p',\bar{p}'+1)})(s,s)]^{22}_{\frac{1}{2}(X^c+\bar{X}^c)},
\end{aligned}$$

and

$$
\begin{aligned}
(p+1)\boldsymbol{C}^{(p+1)}(s) \;&= [\boldsymbol{Q},\boldsymbol{\nu}^{(p+1)}(s)] + \sum_{p'=0}^{p}[\boldsymbol{C}^{(p')}(s),\boldsymbol{\nu}^{(p-p'+1)}(s)]_{\mathfrak{o}_1(s)} \\
&+ \sum_{p'\bar{p}'=0}^{p'+\bar{p}'\le p}[\boldsymbol{B}^{(p',\bar{p}')}(s,s),\boldsymbol{\nu}^{(p-p'-\bar{p}'+1)}(s)]_{\mathfrak{c}_1^0(s,s)} \\
&+ \sum_{p'\bar{p}'=0}^{p'+\bar{p}'\le p-1}[\boldsymbol{B}^{(p',\bar{p}')}(s,s),\boldsymbol{\nu}^{(p-p'-\bar{p}'+1)}(s)]_{\mathfrak{c}_1^1(s,s)} \\
&+ \sum_{p'\bar{p}'=0}^{p'+\bar{p}'\le p}[\boldsymbol{C}^{(p-p'-\bar{p}')}(s),(\boldsymbol{\lambda}^{(p'+1,\bar{p}')}+\bar{\boldsymbol{\lambda}}^{(p',\bar{p}'+1)})(s,s)]_{\mathfrak{c}_1^0(s,s)} \\
&+ \sum_{p'\bar{p}'=0}^{p'+\bar{p}'\le p-1}[\boldsymbol{C}^{(p-p'-\bar{p}'-1)}(s),(\boldsymbol{\lambda}^{(p'+1,\bar{p}')}+\bar{\boldsymbol{\lambda}}^{(p',\bar{p}'+1)})(s,s)]_{\mathfrak{c}_1^1(s,s)} \\
&+ \sum_{p'\bar{p}'=0}^{p'+\bar{p}'\le p}[\boldsymbol{B}^{(p',\bar{p}')}(s,s),\boldsymbol{C}^{(p-p'-\bar{p})}(s)]_{\mathfrak{d}(s,s)}\,,
\end{aligned}
$$

where $\mathfrak{c}_1(s,\bar{s},t) \;\equiv\; \mathfrak{c}_1^0(s,\bar{s}) + t\mathfrak{c}_1^1(s,\bar{s})$.

$\big(\mathfrak{c}_1^0(s,s) = \pi^{(0,0)} + s(\pi^{(1,0)}+\pi^{(0,1)}) + s^2\pi^{(1,1)}$, $\mathfrak{c}_1^1(s,s) = s(X^c+\bar{X}^c)\pi^{(1,1)}\big)$

From the first equation at $p = 0$, we have

$$\boldsymbol{\nu}^{(1)}(s) \;=\; \xi_0^o \circ \partial_s \boldsymbol{C}^{(0)}(s)\,,$$

and then obtain by substituting it in the right hand side of the second equation at $p = 0$ with the initial conditions $(*)$:

$$\begin{aligned}\boldsymbol{C}^{(1)}(s) \;&= [\boldsymbol{Q}, \boldsymbol{\nu}^{(1)}(s)] + [\boldsymbol{N}_B(s), \boldsymbol{\nu}^{(1)}(s)]_{\mathfrak{o}_1(s)} \\ &\quad + [\boldsymbol{L}_B(s,s), \boldsymbol{\nu}^{(1)}(s)]_{\mathfrak{c}_1^0(s,s)} + [\boldsymbol{N}_B(s), (\boldsymbol{\lambda}^{(1,0)} + \bar{\boldsymbol{\lambda}}^{(0,1)})(s,s)]_{\mathfrak{c}_1^0(s,s)} \\ &\quad + [\boldsymbol{L}_B(s,s), \boldsymbol{N}_B(s)]_{\mathfrak{d}(s,s)}\,.\end{aligned}$$

Repeating the procedure, we next obtaine, $\boldsymbol{\nu}^{(2)}(s)$ then $\boldsymbol{C}^{(2)}(s)$ using the equations at $p = 1$, $\boldsymbol{\nu}^{(3)}(s)$ then $\boldsymbol{C}^{(3)}(s)$ using the equations at $p = 2$, and so on so forth.

Explicitly, $\boldsymbol{C}^{(p)}(s)$ can be expanded as $\sum_{n,l} \boldsymbol{C}_{n,l}^{(p)}(s)$ and

$$
\begin{aligned}
\boldsymbol{C}_{1,0}^{(0)}(s) &= \boldsymbol{C}_{1,0}^{(0)} \,|^{2} + s\, \boldsymbol{C}_{1,0}^{(0)} \,|^{0} \,, \\
\boldsymbol{C}_{1,0}^{(1)}(s) &= \boldsymbol{C}_{1,0}^{(1)} \,|^{0} \,, \\
\boldsymbol{C}_{1,1}^{(0)}(s) &= \boldsymbol{C}_{1,1}^{(0)} \,|^{4} + s\, \boldsymbol{C}_{1,1}^{(0)} \,|^{2} + s^{2}\, \boldsymbol{C}_{1,1}^{(0)} \,|^{0} \,, \\
\boldsymbol{C}_{1,1}^{(1)}(s) &= \boldsymbol{C}_{1,1}^{(1)} \,|^{2} + s\, \boldsymbol{C}_{1,1}^{(1)} \,|^{0} \,, \\
\boldsymbol{C}_{1,1}^{(2)}(s) &= \boldsymbol{C}_{1,1}^{(2)} \,|^{0} \,, \\
& \cdots
\end{aligned}
$$

The first components $\boldsymbol{C}_{n,l}^{(p)}(0)$ are those we want to know.

Using $\boldsymbol{N}$ we can define the action with OCHA structure

$$I \;=\; \int_0^1 dt\Omega^o\left(\Psi, \pi_1 \boldsymbol{N}\left(e^{\wedge \Phi_{b.g.}} \otimes \frac{1}{1-t\Psi}\right)\right) ,$$

invariant under the gauge transformation

$$\delta\Psi \;=\; \pi_1 \boldsymbol{N}\left(e^{\wedge \Phi_{b.g.}} \otimes \left(\frac{1}{1-t\Psi} \otimes \Lambda \otimes \frac{1}{1-t\Psi}\right)\right) ,$$

where $\Phi_{b.g.}$ is a background closed-superstring field satisfying the nonlinear equations of motion

$$\boldsymbol{L}(e^{\wedge \Phi_{b.g.}}) \;=\; 0\,.$$

The coderivation $\boldsymbol{N}\left(e^{\wedge \Phi_{b.g.}} \otimes \;\cdot\;\right)$ as that acting on $\mathcal{TH}_o^{res}$ defines a weak A_∞-algebra (deformed by $\Phi_{b.g.}$).

Due to the tadpole term $\Omega^o(\Psi, N_{1,0}(\Phi_{b.g.}))$, $\Psi = 0$ is not a solution of Ψ eom and we must shift Ψ for perturbative calculation. cf.[Maccaferri-Vošmera]

Outlook

♠ Quantization

gauge fixing, loop homotopy algebra, quantum BV master action, spurious pole problem, $\cdots$

♠ Concrete analysis using superstring field theory

non-perturbative: classical solutions, dualities, AdS/CFT, D-instanton, $\cdots$
perturbative: off-shell amplitudes, infrared divergence, $\cdots$

We have still many remaining issues to study!