THESIS

Theoretical study on interacting fermionic topological phases

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Abstract

In this thesis, we study interacting fermionic symmetry protected topological (SPT) phases from field theoretical perspective. We discuss a recipe to produce a lattice construction of fermionic phases of matter with or without the time-reversal symmetry, by extending the fermionization and bosonization known in (1+1) dimensions to higher spacetime dimensions in the presence of time-reversal symmetry. As an application, we provide a lattice path integral for a large class of fermionic SPT phases called the Gu-Wen phases, and discuss the symmetry-preserving gapped boundary states of Gu-Wen phases by utilizing the path integral. In addition, we also provide a lattice path integral of (1+1)-dimensional topological superconductors with the time-reversal symmetry generating the $\mathbb{Z}_8$ classification of the SPT phase. Based on the field theoretical description of the (1+1)-dimensional topological superconductor, we study how one can diagnose the SPT classification given by $\mathbb{Z}_8$ for a given wave function of a (1+1)-dimensional system. Finally, we also discuss a lattice description for (3+1)-dimensional topological superconductor with the time-reversal symmetry classified by $\mathbb{Z}_{16}$.

Citations to published works

Chapter 4, 5 are primarily based on the following works by the author partly done with Kantaro Ohmori and Yuji Tachikawa:

- Ryohei Kobayashi, Kantaro Ohmori, Yuji Tachikawa, *"On gapped boundaries for SPT phases beyond group cohomology" Journal of High Energy Physics* **11** (2019) 131 (2019).

- Ryohei Kobayashi, *"Pin TQFT and Grassmann integral" Journal of High Energy Physics* **12** (2019) 014 (2019).

Chapter 6 is primarily based on the following work by the author:

- Ryohei Kobayashi, *"Pin TQFT and Grassmann integral" Journal of High Energy Physics* **12** (2019) 014 (2019),

and also explains the joint work with Kansei Inamura and Shinsei Ryu:

- Kansei Inamura, Ryohei Kobayashi, Shinsei Ryu, *"Non-local Order Parameters and Quantum Entanglement for Fermionic Topological Field Theories" Journal of High Energy Physics* **01** (2020) 121 (2020).

Chapter 8 is primarily based on the following work by the author:

- Ryohei Kobayashi, *"Commuting projector models for (3+1)d topological superconductors via string net of (1+1)d topological superconductors" Physical Review B* **102** 075135 (2020).

Besides the above works, I worked on the following papers during the PhD and master course:

1. Ryohei Kobayashi, *"Lattice construction of exotic invertible topological phases"* arXiv: 2106.10703.

2. Srivatsa Tata, Ryohei Kobayashi, Daniel Bulmash, Maissam Barkeshli, *"Anomalies in (2+1)D fermionic topological phases and (3+1)D path integral state sums for fermionic SPTs"* arXiv: 2104.14567.

3. Ryohei Kobayashi, Yasunori Lee, Ken Shiozaki, Yuya Tanizaki, *"Topological terms of (2+1)d flag-manifold sigma models"* arXiv: 2103.05035.

4. Ryohei Kobayashi, *"Anomaly constraint on chiral central charge of (2+1)d topological order" Physical Review Research* **3** 023107 (2021).

5. Ryohei Kobayashi, Ken Shiozaki, *"Anomaly indicator of rotation symmetry in (3+1)D topological order"* arXiv: 1901.06195.

6. Ryohei Kobayashi, Ken Shiozaki, Yuta Kikuchi, Shinsei Ryu, *"Lieb-Schultz-Mattis type theorem with higher-form symmetry and the quantum dimer models" Physical Review B* **99** 014402 (2019), *Editors' Suggestion.*

7. Ryohei Kobayashi, Yuya O. Nakagawa, Yoshiki Fukusumi, Masaki Oshikawa, *"Scaling of polarization amplitude in quantum many-body systems in one dimension" Physical Review B* **97** 165133 (2018).

8. Kohei Kawabata, Ryohei Kobayashi, Ning Wu, Hosho Katsura, *"Exact zero modes in twisted Kitaev chains" Physical Review B* **95** 195140 (2017).

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Contents

Chapter 1

Introduction

Many important properties of quantum many-body systems with gapped local Hamiltonians are encoded in the entanglement property of its ground state, independent on the local details of the Hamiltonian [1]. In particular, we can determine the gapped phase realized by such systems from the ground state of the Hamiltonian, which is defined as an equivalence class of Hamiltonians under local deformations preserving the energy gap.

The classification of gapped phases is an important problem in condensed matter physics. Though the classification problem is difficult and unsolved in general, one can simplify the problem by considering a simplified class of systems with a unique gapped ground state on arbitrary closed spatial manifolds. Such phases are called invertible topological phases. A classical and important example of an invertible topological phase is the Integer Quantum Hall Effect (IQHE) discovered by von Klitzing in 1980, in silicon-based MOSFET (metal-oxide-semiconductor field-effect transistor) samples [2]. The IQHE is observed for a (2+1)-dimensional electron system subjected to strong magnetic fields at low temperature, and is characterized by the Hall conductance exactly quantized as an integral value. The quantization of the Hall conductance is explained by Laughlin in 1981 based on an argument that utilizes the invariance under $U(1)$ large gauge transformations [3]. The Quantum Hall states are later found to be effectively described by the Chern-Simons theory proportional to $\int AdA$ in terms of the $U(1)$ gauge field. From the perspective of effective field theory, the quantization of the Hall conductance is accounted by the quantized coefficient of the Chern-Simons theory due to its large gauge invariance [4].

When the system possesses a global symmetry G, one can consider a gapped phase with the specified global symmetry, based on the deformations respecting the symmetry. The global symmetry typically enriches the phase diagram, since the global symmetry in general constrains the possible deformations on the ground state. As a result, a single phase can be refined into distinct phases by taking the global symmetry into account. Invertible topological phases with the global symmetry are also called symmetry protected topological (SPT) phases. [1] One of the most well-known examples of SPT phases is the topological insulator

[1] Precisely speaking, it is more common to define SPT phases as invertible phases which become trivial when we forget the global symmetry, following [1]. For example, the (1+1)-dimensional topological superconductor (Kitaev wire) is not counted as an SPT phase based on such a definition, though it is an invertible topological phase. In this thesis, we do not make a careful distinction between the two concepts, namely invertible phases and SPT phases.

with a gapless edge state on its boundary, initially observed in (2+1)-dimensional systems of PbTe/SnTe and HgTe/CdTe heterostructures [5, 6]. Later, Kane and Mele proposed a set of (2+1)-dimensional Hamiltonian models of free fermions for the topological insulator with the time-reversal symmetry, and constructed a $\mathbb{Z}_2$-valued quantized invariant computed from the ground state that diagnoses the topological insulator [7]. They discovered that the $\mathbb{Z}_2$ classification of the topological insulator arises due to the protection of the nontrivial phase by the time-reversal symmetry.

In the case of free fermions including the model for topological insulator proposed by Kane and Mele, a complete classification of SPT phases has been obtained using K-theory [8, 9]. Later, it was recognized that the classification of SPT phases drastically changes in the presence of interactions compared with free phases [1, 10–17].

A famous example for the classification of interacting SPT phases that differ from free phases is the (1+1)-dimensional topological superconductor with the time-reversal symmetry that squares to unit, $T^2 = 1$, which is called the Kitaev wire [10, 18]. The (1+1)-dimensional topological superconductor is realized by a ferromagnetic iron (Fe) atomic wires located on the surface of superconducting lead (Pb), by introducing the Cooper interaction to the iron wire through the perturbative effect (called the proximity effect) from the superconducting bulk [19]. Though the electrons of superconducting wire is under the two-body Cooper interaction, one can also regard the wire as a system of free Majorana fermions. The Kitaev wire is then characterized by an unpaired Majorana zero mode on its boundary, which gives a nontrivial edge state. In the presence of the time-reversal symmetry, any system given by integral copies of the Kitaev wires gives a nontrivial SPT phase within the free fermion framework, since the two-body hopping term of Majorana zero modes that kills the edge state is prohibited due to the time-reversal symmetry. Hence, the free fermion classification of the (1+1)-dimensional topological superconductor is given by $\mathbb{Z}$. Meanwhile, Fidkowski and Kitaev showed in [10] that the Majorana zero modes for eight copies of the Kitaev wires can be completely eliminated by turning on the four-body interactions of Majorana fermions on the boundary, while respecting the time-reversal symmetry. This in particular shows that the classification of the (1+1)d topological superconductor with the time-reversal symmetry breaks down into $\mathbb{Z}_8$ in the presence of interactions.

While the SPT phases were initially discovered and studied for fermionic systems, it was later recognized that bosonic systems can also host nontrivial topological phases in the presence of the interactions [1], and we have to use completely different method to perform the classification. For example, a large class of interacting bosonic SPT phases with the global symmetry can be classified by utilizing group cohomology [1]. For the case of interacting fermionic systems, the SPT phases have a richer classification than the bosonic phases; for example, Gu and Wen found that a subclass of fermionic SPT phases is classified by a pair of cohomological data [11], generalizing the classification of bosonic phases. These fermionic SPT phases are called Gu-Wen phases or super-cohomology phases.

Later, a comprehensive classification scheme of SPT phases utilizing cobordism group is proposed in [12]. The cobordism group is thought to classify invertible field theories which effectively describe SPT phases. The cobordism theory predicts novel interacting phases beyond group cohomology and Gu-Wen phases. In particular, while the Gu-Wen phases cover a large class of fermionic SPT phases, there are phases that are outside of the Gu-Wen subclass, and such "beyond Gu-Wen phases" turn out to contain various important systems.

For example, the Kitaev wire with time-reversal symmetry classified by $\mathbb{Z}_8$ reviewed above is known to be an example of beyond Gu-Wen phases, where the only $\mathbb{Z}_4$ subclass corresponds to the Gu-Wen phases. Moreover, a (3+1)-dimensional topological superconductor with time-reversal symmetry such that $T^2 = (-1)^F$ is classified by $\mathbb{Z}_{16}$ [15, 20], and the only $\mathbb{Z}_4$ subclass corresponds to the Gu-Wen phases.

One important feature of the SPT phase is the presence of nontrivial edge state that appears on the boundary of the SPT phase, which includes a well-known gapless edge state of a (2+1)-dimensional topological insulator reviewed above [7]. The nontrivial spectrum of the edge state originates from an 't Hooft anomaly of the boundary theory of the SPT phase. An 't Hooft anomaly is an obstruction to coupling the global symmetry of the theory with the background gauge fields. As such, a boundary theory is not invariant under a gauge transformation of background gauge fields, but the variation is canceled by coupling with the bulk SPT phase. After all, the bulk-boundary system gives a fully gauge invariant theory. This cancellation of the 't Hooft anomaly by coupling with the SPT phase in the bulk is called anomaly inflow. Importantly, an 't Hooft anomaly has a dramatic consequence on the low-energy spectrum of the boundary theory. In particular, an 't Hooft anomaly implies that, in a gapped phase, the symmetry must either be spontaneously broken, or the theory is described by a symmetry-preserving topologically ordered states matching the anomaly.

The aim of this thesis is to develop field theoretical understanding of fermionic topological phases and their boundaries, covering both Gu-Wen and beyond Gu-Wen phases. We provide a path integral definition of Gu-Wen SPT phases and (1+1)-dimensional superconductor with time-reversal symmetry. Since these phases are fermionic, the theory intrinsically depends on the choice of spin or pin structure of a spacetime. We show how one can construct a topologically invariant lattice path integral coupled with spin/pin structure. This is done by generalizing the bosonization and fermionization in (1+1) dimensions to higher spacetime dimensions in the presence of time-reversal symmetry, which allows us to construct a path integral of fermionic phases starting with a given path integral for a bosonic theory.

This formulation has various physical applications. For instance, we provide a local lattice definition of symmetry-preserving topologically ordered states on boundary of Gu-Wen SPT phases protected by finite group symmetry which can contain time-reversal symmetry. This means that an 't Hooft anomaly that corresponds to the boundary of Gu-Wen phases can be carried by subtle topological degrees of freedom, not by gapless particles and in particular the system having the anomaly can have an energy gap.

In addition, based on the lattice path integral of (1+1)-dimensional topological superconductor with time-reversal symmetry, we study how one can diagnose the SPT classification given by $\mathbb{Z}_8$, for a given wave function of the SPT phase. We show that a non-local operation on the ground state called a "partial time-reversal" can be utilized to diagnose the $\mathbb{Z}_8$ classification of the given wave function, based on our field theoretical formulation of the topological superconductor.

Moreover, we also discuss the construction of beyond Gu-Wen phases in higher spacetime dimensions. For example, we consider a (3+1)-dimensional topological superconductor with time-reversal symmetry $T^2 = (-1)^F$ classified by $\mathbb{Z}_{16}$. We show that a wave function for the (3+1)-dimensional topological superconductor labeled by $2 \in \mathbb{Z}_{16}$ is constructed by decorating the Kitaev wire on the junction of T symmetry defects in codimension 2.

Structure of this thesis

Here we explain the structure of this thesis. In Chapter 2, we review the SPT phases and its classification scheme based on cobordism groups. We also review fermionic SPT phases and its dependence on spin structure, and 't Hooft anomalies that appears on the boundary of SPT phases. In Chapter 3, we review bosonization and fermionization on oriented spacetime manifolds in generic spacetime dimensions, which is utilized to construct fermionic SPT phases starting with a path integral of bosonic phases.

In Chapter 4, we provide a way to construct a path integral of fermionic phases with time-reversal symmetry, coupled with Pin^+ or Pin^- structure of the spacetime. This generalizes a fermionization in oriented spin case proposed in [21] to the unoriented Pin^+ or Pin^- case. this chapter is based on the author's work [22].

In Chapter 5, we provide a lattice path integral for Gu-Wen fermionic SPT phases with or without time-reversal symmetry. We also show that the boundary of a Gu-Wen phase with finite global symmetry admit a symmetry-preserving gapped topological state, by an explicit construction of path integral for the bulk-boundary system. this chapter is based on the author's work [22, 23].

In Chapter 6, we provide a lattice path integral for (1+1)-dimensional topological superconductor with time-reversal symmetry, which generates the $\mathbb{Z}_8$ classification. We show that the "partial time-reversal" on the ground state wave function of the topological superconductor works as the diagnostic of $\mathbb{Z}_8$ classification, based on our field theoretical formulation. this chapter is based on the author's work [22, 24].

In Chapter 7, we review the construction of (2+1)-dimensional beyond Gu-Wen SPT phase by decorating the Kitaev wire on the domain wall of the global symmetry. In Chapter 8, we construct a ground state wave function that describes a (3+1)-dimensional topological superconductor with time-reversal symmetry such that $T^2 = (-1)^F$. This is done by generalizing the decorated domain wall reviewed in the previous chapter to higher spatial dimensions, and the resulting SPT phase corresponds to $2 \in \mathbb{Z}_{16}$ in the $\mathbb{Z}_{16}$ classification. this chapter is based on the author's work [25].

Chapter 2

Review on SPT phases

2.1 Bosonic SPT phases and group cohomology

An important example for an interacting SPT phase is a bosonic phase based on the internal global symmetry G. In that case, a large class of d spacetime dimensional bosonic SPT phases is known to be classified by the group cohomology $H^d(BG, U(1))$ [1]. As reviewed in Appendix D, the element $[\omega] \in H^d(BG, U(1))$ is represented by a map $\omega : G^d \to \mathbb{R}/\mathbb{Z}$ with the property called the cocycle condition,

$$\omega(g_2, \dots, g_{d+1}) + (-1)^d \omega(g_1, \dots g_d) + \sum_{i=1}^{d-1} (-1)^i \omega(g_1, \dots, g_i g_{i+1}, \dots, g_{d+1}) = 0. \tag{2.1}$$

Then, one can use this function ω to define a gauge invariant action for the SPT phase. Suppose a d-dimensional closed spacetime manifold M is equipped with a triangulation, and we have a flat G-gauge field $A \in Z^1(M, G)$ on M. Then, on each d-simplex $\langle 01 \dots d \rangle$ of M, the Boltzmann weight for the action is given by

$$\Omega(01 \dots d) = \omega(A(01), A(12) \dots, A(d-1, d)). \tag{2.2}$$

Here, the flat gauge field A is known to be specified by a map $g : M \to BG$, where BG denotes the classifying space (see Appendix D for a review). One can also write $\Omega = g^* \omega$ using the pullback by g. Now the partition function is given by integrating the weight over the spacetime,

$$Z(M, g) = \exp\left(2\pi i \int_M g^* \omega\right). \tag{2.3}$$

This partition function is $U(1)$-valued, and shown to be invariant under gauge transformations. Concretely, the G gauge transformation shifts Ω by a coboundary $\Omega \to \Omega + \delta\chi$ for some $\chi \in C^{d-1}(M, U(1))$, which does not shift the partition function. The generic review chains and cochains is found in Appendix A.

2.2 Bordism and cobordism groups

Though the group cohomology [1] captures a large class of bosonic SPT phases, it still leaves out some topological phases which are called "beyond cohomology" phases [26]. A comprehensive classification scheme of interacting SPT phases including the beyond cohomology phases, utilizing bordism groups has been proposed by [12], supposing that the partition function of a $(d+1)$-dimensional SPT phase is a bordism invariant of the $(d+1)$-dimensional spacetime manifold. Then, the understanding of interacting SPT phases has been developed based on effective quantum field theory (QFT), whose classification is directly understood in terms of bordism groups. That is, an SPT phase is thought to be described by an invertible field theory at low energy [27], which is a class of QFTs whose dimension of the Hilbert space is one for an arbitrary spatial manifold. The action for the bosonic SPT phase (2.3) is also an invertible field theory.

It has been argued that the elements of the cobordism group $\Omega^d_S(BG)$, which is the suitable dual of the bordism group $\Omega_d^S(BG)$ is identified with partition functions of invertible QFTs for spacetime d-dimensional manifolds [27]. Here, G is the global symmetry group and S stands for the choice of the space-time symmetry such as fermion parity and/or time-reversal. We review the bordism groups and their dual groups utilized to classify the invertible QFT.

The bordism group $\Omega_d^S(BG)$ consists of equivalence classes $[M, g, \eta]$, where M is a closed d-dimensional manifold, η is an S-structure on TM, and $g : M \to BG$ is a flat G gauge field. Here, (M, g, η) and (M', g', η') are said to be equivalent if there exists a $(d+1)$-dimensional manifold $(W, \widetilde{g}, \widetilde{\eta})$ equipped with $\widetilde{g} : W \to BG$ and the S-structure $\widetilde{\eta}$, where $\partial W = M \sqcup \overline{M}'$ and $(\widetilde{g}, \widetilde{\eta})$ restrict to (g, η) and (g', η') on the boundary M and $\overline{M}'$, respectively (see Fig. 2.1). These classes form an abelian group with its multiplication law given by the disjoint union, and with inverses given by orientation reversal.

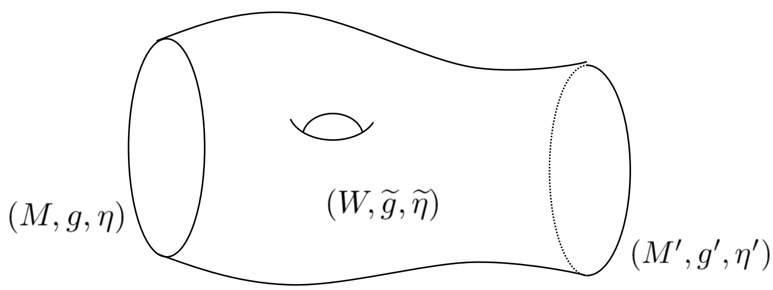

Figure 2.1: The bordism between manifolds equipped with a S-structure and a flat G-gauge field, (M, g, η) and (M', g', η').

Then, the cobordism group $\Omega^d_S(BG)$ is given by

$$\Omega^d_S(BG) = \mathrm{Tor}(\Omega_d^S(BG)) \oplus \mathrm{Free}(\Omega_{d+1}^S(BG)). \tag{2.4}$$

Mathematically, the cobordism group is defined as the so-called Anderson dual of the bordism group, [1] characterized by the following split exact sequence

$$0 \to \mathrm{Ext}(\Omega_d^S(BG), \mathbb{Z}) \to \Omega^d_S(BG) \to \mathrm{Hom}(\Omega_{d+1}^S(BG), \mathbb{Z}) \to 0, \tag{2.5}$$

[1] Mathematically, the exact sequence (2.5) defines a cohomology of degree $(d+1)$, so we should have expressed the middle of the sequence as $\Omega^{d+1}_S(BG)$. However, since this object physically classifies phases in d spacetime dimensions, we write the middle as $\Omega^d_S(BG)$ in this thesis instead. This notation of dimensions matches with e.g., [12].

which splits as (2.4), since $\mathrm{Ext}(\Omega_d^S(BG), \mathbb{Z}) = \mathrm{Tor}(\Omega_d^S(BG))$ and $\mathrm{Hom}(\Omega_{d+1}^S(BG), \mathbb{Z}) = \mathrm{Free}(\Omega_{d+1}^S(BG))$.

Now let us see how the elements of the cobordism group are regarded as the partition functions of invertible QFTs. Firstly, an element of torsion part $\mathrm{Tor}(\Omega_d^S(BG))$ in (2.4) corresponds to a partition function of a d-dimensional invertible topological quantum field theory (TQFT). It is known that a partition function of an invertible TQFT evaluated on a closed manifold (M, g, η) only depends on the bordism class of $\Omega_d^S(BG)$ [28]. Hence, the partition function of an invertible TQFT gives the representation of the bordism group

$$\chi : \Omega_d^S(BG) \to U(1), \tag{2.6}$$

which lives in $\mathrm{Hom}(\Omega_d^S(BG), U(1))$. Further, it is shown in [28] that the invertible TQFT up to isomorphism is in 1-1 correspondence to the element of $\mathrm{Hom}(\Omega_d^S(BG), U(1))$. Here, when $\Omega_d^S(BG)$ has a free part, $\mathrm{Hom}(\Omega_d^S(BG), U(1))$ contains the continuous $U(1)$ groups. The element of the $U(1)$ corresponds to a topological theory labeled by a continuous parameter, e.g., the two-dimensional theta angle $\int \theta F/(2\pi)$ for the $U(1)$ gauge field. Once we are interested in the deformation class of the invertible theories, we drop off such continuous $U(1)$ factors, since the theta parameter can be continuously varied. In this sense, the discrete part $\mathrm{Hom}(\mathrm{Tor}(\Omega_d^S(BG)), U(1)) = \mathrm{Tor}(\Omega_d^S(BG))$ in (2.4) classifies the deformation class of the invertible TQFT.

Secondly, the elements of the free part (2.4) classifies the deformation class of d-dimensional invertible field theories which are not topological [29]. For example, the free part of the Anderson dual $\mathrm{Free}(\Omega_{3+1}^S(BG))$ for $d = 3$ accounts for the three-dimensional Chern-Simons term $k \int AdA/(4\pi)$, which is not topological in the sense that the partition function depends continuously on gauge transformations of the background gauge field. Since the Chern-Simons term in d dimensions is defined via the continuous theta term in $(d+1)$ dimensions, the free part in $\Omega_{d+1}^S(BG)$ gives the classification for such a non-topological invertible theories in d dimension.

Summarizing, the cobordism group (2.4) classifies the invertible QFTs up to deformations, and in turn conjectured to classify the SPT phases. We note, however, it still remains unsolved if the classification of invertible QFTs matches with that of SPT phases, which is formulated as the equivalence class of lattice Hamiltonians up to local deformations.

Before proceeding, let us observe how the classification of SPT phases works using several examples. In Table 2.1 and 2.2, we show the bordism and cobordism groups in various dimensions, based on the global symmetry relevant in physics. A similar table can be found in [12]. Each element of the table 2.2 corresponds to the classification of SPT phases based on some symmetry in d spacetime dimensions.

For example, as we will review in Sec. 2.3 the spin structure (the leftmost column of the table) is a spacetime structure required for fermionic systems, where the only global symmetry (except for the Lorentz symmetry) is the fermion parity $(-1)^F$. In this case, for one dimensional spacetime $d = 1$ we observe the cobordism group $\Omega^1_{\mathrm{spin}} = \mathbb{Z}_2$. The nontrivial phase is generated by a single complex fermion on a point. For two dimensional spacetime $d = 2$, we have the cobordism group $\Omega^2_{\mathrm{spin}} = \mathbb{Z}_2$. This implies a nontrivial invertible topological phase based on the the fermion parity $(-1)^F$. This corresponds to the well-known (1+1)-dimensional topological superconductor, also known as the Kitaev wire [18]. If we incorporate the time-reversal symmetry with $T^2 = 1$, the Kitaev wire instead generates the

	no symmetry	$T^2=(-1)^F$	$T^2=1$	unitary $\mathbb{Z}_2$
d	Spin	Pin^+	Pin^-	$\mathrm{Spin}\times\mathbb{Z}_2$
1	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2^2$
3	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_8$
4	$\mathbb{Z}$	$\mathbb{Z}_{16}$	0	$\mathbb{Z}$
5	0	0	0	0
6	0	0	$\mathbb{Z}_{16}$	0

Table 2.1: Bordism groups Ω_d^S.

	no symmetry	$T^2=(-1)^F$	$T^2=1$	unitary $\mathbb{Z}_2$
d	Spin	Pin^+	Pin^-	$\mathrm{Spin}\times\mathbb{Z}_2$
1	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_8\oplus\mathbb{Z}$
4	0	$\mathbb{Z}_{16}$	0	0
5	0	0	0	0
6	0	0	$\mathbb{Z}_{16}$	0

Table 2.2: Cobordism groups Ω_S^d.

$\mathbb{Z}_8$ classification [10], represented by $\Omega^2_{\mathrm{pin}^-}=\mathbb{Z}_8$. This $\mathbb{Z}_8$ classification will be studied in detail in Chapter 6.

For the spacetime dimension $d=3$, we have $\Omega^3_{\mathrm{spin}}=\mathbb{Z}$. This free group is generated by the gravitational Chern-Simons theory

$$\int_{M=\partial W}\mathrm{CS}_{\mathrm{grav}}=\pi\int_W \hat{A}(R)=\frac{1}{192\pi}\int_W \mathrm{Tr}(R\wedge R), \tag{2.7}$$

which effectively describes the $p+ip$ superconductor. The spin cobordism group and the corresponding SPT phases are summarized in Table 2.3.

For the cobordism group $\Omega^3_{\mathrm{spin}}(B\mathbb{Z}_2)=\mathbb{Z}_8\oplus\mathbb{Z}$, the free group is generated by the $p+ip$ superconductor. The torsion group $\mathbb{Z}_8$ corresponds to the $(2+1)$-dimensional SPT phase intrinsically based on the $\mathbb{Z}_2$ symmetry, whose structure will be explained in Sec. 2.4. The cobordism group $\Omega^4_{\mathrm{pin}^+}=\mathbb{Z}_{16}$ classifies the SPT phase based on time-reversal symmetry with $T^2=(-1)^F$. The structure of $\mathbb{Z}_{16}$ is also outlined in Sec. 2.4.

2.3 Fermionic phases and spin/pin structure

In order to define a Lorentz invariant QFT that describes a fermionic system, we require a choice of spin structure η. Mathematically, a spin structure η is a trivialization of the 2nd Stiefel-Whitney class, $\delta\eta=w_2$, and two distinct spin structures on the spacetime manifold M are related to each other by $H^1(M,\mathbb{Z}_2)$. Namely, for a given $\chi\in H^1(M,\mathbb{Z}_2)$, we can shift η by χ to define another spin structure $\eta+\chi$. The need for a spin structure arises for the following reason.

dimensions	$(0+1)$d	$(1+1)$d	$(2+1)$d
SPT phases	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
generator	complex fermion	Kitaev wire	$p+ip$ superconductor

Table 2.3: spin cobordism groups Ω^d_{spin} and the corresponding fermionic SPT phases.

A relativistic quantum field theory in d spacetime dimensions possesses the Lorentz $SO(d)$ symmetry. However, since fermions are spinors, fermions transform according to the double cover of $SO(d)$, which is $Spin(d)$. To define the field theory on a generic spacetime manifold, one needs to consider a $SO(d)$ bundle $\phi : M \to BSO(d)$, which can be thought of as the tangent bundle TM of an oriented manifold M. In order to have fermions, the transition functions $\phi_{ij} \in SO(d)$ between overlapping patches U_i and U_j must be lifted to $\widetilde{\phi}_{ij} \in Spin(d)$. Since $Spin(d)$ is the group extension

$$\mathbb{Z}_2 \to Spin(d) \to SO(d) \tag{2.8}$$

whose extension is given by $w_2 \in H^2(BSO(d), \mathbb{Z}_2)$. Then, $Spin(d)$ is identified as $SO(d)\times\mathbb{Z}_2$ *as a set*, so we can express $\widetilde{\phi}_{ij}$ as a pair $(\phi_{ij}, \eta_{ij}) \in SO(d)\times\mathbb{Z}_2$. The nontrivial group extension is reflected in the multiplication law of $\mathbb{Z}_2$ elements η_{ij} twisted by w_2. Namely, for transition functions $\widetilde{\phi}_{ij} \in Spin(d)$, we have the multiplication law

$$\widetilde{\phi}_{ij}\widetilde{\phi}_{jk} = (\phi_{ij}\phi_{jk}, \eta_{ij} + \eta_{jk} + w_2(\phi_{ij}, \phi_{jk})). \tag{2.9}$$

Due to the cocycle condition $\widetilde{\phi}_{ij}\widetilde{\phi}_{jk} = \widetilde{\phi}_{ik}$, we find

$$\eta_{ij} + \eta_{jk} + \eta_{ik} = w_2(\phi_{ij}, \phi_{jk}) \tag{2.10}$$

In coordinate-free notation, this is precisely the equation $\delta\eta = \phi^* w_2 = w_2(TM)$.

When the field theory possesses a global symmetry that reverses the orientation of the spacetime, e.g., time-reversal (T) symmetry, we may put the theory on an unoriented spacetime manifold. In that case, the Lorentz symmetry is now expressed as $O(d)$, where the time-reversal symmetry corresponds to the $\mathbb{Z}_2$ subgroup $\mathbb{Z}_2^T \subset O(d)$ generated by the orientation-reversing element. Then, we have the tangent bundle $\phi : M \to BO(d)$, and the transition function $\phi_{ij} \in O(d)$ will be lifted to its double cover via the group extension

$$\mathbb{Z}_2 \to Pin^{\pm}(d) \to O(d), \tag{2.11}$$

whose extension is characterized by the element of $H^2(BO(d), \mathbb{Z}_2)$. Since $H^2(BO(d), \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by w_1^2 and w_2, there are four possible choices of the symmetry extension of $O(d)$ by $\mathbb{Z}_2$. In addition, $SO(d) \subset O(d)$ must be lifted to $Spin(d)$ for fermionic theories, so the extension class in (2.11) must be w_2 or $w_2 + w_1^2$. This amounts to two choices of the double cover $Pin^+(d)$ or $Pin^-(d)$;

- When the extension class is chosen as w_2, we have the Pin^+ group by the extension (2.11). In that case, analogously to (2.10) the Pin$^+$ structure is specified by a choice of $\eta \in C^1(M, \mathbb{Z}_2)$ with $\delta\eta = w_2$.

- When the extension class is chosen as $w_2 + w_1^2$, we have the Pin^- group by the extension (2.11). In that case, the Pin$^-$ structure is specified by a choice of $\eta \in C^1(M, \mathbb{Z}_2)$ with $\delta\eta = w_2 + w_1^2$.

Physically, Pin$^-$ structure differs from Pin$^+$ by the action of the time-reversal symmetry on a fermion. That is, the subgroup $\mathbb{Z}_2^T \in O(d)$ is extended to the $\mathbb{Z}_4$ in the case of $Pin^-(d)$,

$$\mathbb{Z}_2 \to \mathbb{Z}_4^T \to \mathbb{Z}_2^T, \tag{2.12}$$

while $\mathbb{Z}_2^T$ is not extended for $Pin^+(d)$. This means *in the Euclidean spacetime* that the time-reversal symmetry acts as $T^2 = (-1)^F$ for the Pin$^-$ structure, while $T^2 = +1$ for the Pin$^+$ structure, where $(-1)^F$ represents the fermion parity.

When we are interested on the action of T in the Minkowski signature, the above T action should be Wick-rotated, meaning $T^2 = (-1)^F$ for Pin$^+$ structure, while $T^2 = +1$ for Pin$^-$ structure. These T symmetry actions are represented in Table 2.1 and Table 2.2.

2.4 The layer structure of fermionic SPT phases

The cobordism group $\Omega^d_{\text{spin}}(BG)$ can in principle be computed using a mathematical toolkit called the Atiyah-Hirzebruch spectral sequence (AHSS) [30, 31]. The AHSS provides a way to evaluate $\Omega^d_{\text{spin}}(BG)$ using the cohomological data

$$E_2^{p,q} := H^p(BG, \Omega^q_{\text{spin}}), \quad p + q = d. \tag{2.13}$$

Each cohomological data $E_2^{p,q}$ is usually associated with a physical interpretation. Namely, $E_2^{p,q}$ corresponds to decorating a q-dimensional fermionic SPT phase classified by Ω^q_{spin}, on a spacetime q-dimensional junction of G-symmetry defects [32]. For low dimensions p, the decorations are described as follows:

- For $p = 1$, we decorate a codimension-1 symmetry defect labeled by a group element $g \in G$ with a $(d-1)$-dimensional SPT phase. The rule for the decoration is controlled by $n_1 \in H^1(BG, \Omega^{d-1}_{\text{spin}})$, i.e., for a given G-gauge field $g : M \to BG$, $g^* n_1 \in H^1(M, \Omega^{d-1}_{\text{spin}})$ indicates that we assign a $(d-1)$-dimensional fermionic SPT phase $g^* n_1(ij) \in \Omega^{d-1}_{\text{spin}}$ on the Poincaré dual of a 1-simplex $\langle ij \rangle$.

- For $p = 2$, we consider a codimension-2 junction of two G-symmetry defects labeled by group elements $g_1, g_2 \in G$ that fuses into a single defect $g_1 g_2$ at the junction. We decorate the junction according to $n_2 \in H^2(BG, \Omega^{d-2}_{\text{spin}})$ with a $(d-2)$-dimensional SPT phase, i.e., for a given G-gauge field $g : M \to BG$, $g^* n_2 \in H^2(M, \Omega^{d-2}_{\text{spin}})$ indicates that we assign a $(d-2)$-dimensional fermionic SPT phase $g^* n_1(ijk) \in \Omega^{d-2}_{\text{spin}}$ on the Poincaré dual of the junction of g_{ij}, g_{jk} at a 2-simplex $\langle ijk \rangle$.

- For $p = 3$, we consider a codimension-3 junction of three G-symmetry defects labeled by $g_1, g_2, g_3 \in G$ respectively. This junction can be understood as follows. We take a spacetime around the junction as I^d with $I = [0, 1]$ locally, and regard one coordinate as the time direction $t \in [0, 1]$. Then, for the snapshot at the initial time $t = 0$, we consider a F-like configuration of three symmetry defects g_1, g_2, g_3 with two junctions as Fig. 2.2. Then, we think of a movie that transforms the initial configuration at $t = 0$ to the final one at $t = 1$, which changes the way to fuse the three defects. At the middle of the movie $t = 0.5$, we observe a junction where three junctions fuses into a point, which is regarded as the codimension-3 defect involving the three symmetry defects, see Fig. 2.2. We decorate the junction according to $n_3 \in H^3(BG, \Omega^{d-3}_{\text{spin}})$ with a $(d-3)$-dimensional SPT phase, i.e., for a given G-gauge field $g : M \to BG$, $g^*n_3 \in H^3(M, \Omega^{d-3}_{\text{spin}})$ indicates that we assign a $(d-3)$-dimensional fermionic SPT phase $g^*n_3(ijkl) \in \Omega^{d-3}_{\text{spin}}$ on the Poincaré dual of the junction of g_{ij}, g_{jk}, g_{kl} at a 3-simplex $\langle ijkl \rangle$.

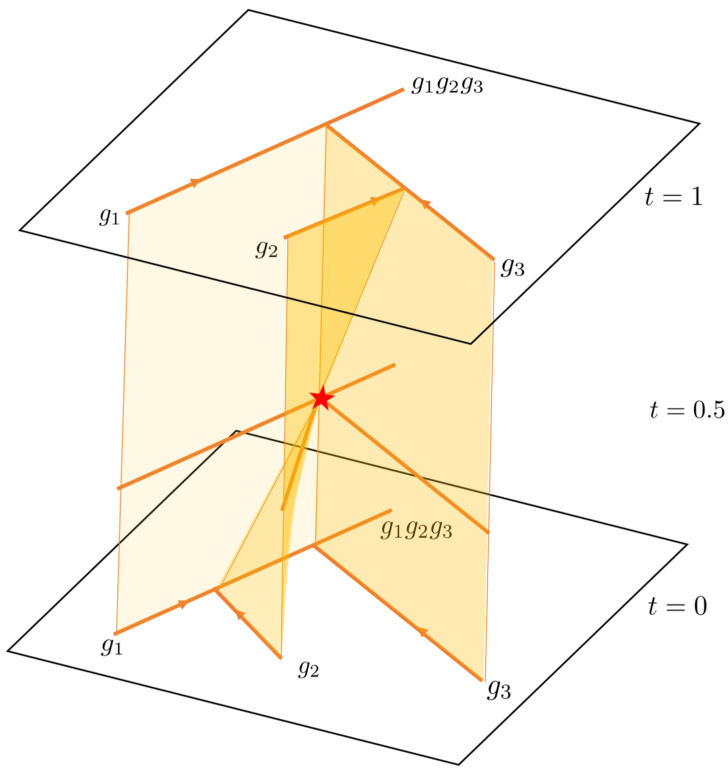

Figure 2.2: The movie for the network of symmetry defects. At the middle of the movie all the defects for $g_1, g_2, g_3 \in G$ meets at a point, which gives a codimension-3 junction.

Let us observe how the above physical intuition works for the example $\Omega^3_{\text{spin}}(B\mathbb{Z}_2) = \mathbb{Z}_8$. In that case, the cobordism group is constructed by three layers

$$\begin{aligned} n_1 &\in H^1(B\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \qquad \text{Kitaev layer,} \\ n_2 &\in H^2(B\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \qquad \text{Complex fermion layer,} \\ \nu_3 &\in H^3(B\mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \qquad \text{Bosonic layer,} \end{aligned} \tag{2.14}$$

where the third layer represents $E_2^{4,-1} = H^4(B\mathbb{Z}_2, \mathbb{Z}) = H^3(B\mathbb{Z}_2, U(1))$. According to the AHSS computation, the SPT classification $\nu \in \mathbb{Z}_8$ is obtained by three layers as the binary expansion

$$\nu = 4\nu_3 + 2n_2 + n_1 \qquad \text{mod } 8, \tag{2.15}$$

which implies the following description of $(2+1)$-dimensional $\mathbb{Z}_2$ SPT phases. Firstly, the $\mathbb{Z}_2$ subgroup of the $\mathbb{Z}_8$ classification is solely labeled by $\nu_3 \in H^3(B\mathbb{Z}_2, U(1))$ with $n_1 = n_2 = 0$. This corresponds to a bosonic $\mathbb{Z}_2$ SPT phase classified by group cohomology, and the $\mathbb{Z}_2$ subgroup is independent of spin structure of the spacetime. Secondly, the $\mathbb{Z}_4$ subgroup corresponds to turning on (n_2, ν_3). This $\mathbb{Z}_4$ subclass is called the Gu-Wen SPT phase, the simplest intrinsically fermionic phase in the sense of intrinsic dependence on the spin structure. The nontrivial n_2 assigns a complex fermion on the codimension-2 junction of symmetry defects, so the junction can carry nontrivial fermion parity for the generator of $\mathbb{Z}_4$ subclass. The Gu-Wen SPT phase will be studied in Sec. 5. Finally, the full $\mathbb{Z}_8$ group can have nontrivial n_1, so the generator of $\mathbb{Z}_8$ has the Kitaev wire on the $\mathbb{Z}_2$ symmetry defect.

Using the AHSS, we can also compute the Pin$^+$, Pin$^-$ cobordism group in d dimension based on the data

$$E_2^{p,q} := H^p_\rho(B\mathbb{Z}_2^T, \Omega^q_{\mathrm{spin}}), \quad p+q=d. \tag{2.16}$$

Here, H^*_ρ denotes the twisted cohomology, see Appendix D for the definition. For example, the Pin$^+$ cobordism group $\Omega^4_{\mathrm{pin}^+} = \mathbb{Z}_{16}$ consists of four layers

$$\begin{aligned}
n_1 &\in H^1_\rho(B\mathbb{Z}_2^T, \mathbb{Z}) = \mathbb{Z}_2 \qquad p+ip \text{ layer},\\
n_2 &\in H^2(B\mathbb{Z}_2^T, \mathbb{Z}_2) = \mathbb{Z}_2 \qquad \text{Kitaev layer},\\
n_3 &\in H^3(B\mathbb{Z}_2^T, \mathbb{Z}_2) = \mathbb{Z}_2 \qquad \text{Complex fermion layer},\\
\nu_4 &\in H^4_\rho(B\mathbb{Z}_2^T, U(1)) = \mathbb{Z}_2 \qquad \text{Bosonic layer},
\end{aligned} \tag{2.17}$$

which again compiles into the SPT classification $\nu \in \mathbb{Z}_{16}$ by the binary expansion

$$\nu = 8\nu_4 + 4n_3 + 2n_2 + n_1 \qquad \mathrm{mod}\ 16. \tag{2.18}$$

This implies the layer structure of the $\mathbb{Z}_{16}$ classification; the $\mathbb{Z}_2$ subgroup gives the bosonic SPT phase classified by $H^4_\rho(B\mathbb{Z}_2^T, U(1)) = \mathbb{Z}_2$, the $\mathbb{Z}_4$ subgroup gives the Gu-Wen SPT phases, the $\mathbb{Z}_8$ subgroup involves the Kitaev wire assigned on the junction of T symmetry defects, and finally the full $\mathbb{Z}_{16}$ can have a $p+ip$ superconductor assigned on the T symmetry defect.

2.5 SPT phases and 't Hooft anomalies: anomaly inflow

An important property of an SPT phase is the presence of an edge state which appears on the boundary of SPT phase. In particular, a state on the boundary of an SPT phase must have a nontrivial spectrum at low energy, e.g., a well-known gapless edge state on the boundary of a (2+1)-dimensional topological insulator. Such a nontrivial spectral property on the boundary of an SPT phase originates from a quantum anomaly of the edge state called an 't Hooft anomaly.

In general, a QFT with a global symmetry G is said to have an 't Hooft anomaly, when the QFT is coupled with a background G gauge field A, and its partition has a phase ambiguity under the G gauge transformation, $Z(A+\delta\chi) = Z(A)\cdot e^{2\pi i\omega(A,\chi)}$. Importantly, a theory with an 't Hooft anomaly becomes gauge invariant after coupling with an SPT phase in one more dimension, and consider it as a bulk-boundary system. This is called an anomaly inflow mechanism. To see how the 't Hooft anomaly is accounted for by coupling the theory with the bulk SPT phase, let us consider a two-dimensional bosonic system with an internal symmetry G. Then, the configuration of background gauge field is specified by a network of symmetry defects on a spacetime manifold M. In this description, the gauge transformation is understood as changing the presentation of defect networks locally, as shown in Fig. 2.3.

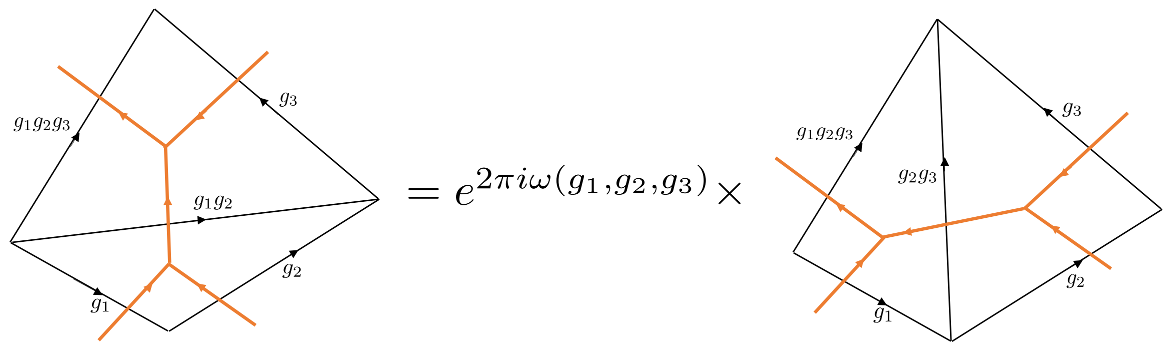

Figure 2.3: The partition function shifts under changing the configuration of the symmetry defects locally.

When the theory has an 't Hooft anomaly, the change of defect networks is associated with a phase variation ω of the partition function. ω is a function $G^3 \to U(1)$, so it is an element of $C^3(BG, U(1))$. Further, ω is subject to the consistency equation represented in Fig. 2.4. The consistency condition compiles into a simple form $\delta\omega = 0$ using the coboundary operation for the group cochain (see Appendix D). Hence, $\omega \in Z^3(BG, U(1))$.

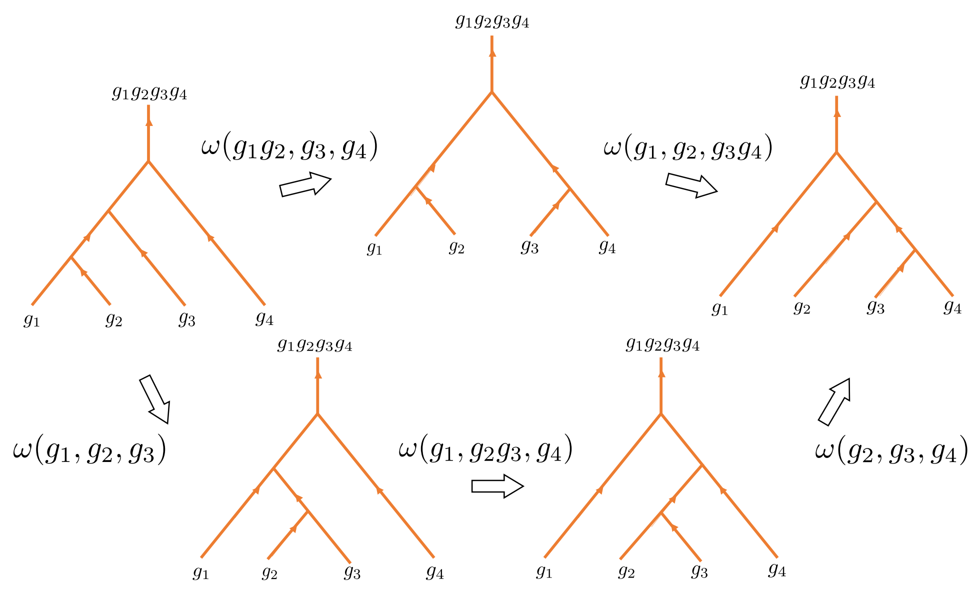

Figure 2.4: The consistency equation for the phase ambiguity ω. Since the overall phase shift should be independent of the path in this pentagon, it leads to the condition $\delta\omega = 0$.

Note that we can add a term $\chi(g_1, g_2)$ on each junction of $g_1, g_2 \in G$. This is regarded as introducing a local counterterm described by $\chi \in C^2(BG, U(1))$. This redefinition of the partition function amounts to shifting ω by a coboundary, $\omega \to \omega + \delta\chi$. Thus, the 't Hooft anomaly up to the addition of local counterterms are said to be characterized by the group cohomology $[\omega] \in H^3(BG, U(1))$.

The above theory with an 't Hooft anomaly characterized by $\omega \in H^3(BG, U(1))$ is made gauge invariant by coupling with the $(2+1)$-dimensional bosonic SPT phase $\omega \in H^3(BG, U(1))$ reviewed in Sec. 2.1. Actually, the change of the defect network represented in Fig. 2.3 corresponds to attaching a single 3-simplex on the boundary, see Fig. 2.5. When the bulk 3-simplex has the Boltzmann weight $\omega \in Z^3(BG, U(1))$, it cancels the gauge variation from the boundary, and constitutes a gauge-invariant bulk-boundary system after all.

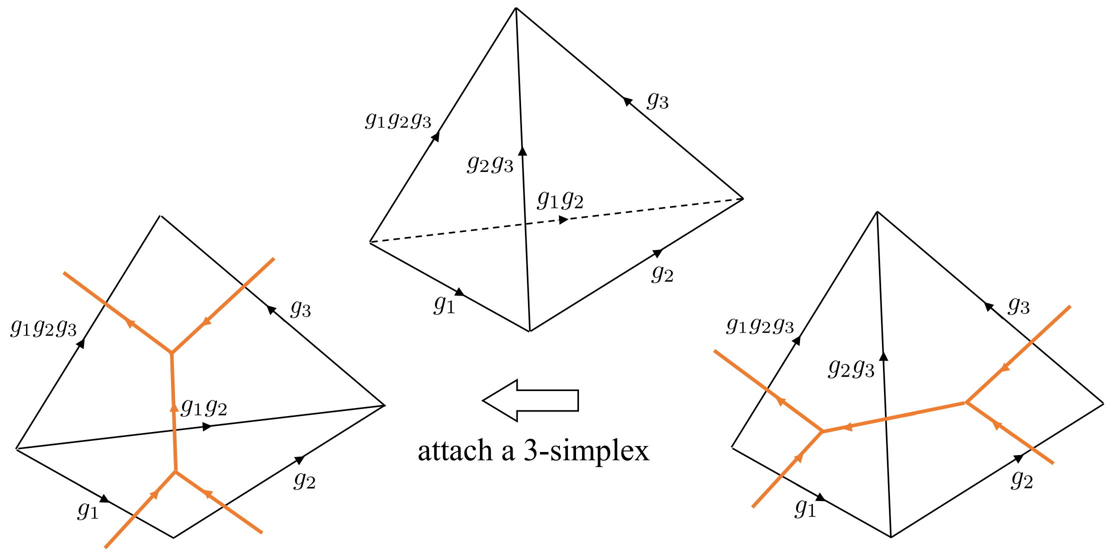

Figure 2.5: The phase ambiguity under the change of defect networks can be canceled by attaching a 3-simplex with the Boltzmann weight ω.

Chapter 3

Review on bosonization

So far, we have illustrated an abstract classification scheme of SPT phases based on the bordism groups. In the rest of this thesis, we study explicit constructions of fermionic SPT phases on lattice that embody the layer structure of cobordism groups, and explore their physical properties. Here, we provide a general overview for constructing a path integral for a fermionic SPT phase by starting with a path integral for a bosonic topological phase in d spacetime dimensions and condensing fermionic particles of the bosonic theory by gauging a specific higher-form global symmetry. Such a construction of fermionic theories from a bosonic dual theory is regarded as a generalization of the celebrated Jordan-Wigner transformation in $(1+1)$ dimensions to higher spacetime dimensions.

3.1 $(1+1)$d bosonization

We start with recalling the Jordan-Wigner transformation and bosonization of $(1+1)$-dimensional quantum systems from the field theoretical perspective. The Jordan-Wigner transformation maps a bosonic system with a global $\mathbb{Z}_2$ symmetry to a fermionic system with the fermion parity symmetry $(-1)^F$. For example, the critical Ising model with the spin-flip $\mathbb{Z}_2$ symmetry in $(1+1)$ dimensions can be transformed to a free massless Majorana fermion.

We can also define the bosonization as the inverse of the Jordan-Wigner transformation, which maps a fermionic theory to a bosonic one. At the QFT level, bosonization generates a bosonic theory independent of spin structure of the spacetime, from a given fermionic theory intrinsically coupled with spin structure. If we write a fermionic partition function $Z_f(M,\eta)$ on an oriented manifold M equipped with a specific spin structure η, the following transformation gives the simplest form of the bosonization,

$$Z_b(M) := \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_2)|}} \sum_{\eta} Z_f(M,\eta), \tag{3.1}$$

where Z_b is a bosonic theory given by summing over all possible choices of spin structure on M. Since the distinct spin structures are related by shifting $\eta \to \eta+\chi$ with $[\chi] \in H^1(M,\mathbb{Z}_2)$, the summation is thought of as running over $[\chi] \in H^1(M,\mathbb{Z}_2)$. Physically, the shift $\eta \to \eta+\chi$ corresponds to turning on flat background gauge field for the $(-1)^F$ symmetry, and the summation in (3.1) amounts to promoting the background gauge field of $(-1)^F$ to the dynamical gauge field. In $(1+1)$ dimensions, gauging a $\mathbb{Z}_2$ (in our case $(-1)^F$) symmetry gives rise to a dual $\hat{\mathbb{Z}}_2$ symmetry. We can couple the bosonic theory with the background gauge field of the $\hat{\mathbb{Z}}_2$ symmetry as

$$Z_b(M,\alpha) := \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_2)|}} \sum_{\eta} z(M,\alpha,\eta) Z_f(M,\eta). \tag{3.2}$$

On an oriented manifold, $z(M,\alpha,\eta)$ is a phase that defines a way to couple Z_b with the $\hat{\mathbb{Z}}_2$ background gauge field α. This phase can be regarded as a partition function of an invertible topological quantum field theory (TQFT) based on the spin structure η and the $\hat{\mathbb{Z}}_2$ symmetry. We will discuss in detail the properties of $z(M,\alpha,\eta)$ in the following sections. Here, we refer to crucial properties of this theory. Firstly, the partition function is a $\mathbb{Z}_2 = \{\pm 1\}$ valued quadratic function of $\alpha \in H^1(M,\mathbb{Z}_2)$, with the quadratic property

$$z(M,\alpha,\eta) z(M,\alpha',\eta) = z(M,\alpha+\alpha',\eta)(-1)^{\int \alpha\cup\alpha'}. \tag{3.3}$$

Secondly, if we shift the spin structure as $\eta \to \eta+\chi$ with $\chi \in Z^1(M,\mathbb{Z}_2)$, $z(M,\alpha,\eta)$ transforms as

$$z(M,\alpha,\eta+\chi) = z(M,\alpha,\eta)(-1)^{\int \chi\cup\alpha}. \tag{3.4}$$

(3.4) immediately leads to

$$\sum_{[\alpha]\in H^1(M,\mathbb{Z}_2)} z(M,\alpha,\eta') z^{-1}(M,\alpha,\eta) = |H^1(M,\mathbb{Z}_2)| \cdot \delta(\eta'-\eta), \tag{3.5}$$

where $\delta(\eta'-\eta) = 1$ if $\eta'-\eta$ is cohomologically trivial, otherwise zero. This relation allows us to define the inverse transformation of the bosonization defined in (3.2),

$$Z_f(M,\eta) = \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_2)|}} \sum_{[\alpha]\in H^1(M,\mathbb{Z}_2)} z(M,\alpha,\eta) Z_b(M,\alpha). \tag{3.6}$$

This is regarded as gauging the $\mathbb{Z}_2$ symmetry of the bosonic theory after coupling with the spin theory $z(M,\alpha,\eta)$. This procedure is called fermionization, and especially turns out to provide a field theoretical description of the Jordan-Wigner transformation. To see how it works, we illustrate an example of (1+1)-dimensional fermionization in Sec. 3.2, focusing on (1+1)d conformal field theory (CFT).

3.2 Example: $(1+1)$d CFT

In a rational conformal field theory (RCFT) with diagonal modular invariance, there is a family of topological line operators known as the Verlinde lines, which are in one-to-one correspondence with the chiral primaries of the diagonal RCFT.

In the diagonal RCFT, some Verlinde lines are regarded as generators of global symmetries of the theory. To see this, let us explain how the Verlinde lines acts on the Hilbert space of the CFT. The torus partition function of the diagonal RCFT takes the form of

$$Z(\tau,\overline{\tau}) = \sum_i \chi_i(\tau)\overline{\chi}_i(\overline{\tau}), \tag{3.7}$$

where the sum is over the labels of chiral primaries. In such a diagonal modular invariant theory, the states of the Hilbert space are denoted by $|\phi_i\rangle$, in correspondence with the chiral primaries ϕ_i. Then, the action of the Verlinde line $\widehat{\mathcal{L}}_k$ corresponding to the chiral primary ϕ_k is given by

$$\widehat{\mathcal{L}}_k \left|\phi_i\right\rangle = \frac{S_{ki}}{S_{0i}} \left|\phi_i\right\rangle . \tag{3.8}$$

Here, S_{ij} is the elements of the modular S-matrix, which is unitary and symmetric. To discuss the algebra of Verlinde line, we use the Verlinde formula

$$N_{ij}^k = \sum_l \frac{S_{il}S_{jl}S^*_{kl}}{S_{0l}}, \tag{3.9}$$

where N_{ij}^k is the fusion coefficient of the chiral algebra,

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k. \tag{3.10}$$

Then, we can immediately see that the Verlinde lines also obey the fusion rule of the chiral primaries,

$$\widehat{\mathcal{L}}_i \times \widehat{\mathcal{L}}_j = \sum_k N_{ij}^k \widehat{\mathcal{L}}_k. \tag{3.11}$$

Hence, the Verlinde line $\widehat{\mathcal{L}}_i$ corresponding to ϕ_i with the property $\phi_i \times \phi_i = \phi_0$ generates the $\mathbb{Z}_2$ symmetry, where ϕ_0 is the identity operator. If there is a global $\mathbb{Z}_2$ symmetry free of an 't Hooft anomaly, we can gauge it to generate a fermionic CFT coupled with the spin structure, via the fermionization (3.6). The fermionization of (1+1)-dimensional CFT is also discussed in [31, 33–35].

For later convenience, we describe the torus partition function in the presence of Verlinde lines inserted in the torus. The torus partition function with the action of the $\widehat{\mathcal{L}}_i$ line (i.e., the $\widehat{\mathcal{L}}_k$ line inserted along the spatial direction) is expressed as

$$Z^{\mathcal{L}_k}(\tau,\overline{\tau}) = \sum_l \frac{S_{kl}}{S_{0l}} \chi_l(\tau)\overline{\chi}_l(\overline{\tau}). \tag{3.12}$$

Then, we can obtain the twisted partition function $Z_{\mathcal{L}_k}$ with the $\widehat{\mathcal{L}}_k$ line inserted along the temporal direction, by the modular S transformation on $Z^{\mathcal{L}_k}(\tau,\overline{\tau})$,

$$Z_{\mathcal{L}_k}(\tau,\overline{\tau}) = \sum_{i,j} N_{ki}^j \chi_i(\tau)\overline{\chi}_j(\overline{\tau}). \tag{3.13}$$

where we used the modular property of characters $\chi_i(-1/\tau) = \sum_j S_{ij}\chi_j(\tau)$.

3.2.1 Ising CFT

The two dimensional Ising CFT has three chiral primaries $\{1, \epsilon, \sigma\}$ with $h = 0, h = 1/2, h = 1/16$, respectively. Due to the fusion rule $\epsilon \times \epsilon = 1$, the Verlinde line $\widehat{\mathcal{L}}_\epsilon$ generates the $\mathbb{Z}_2$ symmetry of the diagonal theory

$$Z(\tau, \overline{\tau}) = |\chi_1(\tau)|^2 + |\chi_\epsilon(\tau)|^2 + |\chi_\sigma(\tau)|^2. \tag{3.14}$$

The modular S-matrix is given by

$$S_{\text{Ising}} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}, \tag{3.15}$$

from which we can read off the action of $\widehat{\mathcal{L}}_\epsilon$ on the Hilbert space (3.8) as

$$\widehat{\mathcal{L}}_\epsilon : |1\rangle \to |1\rangle \,, \; |\epsilon\rangle \to |\epsilon\rangle \,, \; |\sigma\rangle \to -|\sigma\rangle \,. \tag{3.16}$$

The partition functions with the $\mathbb{Z}_2$ defect inserted along the spatial and temporal direction are given by

$$Z^{\mathcal{L}}(\tau, \overline{\tau}) = |\chi_1(\tau)|^2 + |\chi_\epsilon(\tau)|^2 - |\chi_\sigma(\tau)|^2, \tag{3.17}$$

$$Z_{\mathcal{L}}(\tau, \overline{\tau}) = \chi_\epsilon(\tau)\overline{\chi}_0(\overline{\tau}) + \chi_0(\tau)\overline{\chi}_\epsilon(\overline{\tau}) + |\chi_\sigma(\tau)|^2, \tag{3.18}$$

respectively. We can also obtain the partition function $Z_{\mathcal{L}}^{\mathcal{L}}$ with the $\mathbb{Z}_2$ defect inserted in both spatial and temporal directions, by applying the modular T transformation on $Z_{\mathcal{L}}$,

$$Z_{\mathcal{L}}^{\mathcal{L}}(\tau, \overline{\tau}) = -\chi_\epsilon(\tau)\overline{\chi}_0(\overline{\tau}) - \chi_0(\tau)\overline{\chi}_\epsilon(\overline{\tau}) + |\chi_\sigma(\tau)|^2, \tag{3.19}$$

where the modular T-matrix is given by

$$T_{\text{Ising}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{\frac{2\pi i}{16}} \end{pmatrix}. \tag{3.20}$$

3.2.2 Fermionization of the Ising CFT: Majorana fermion

Now, we can perform the fermionization on the Ising CFT. On the fermionic side, we want to obtain partition functions of the fermionized theory coupled with the spin structure on a torus. In general, the spin structure η on an oriented two dimensional surface M is in one-to-one correspondence to the quadratic function $z(M, \alpha, \eta)$ of $\alpha \in H^1(M, \mathbb{Z}_2)$ with the quadratic property (3.3). We label the spin structure on M as

$$z(M, \alpha, \eta) = \begin{cases} 1 & \text{the spin structure is NS around } C_\alpha \,, \\ -1 & \text{the spin structure is R around } C_\alpha \end{cases} \tag{3.21}$$

where C_α is a single, non-self-intersecting closed curve Poincaré dual to $\alpha \in H^1(M, \mathbb{Z}_2)$. The spin structure on a torus is represented by a pair of the above labels e.g., (NS, NS), which means the spin structure in the spatial and temporal cycle, respectively. By the fermionization (3.6), the fermionic partition function on a torus with (NS, NS) spin structure is given by [31, 33, 34]

$$\begin{aligned} Z_{(\mathrm{NS},\mathrm{NS})} &= \frac{1}{2}\left(Z + Z^{\mathcal{L}} + Z_{\mathcal{L}} - Z_{\mathcal{L}}^{\mathcal{L}}\right) \\ &= |\chi_1(\tau) + \chi_\epsilon(\tau)|^2. \end{aligned} \tag{3.22}$$

We can also obtain partition function for other spin structures as

$$\begin{aligned} Z_{(\mathrm{NS},\mathrm{R})} &= \frac{1}{2}\left(Z + Z^{\mathcal{L}} - Z_{\mathcal{L}} + Z_{\mathcal{L}}^{\mathcal{L}}\right) \\ &= |\chi_1(\tau) - \chi_\epsilon(\tau)|^2, \end{aligned} \tag{3.23}$$

$$\begin{aligned} Z_{(\mathrm{R},\mathrm{NS})} &= \frac{1}{2}\left(Z - Z^{\mathcal{L}} + Z_{\mathcal{L}} + Z_{\mathcal{L}}^{\mathcal{L}}\right) \\ &= 2|\chi_\sigma(\tau)|^2, \end{aligned} \tag{3.24}$$

$$\begin{aligned} Z_{(\mathrm{R},\mathrm{R})} &= \frac{1}{2}\left(Z - Z^{\mathcal{L}} - Z_{\mathcal{L}} - Z_{\mathcal{L}}^{\mathcal{L}}\right) \\ &= 0. \end{aligned} \tag{3.25}$$

They are nothing but the partition function of a free Majorana fermion on a torus.

3.3 Bosonization in higher dimensions

We can also define the bosonization/fermionization in generic spacetime dimensions, analogously to what we have done in (1+1) dimensions. In higher dimensions, we consider a phase factor $z(M, \alpha, \eta)$ on an oriented d-dimensional manifold M with a spin structure η and the $\mathbb{Z}_2$ $(d-2)$-form symmetry, as proposed in [21]. $\alpha \in Z^{d-1}(M, \mathbb{Z}_2)$ denotes the background gauge field of the $(d-2)$-form symmetry. Then, the bosonization (3.2) is generalized as

$$Z_b(M, \alpha) := \frac{1}{\sqrt{|H^{d-1}(M, \mathbb{Z}_2)|}} \sum_\eta z(M, \alpha, \eta) Z_f(M, \eta). \tag{3.26}$$

This produces the bosonic theory Z_b with the $\mathbb{Z}_2$ $(d-2)$-form symmetry.

Now, we illustrate the formal properties of $z(M, \alpha, \eta)$ in detail. Let M be a d-dimensional oriented manifold M equipped with a triangulation. In generic spacetime dimensions higher than (1+1) dimensions, $z(M, \alpha, \eta)$ depends on the choice of the triangulation on M, and also depends on a specific cochain representative of $\alpha \in Z^{d-1}(M, \mathbb{Z}_2)$.

To construct $z(M, \alpha, \eta)$, we also want to equip M with the spin structure η. To do this, let $w_i \in H^i(M, \mathbb{Z}_2)$ be the i-th Stiefel-Whitney class of M. Assume that we have a cochain representative of w_2 on M. As reviewed in Sec. 2.3, for a given representative of w_2, we define the spin structure of M as a cochain $\eta \in C^1(M, \mathbb{Z}_2)$ with $\delta\eta = w_2$. Later, we will see how to give an explicit representative of w_2 in Sec. 3.4.

Then, $z(M, \alpha, \eta)$ has the form of

$$z(M, \alpha, \eta) = (-1)^{\int \eta \cup \alpha} \sigma(M, \alpha), \tag{3.27}$$

where $\sigma(M, \alpha)$ is independent of the spin structure η. On an oriented manifold, $\sigma(M, \alpha)$ takes the value in $\mathbb{Z}_2 = \{\pm 1\}$. Then, the main properties of $\sigma(M, \alpha)$ is listed as follows:

1. The quadratic property

$$\sigma(\alpha)\sigma(\alpha') = \sigma(\alpha + \alpha')(-1)^{\int \alpha \cup_{d-2} \alpha'}. \tag{3.28}$$

2. The change of $\sigma(\alpha)$ under the gauge transformation $\alpha \to \alpha + \delta\lambda$ or under the change of the triangulation is controlled by the formula

$$\sigma(\widetilde{M}, \widetilde{\alpha}) = (-1)^{\int_K (\mathrm{Sq}^2(\alpha) + w_2 \cup \alpha)} \sigma(M, \alpha), \tag{3.29}$$

   where $\widetilde{M}$ is the same manifold M with a different triangulation, $\widetilde{\alpha}$ is a cocycle such that $[\alpha] = [\widetilde{\alpha}]$ in cohomology, and $K = M \times [0, 1]$ such that the two boundaries are given by M and $\widetilde{M}$, and finally α is extended to K so that it restricts to α and $\widetilde{\alpha}$ on the boundaries.

Due to the variation $(-1)^{\int_K (\mathrm{Sq}^2(\alpha) + w_2 \cup \alpha)}$ of $\sigma(M, \alpha)$ under a bordism, we can see that the combination $z(M, \alpha, \eta) = (-1)^{\int \eta \cup \alpha} \sigma(M, \alpha)$ changes by $(-1)^{\int_K \mathrm{Sq}^2(\alpha)}$ under a bordism. In particular, $z(M, \alpha, \eta)$ works as a bordism invariant in two dimensions, since the 1-cochain $\alpha \in Z^1(M, \mathbb{Z}_2)$ is killed by the Steenrod square, $\mathrm{Sq}^2(\alpha) = 0$. Hence, $z(M, \alpha, \eta)$ in two dimensions is regarded as the partition function of a (1+1)-dimensional spin invertible TQFT with the $\mathbb{Z}_2$ symmetry, and reproduces the descriptions in Sec. 3.1. For higher dimensions $d > 2$, $z(M, \alpha, \eta)$ is no longer a bordism invariant, and the variation $\mathrm{Sq}^2(\alpha)$ is regarded as an 't Hooft anomaly of the $(d-2)$-form $\mathbb{Z}_2$ symmetry of the theory.

Finally, we illustrate the fermionization as the inverse of the bosonization (3.26). The form (3.27) immediately leads to $z(M, \alpha, \eta + \chi) = z(M, \alpha, \eta)(-1)^{\int \chi \cup \alpha}$ under the shift $\eta \to \eta + \chi$, and

$$\sum_{[\alpha] \in H^{d-1}(M, \mathbb{Z}_2)} z(M, \alpha, \eta') z^{-1}(M, \alpha, \eta) = |H^{d-1}(M, \mathbb{Z}_2)| \cdot \delta(\eta' - \eta). \tag{3.30}$$

Here, we note that the combination $z(M, \alpha, \eta') z^{-1}(M, \alpha, \eta)$ is gauge invariant under $\alpha \to \alpha + \delta\lambda$ since the anomaly of the two theories are canceled out, and works as a function of the cohomology $[\alpha] \in H^{d-1}(M, \mathbb{Z}_2)$. Then, the inverse of the bosonization (3.26) is given by

$$Z_f(M, \eta) = \frac{1}{\sqrt{|H^{d-1}(M, \mathbb{Z}_2)|}} \sum_{[\alpha] \in H^{d-1}(M, \mathbb{Z}_2)} z(M, \alpha, \eta) Z_b(M, \alpha). \tag{3.31}$$

Since $z(M, \alpha, \eta)$ has an 't Hooft anomaly controlled by $\mathrm{Sq}^2(\alpha)$, $Z_b(M, \alpha)$ in (3.26) must also have the same anomaly to cancel it out. We remark that it is required that M admits a spin structure when we perform fermionization starting with a bosonic theory $Z_b(M, \alpha)$, though $Z_b(M, \alpha)$ itself is bosonic and can be defined on non-spin manifolds.

3.4 Grassmann integral

So far, we have introduced a spin theory $z(M, \alpha, \eta)$ to fermionize a bosonic theory with the anomalous $(d-2)$-form $\mathbb{Z}_2$ symmetry $Z_b(M, \alpha)$. In this section, we will see that $z(M, \alpha, \eta)$ admits a local definition defined on a lattice (i.e., triangulation of M). This is particularly useful for constructing a lattice model of fermionic topological phases from a bosonic dual defined on a lattice.

To do this, we first endow M with a triangulation. In addition, we take the barycentric subdivision for the triangulation of M. Namely, each d-simplex in the initial triangulation of M is subdivided into $(d+1)!$ simplices, whose vertices are barycenters of the subsets of vertices in the d-simplex. We further assign a local ordering to vertices of the barycentric subdivision, such that a vertex on the barycenter of i vertices is labeled as i. Each simplex can then be either a $+$ simplex or a $-$ simplex, depending on whether the ordering agrees with the orientation or not.

There is an important property for the barycentric subdivision. That is, a set of all $(d-i)$-simplices of the barycentric subdivision of a triangulation of M gives a representative of Poincaré dual of the i-th Stiefel-Whitney class w_i [36–38]. In particular, a set of all $(d-2)$-simplices of the barycentric subdivision will be used to represent w_2 required for the definition of spin structure η with $\delta\eta = w_2$.

We assign a pair of Grassmann variables $\theta_e, \overline{\theta}_e$ on each $(d-1)$-simplex e of M such that $\alpha(e) = 1$, we associate θ_e on one side of e contained in one of d-simplices neighboring e (which will be specified later), $\overline{\theta}_e$ on the other side. Then, we define $\sigma(M, \alpha)$ introduced in (3.27) as

$$\sigma(M, \alpha) = \int \prod_{e|\alpha(e)=1} d\theta_e d\overline{\theta}_e \prod_t u(t), \tag{3.32}$$

where t denotes a d-simplex, and $u(t)$ is the product of Grassmann variables contained in t. For instance, for $d = 2$, $u(t)$ on $t = (012)$ is the product of $\vartheta_{12}^{\alpha(12)}, \vartheta_{01}^{\alpha(01)}, \vartheta_{02}^{\alpha(02)}$. Here, ϑ denotes θ or $\overline{\theta}$ depending on the choice of the assigning rule, which will be introduced later. The order of Grassmann variables in $u(t)$ will also be defined shortly. We note that $u(t)$ is ensured to be Grassmann-even when α is closed.

Due to the fermionic sign of Grassmann variables, $\sigma(\alpha)$ becomes a quadratic function, whose quadratic property depends on the order of Grassmann variables in $u(t)$. We will adopt the order used in Gaiotto-Kapustin [21], which is defined as follows.

- For $t = (01\dots d)$, we label a $(d-1)$-simplex $(01\dots\hat{i}\dots d)$ (i.e., a $(d-1)$-simplex given by omitting a vertex i) simply as i.
- Then, the order of ϑ_i for $+$ d-simplex t is defined by first assigning even $(d-1)$-simplices in ascending order, then odd simplices in ascending order again:

$$0 \to 2 \to 4 \to \dots \to 1 \to 3 \to 5 \to \dots \tag{3.33}$$

- For $-$ d-simplices, the order is defined in opposite way:

$$\dots \to 5 \to 3 \to 1 \to \dots \to 4 \to 2 \to 0. \tag{3.34}$$

For example, for $d=2$, $u(012)=\vartheta_{12}^{\alpha(12)}\vartheta_{01}^{\alpha(01)}\vartheta_{02}^{\alpha(02)}$ when (012) is a + triangle, and $u(012)=\vartheta_{02}^{\alpha(02)}\vartheta_{01}^{\alpha(01)}\vartheta_{12}^{\alpha(12)}$ for a $-$ triangle. Then, we choose the assignment of θ and $\overline{\theta}$ on each e such that, if t is a + (resp. $-$) simplex, $u(t)$ includes $\overline{\theta}_e$ when e is labeled by an odd (resp. even) number, see Fig. 3.1. Based on the above definition of $u(t)$, the quadratic property of $u(t)$ is

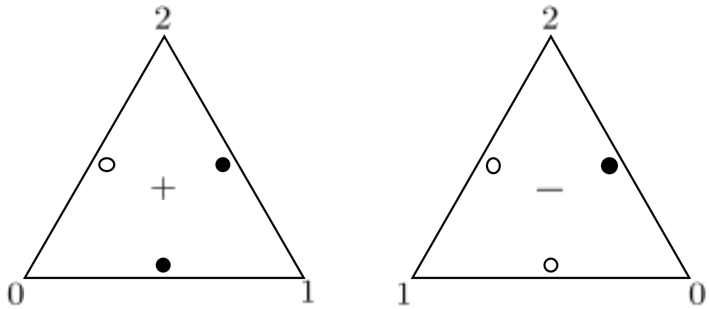

Figure 3.1: Assignment of Grassmann variables on 1-simplices in the case of $d=2$. θ (resp. $\overline{\theta}$) is represented as a black (resp. white) dot.

given by (3.28),

$$\sigma(\alpha)\sigma(\alpha')=\sigma(\alpha+\alpha')(-1)^{\int \alpha\cup_{d-2}\alpha'}, \tag{3.35}$$

for closed α,α'. To see this, we just have to bring the product of two Grassmann integrals

$$\sigma(\alpha)\sigma(\alpha')=\int \prod_{e|\alpha(e)=1} d\theta_e d\overline{\theta}_e \prod_{e|\alpha'(e)=1} d\theta_e d\overline{\theta}_e \prod_t u(t)[\alpha] \prod_t u(t)[\alpha'] \tag{3.36}$$

into the form of $\sigma(\alpha+\alpha')$ by permuting Grassmann variables, and count the net fermionic sign. First of all, each path integral measure on e picks up a sign $(-1)^{\alpha(e)\alpha'(e)}$ by permuting $d\overline{\theta}_e^{\alpha(e)}$ and $d\theta_e^{\alpha'(e)}$. For integrands, $u(t)$ on different d-simplices commute with each other for closed α, so nontrivial signs occur only by reordering $u(t)[\alpha]u(t)[\alpha']$ to $u(t)[\alpha+\alpha']$ on a single d-simplex. The sign on t is explicitly written as

$$(-1)^{\sum_{e,e'\in t}^{e>e'}\alpha(e)\alpha'(e')}, \tag{3.37}$$

where the order $e>e'$ is determined by $u(t)$. Hence, the net fermionic sign is given by

$$\sigma(\alpha)\sigma(\alpha')=\sigma(\alpha+\alpha')\prod_t (-1)^{\epsilon[t,\alpha,\alpha']}, \tag{3.38}$$

with

$$\epsilon[t,\alpha,\alpha']=\sum_{e,e'\in t, e>e'}\alpha(e)\alpha'(e')+\sum_{e\in t, e>0}\alpha(e)\alpha'(e), \tag{3.39}$$

where $e>0$ if $u[t]$ includes a $\overline{\theta}_e$ variable. The sign $\epsilon[t,\alpha,\alpha']$ turns out to have a neat expression in terms of the higher cup product. For later convenience, we compute $\epsilon[t,\alpha,\alpha']$ including the case that α,α' are not closed.

At a + simplex, after some efforts we can rewrite $\epsilon[t,\alpha,\alpha']$ as

$$\begin{aligned}\epsilon[t,\alpha,\alpha'] &= \sum_i \alpha_{2i+1}\cdot\delta\alpha'(t)+\sum_{i<j}\alpha_{2i+1}\alpha'_{2j+1}+\sum_{i>j}\alpha_{2i}\alpha'_{2j} \\ &= \alpha\cup_{d-2}\alpha'+\alpha\cup_{d-1}\delta\alpha'.\end{aligned} \tag{3.40}$$

At a $-$ simplex, similarly we have

$$\begin{aligned}\epsilon[t,\alpha,\alpha'] &= \sum_i \alpha_{2i}\cdot\delta\alpha'(t) + \sum_{i<j}\alpha_{2i+1}\alpha'_{2j+1} + \sum_{i>j}\alpha_{2i}\alpha'_{2j} \\ &= \delta\alpha(t)\delta\alpha'(t) + \alpha\cup_{d-2}\alpha' + \alpha\cup_{d-1}\delta\alpha'.\end{aligned} \tag{3.41}$$

We can see the quadratic property (3.35) when α,α' are closed.

The change of $\sigma(\alpha)$ under the gauge transformation $\alpha\to\alpha+\delta\gamma$ or under the change of the triangulation is controlled by the formula

$$\sigma(\widetilde{M},\widetilde{\alpha}) = (-1)^{\int_K(\mathrm{Sq}^2(\alpha)+w_2\cup\alpha)}\sigma(M,\alpha), \tag{3.42}$$

where $\widetilde{M}$ is the same manifold M with a different triangulation, $\widetilde{\alpha}$ is a cocycle such that $[\alpha]=[\widetilde{\alpha}]$ in cohomology, and $K=M\times[0,1]$ such that the two boundaries are given by M and $\widetilde{M}$, and finally α is extended to K so that it restricts to α and $\widetilde{\alpha}$ on the boundaries. The derivation was given in [21].

We note that due to the Wu relation [39], we have

$$(-1)^{\int_K(\mathrm{Sq}^2(\alpha)+w_2\cup\alpha)} = +1, \tag{3.43}$$

when K is an oriented closed manifold and α is a cocycle. This means that $\int_K(\mathrm{Sq}^2(\alpha)+w_2\cup\alpha)$ represents a trivial phase in $(d+1)$ dimensions, and therefore there should be a trivial boundary in d dimensions. We can think of the Gu-Wen Grassmann integral $\sigma(M,\alpha)$ as providing an explicit formula for such a trivial boundary.

Chapter 4

Fermionic phases with time-reversal symmetry

The content of this chapter is based on the author's work [22].

In Chapter 3, we reviewed a way to construct lattice path integral for fermionic topological phases on oriented spacetime manifolds. Meanwhile, it is sometimes useful to consider a fermionic topological phase on an unoriented spacetime manifold [12, 40–43], when the system has a symmetry that reverses the orientation of spacetime. In particular, an unoriented manifold naturally encodes orientation reversing symmetry defects inserted in the spacetime regarded as "background gauge field" of the orientation reversing symmetry. Thus it enables us to see the response of the system to the gauge field of the symmetry, which will be crucial for studying the SPT phases or anomalies based on time-reversal symmetry.

As we have reviewed in Sec. 2.3, the fermionic theory on an unoriented spacetime requires a $\mathrm{Pin}^{\pm}$ structure. For instance, let us think of a $(1+1)$-dimensional topological superconductor with $T^2 = 1$, which follows a $\mathbb{Z}_8$ classification [10]. Cobordism theory reviewed in Sec. 2.2 predicts that the $\mathbb{Z}_8$ classification is diagnosed by computing the partition function of the corresponding theory on a surface generating a Pin^- bordism group $\Omega_2^{\mathrm{pin}^-} = \mathbb{Z}_8$, which is $\mathbb{RP}^2$ equipped with a Pin^- structure. As another example, the $(3+1)$-dimensional topological superconductor with $T^2 = (-1)^F$ is known to be classified by the Pin^+ cobordism group $\Omega^4_{\mathrm{pin}^+} = \mathbb{Z}_{16}$ [15, 20, 44, 45]. The $\mathbb{Z}_{16}$ classification is detected by the partition function on $\mathbb{RP}^4$ equipped with a Pin^+ structure, reflecting that the Pin^+ bordism group is generated by that manifold [46]. In this context, it is important to ask how to formulate the $\mathrm{Pin}^{\pm}$ quantum field theory on a manifold which is not necessarily oriented.

Here, we propose a strategy to produce a lattice definition of $\mathrm{Pin}^{\pm}$ field theory in general dimensions, by extending the recipe reviewed in Chapter 3. Concretely, we obtain the extended Grassmann integral $\sigma(M, \alpha)$ on an unoriented d-manifold M, with a $(d-2)$-form $\mathbb{Z}_2$ symmetry whose background gauge field is $\alpha \in Z^{d-1}(M, \mathbb{Z}_2)$. This is done by modifying the definition of the Grassmann integral properly, in the vicinity of the orientation reversing wall in M, which flips the orientation as we go across the wall. We will show that the Grassmann integral possesses an 't Hooft anomaly for the $(d-2)$-form $\mathbb{Z}_2$ symmetry controlled by the d-dimensional response action

$$(-1)^{\int (\mathrm{Sq}^2(\alpha) + (w_2 + w_1^2)\cup\alpha)}. \tag{4.1}$$

Then, we can define the Pin$^-$ QFT when M admits a Pin$^-$ structure, by coupling with a bosonic theory $\widetilde{Z}_-[\alpha]$ which possesses an anomaly $(-1)^{\int \mathrm{Sq}^2(\alpha)}$,

$$Z_{\mathrm{pin}^-}(M,\eta) = \sum_{\alpha} \widetilde{Z}_-[\alpha]\sigma(M,\alpha)(-1)^{\int_M \eta\cup\alpha}, \tag{4.2}$$

where η specifies a Pin$^-$ structure that satisfies $\delta\eta = w_2 + w_1^2$. We can also construct the Pin$^+$ QFT when M admits a Pin$^+$ structure, by coupling with a bosonic theory $\widetilde{Z}_+[\alpha]$ with an anomaly $(-1)^{\int \mathrm{Sq}^2(\alpha)+w_1^2\cup\alpha}$,

$$Z_{\mathrm{pin}^+}(M,\eta) = \sum_{\alpha} \widetilde{Z}_+[\alpha]\sigma(M,\alpha)(-1)^{\int_M \eta\cup\alpha}, \tag{4.3}$$

where η specifies a Pin$^+$ structure that satisfies $\delta\eta = w_2$.

Unoriented Grassmann integral

Now let us construct the Grassmann integral $\sigma(M,\alpha)$ on a d-manifold M which might be unoriented. We construct an unoriented manifold by picking locally oriented patches, and then gluing them along codimension one loci by transition functions. The locus where the transition functions are orientation reversing, constitutes a representative of the dual of first Stiefel-Whitney class w_1. We will sometimes call the locus an orientation reversing wall. Again, we endow M with a barycentric subdivision for the triangulation of M. We then assign a local ordering to vertices of the barycentric subdivision, such that a vertex on the barycenter of i vertices is labeled as i.

For the oriented case, we have placed a pair of Grassmann variables $\theta_e, \overline{\theta}_e$ on each $(d-1)$-simplex e, whose assignment is determined by the sign of d-simplices $(+,-)$ sharing e. We remark that the assigning rule fails, when e lies on the wall where we glue patches of M by the orientation reversing map. In this case, we would have to assign Grassmann variables of the same color on both sides of e (i.e., both are black (θ) or white ($\overline{\theta}$)), since the two d-simplices sharing e have the identical sign when e is on the orientation reversing wall, see Fig. 4.1 (a). Hence, we need to slightly modify the construction of the Grassmann integral on the orientation reversing wall. To do this, instead of specifying a canonical rule to assign Grassmann variables on the wall, we just place a pair θ_e, $\overline{\theta}_e$ on the wall in an arbitrary fashion. Then, we define the Grassmann integral as

$$\sigma(M,\alpha) = \int \prod_{e|\alpha(e)=1} d\theta_e d\overline{\theta}_e \prod_t u(t) \prod_{e|\mathrm{wall}} (\pm i)^{\alpha(e)}, \tag{4.4}$$

where the $\prod_{e|\mathrm{wall}}(\pm i)^{\alpha(e)}$ term assigns weight $+i^{\alpha(e)}$ (resp. $-i^{\alpha(e)}$) on each $(d-1)$-simplex e on the orientation reversing wall, when e is shared with $+$ (resp. $-$) d-simplices. There is no ambiguity in such definition, since both d-simplices on the side of e have the same sign. This factor makes the Grassmann integral a $\mathbb{Z}_4$ valued quadratic function. The quadratic property is expressed as

$$\sigma(\alpha)\sigma(\alpha') = \sigma(\alpha+\alpha')(-1)^{\int \alpha\cup_{d-2}\alpha'}. \tag{4.5}$$

Basically, the quadratic property is derived in the similar fashion to the oriented spin case. In this case, the net sign consists of three parts;

- the fermionic sign that occurs when reordering $u(t)[\alpha]u(t)[\alpha']$ to $u(t)[\alpha+\alpha']$ on a single d-simplex. The sign on t is expressed as

$$(-1)^{\sum_{e,e'\in t}^{e>e'} \alpha(e)\alpha'(e')}. \tag{4.6}$$

- the fermionic sign by permuting the path integral measure, $(-1)^{\alpha(e)\alpha'(e)}$ on each $(d-1)$-simplex.

- the sign that comes from $i^{\alpha(e)}$ factor on the wall, which is given by comparing $i^{\alpha(e)}i^{\alpha'(e)}$ with $i^{\alpha(e)+\alpha'(e)}$, with the sum of α taken mod 2. This part counts $(-1)^{\alpha(e)\alpha'(e)}$ on the orientation reversing wall.

Analogously to what we did to the second term in (3.39) for the oriented case, we try to re-distribute the fermionic sign from the measure $\prod_e(-1)^{\alpha(e)\alpha'(e)}$ to d-simplices, by assigning $(-1)^{\alpha(e)\alpha'(e)}$ to a $+$ simplex (resp. $-$ simplex) t sharing e, when e is labeled by an odd (resp. even) number. However, such a distribution fails when e is on the orientation reversing wall, due to the mismatch of the sign of two d-simplices on the side of e. Such a distribution counts no sign on the orientation reversing wall. But, this lack of the sign on the wall is complemented by the factor $(-1)^{\alpha(e)\alpha'(e)}$ from the contribution of the $i^{\alpha(e)}$ term, making the re-distribution possible after all. Hence, we can express the net sign in exactly the same fashion as the oriented case (3.38), which proves (4.5).

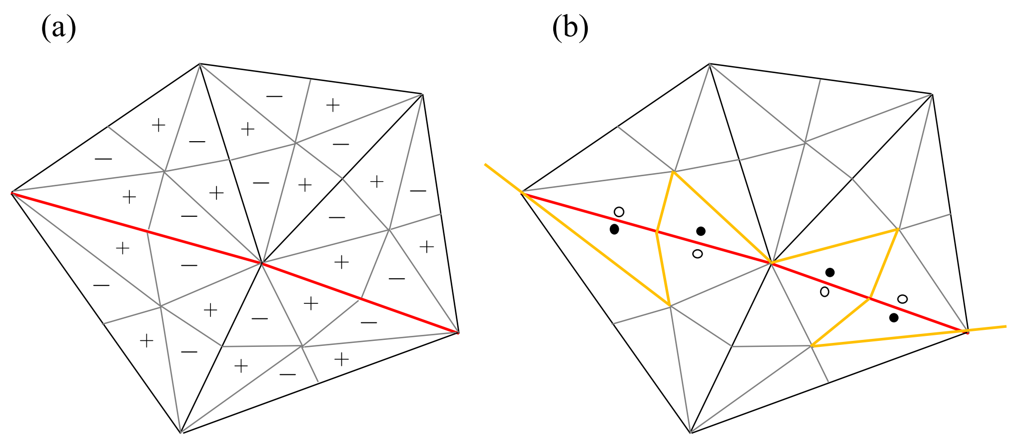

Figure 4.1: (a): The signs of d-simplices near the orientation reversing wall, which is represented as a red line. (b): Assignment of Grassmann variables on the wall specifies a deformation of the wall that intersects the wall transversally at $(d-2)$-simplices.

’t Hooft anomaly

Next, we move on to discuss the ’t Hooft anomaly of the Grassmann integral under the gauge transformation of $(d-2)$-form $\mathbb{Z}_2$ symmetry. The effect of the gauge transformation is determined by a couple of key formulae,

- The quadratic property that we proved in Sec. 4,

$$\sigma(\alpha)\sigma(\alpha') = \sigma(\alpha+\alpha')(-1)^{\int \alpha\cup_{d-2}\alpha'}. \tag{4.7}$$

- When $\alpha = \delta\lambda$ for some $\lambda \in C^{d-2}(M, \mathbb{Z}_2)$, the Grassmann integral is explicitly computed as

$$\sigma(\delta\lambda) = (-1)^{\int_{w_2+w_1^2}\lambda}(-1)^{\int_M \lambda\cup_{d-3}\delta\lambda+\lambda\cup_{d-4}\lambda}, \tag{4.8}$$

where the integral over $w_2 + w_1^2$ means that we sum λ over a $(d-2)$-cycle $S_M \in Z_{d-2}(M, \mathbb{Z}_2)$ Poincaré dual of the Stiefel-Whitney class $w_2 + w_1^2$.

Based on the above relations, the variation of $\sigma(\alpha)$ under the gauge transformation is obtained as

$$\begin{aligned}\sigma(\alpha+\delta\lambda) &= \sigma(\alpha)(-1)^{\int \alpha\cup_{d-2}\delta\lambda}(-1)^{\int_M \lambda\cup_{d-3}\delta\lambda+\lambda\cup_{d-4}\lambda}(-1)^{\int_{w_2+w_1^2}\lambda} \\ &= \sigma(\alpha)(-1)^{\int \alpha\cup_{d-3}\lambda+\lambda\cup_{d-3}\alpha+\lambda\cup_{d-3}\delta\lambda+\lambda\cup_{d-4}\lambda}(-1)^{\int_{w_2+w_1^2}\lambda},\end{aligned} \tag{4.9}$$

where we used the generalized Leibniz rule of the higher cup product, which is reviewed in Appendix B. The part $(-1)^{\int_{w_2+w_1^2}\lambda}$ gives an 't Hooft anomaly with the bulk response action expressed as $(-1)^{\int_{w_2+w_1^2}\alpha}$. Importantly, if we are given a Pin^- structure η of the spacetime, this anomaly can be canceled by coupling $\sigma(\alpha)$ with the term in the form of $(-1)^{\int_M \eta\cup\alpha}$. Concretely, this term is written as

$$(-1)^{\sum_E \alpha} \tag{4.10}$$

using $E \in C_{d-1}(M, \mathbb{Z}_2)$ with $\partial E = S_M$, where E is regarded as the Poincaré dual of η with $\delta\eta = w_2 + w_1^2$, and specifies a choice of a Pin^- structure. Under the gauge transformation $\alpha \mapsto \alpha + \delta\lambda$, this term transforms as

$$(-1)^{\sum_E \alpha+\delta\lambda} = (-1)^{\sum_E \alpha}(-1)^{\sum_{S_M}\lambda} \tag{4.11}$$

and hence cancels the $(-1)^{\int_{w_2+w_1^2}\alpha}$ part of the anomaly. The rest of the anomaly in (4.9) corresponds to the variation of a $(d+1)$-dimensional response action $(-1)^{\int \mathrm{Sq}^2(\alpha)}$, since

$$\begin{aligned}\mathrm{Sq}^2(\alpha+\delta\lambda) - \mathrm{Sq}^2(\alpha) &= \alpha\cup_{d-3}\delta\lambda + \delta\lambda\cup_{d-3}\alpha + \delta\lambda\cup_{d-3}\delta\lambda \\ &= \delta(\alpha\cup_{d-3}\lambda + \lambda\cup_{d-3}\alpha + \lambda\cup_{d-3}\delta\lambda + \lambda\cup_{d-4}\lambda).\end{aligned} \tag{4.12}$$

Thus, the action $\sigma(\alpha)(-1)^{\sum_E \alpha}$ coupled with the $(d+1)$-dimensional response action $(-1)^{\int \mathrm{Sq}^2(\alpha)}$ in the bulk gives a gauge-invariant theory.

Here let us demonstrate (4.8). First, we note that $\sigma(\delta\lambda)$ is a quadratic function of λ, and its quadratic part can be determined by solving the quadratic property

$$\sigma(\delta\lambda+\delta\lambda') = \sigma(\delta\lambda)\sigma(\delta\lambda')(-1)^{\int_M \lambda\cup_{d-3}\delta\lambda'+\lambda'\cup_{d-3}\delta\lambda+\lambda\cup_{d-4}\lambda'+\lambda'\cup_{d-4}\lambda}. \tag{4.13}$$

This is satisfied by an ansatz $(-1)^{\int_M \lambda\cup_{d-3}\delta\lambda+\lambda\cup_{d-4}\lambda}$, so we can fix the form of $\sigma(\delta\lambda)$ up to linear term as

$$\sigma(\delta\lambda) = (-1)^{\sum_S \lambda}(-1)^{\int_M \lambda\cup_{d-3}\delta\lambda+\lambda\cup_{d-4}\lambda}, \tag{4.14}$$

with a $(d-2)$-chain S to be determined. Next, the linear term is obtained by computing $\sigma(\alpha)$ in the simplest case; $\alpha = \delta\lambda$, and $\lambda(v) = 1$ on a single $(d-2)$-simplex of M, otherwise 0. We can see that the quadratic part of $\sigma(\delta\lambda)$ vanishes for such a choice of λ. Once we take a barycentric subdivision, when λ is nonzero away from the orientation reversing wall, one can see that $\sigma(\delta\lambda) = -1$, by imitating the logic of Sec. 4.1. of Gaiotto-Kapustin [21]. See also Fig. 4.2 (a). In the case that λ is nonzero on the orientation reversing wall, the value of $\sigma(\delta\lambda)$ depends on the way of assigning Grassmann variables to $(d-1)$-simplices on the wall such that $\delta\lambda = 1$. For simplicity, we examine the case that $\delta\lambda$ is nonzero on two $(d-1)$-simplices on the wall. (In general, there are even number of such $(d-1)$-simplices. It is not hard to generalize for these situations.) Then, we have two Grassmann variables attached on each side of the orientation reversing wall. When the two Grassmann variables on one side of the wall share the same color (i.e., both are black (θ) or white ($\overline{\theta}$)), we can show that $\sigma(\delta\lambda) = -1$ (see Fig. 4.2 (b')).

On the other hand, if the Grassmann variables on one side have different colors (i.e., one θ and one $\overline{\theta}$), we have $\sigma(\delta\lambda) = +1$ (see Fig. 4.2 (b)). (In these computations, the $\prod(\pm i)^{\delta\lambda(e)}$ term emits no sign, $(+i)\cdot(-i) = 1$.)

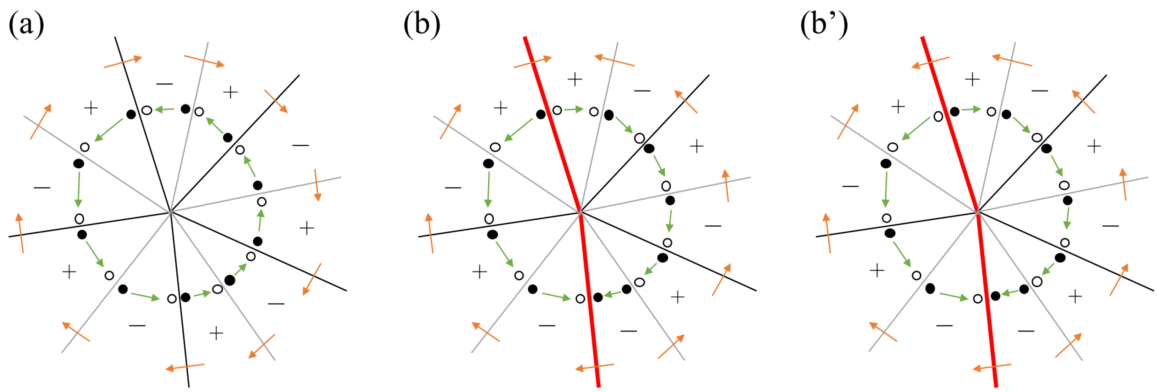

Figure 4.2: When $\lambda(v) = 1$ on a single $(d-2)$ simplex v, Grassmann variables on $(d-1)$-simplices surrounding v are counted in the integral. In the expression of the integral, we encounter $\pm d\vartheta_{2i} d\vartheta_{2i+1}$ measure factors from $(d-1)$-simplices, and $\pm\vartheta_{2i+1}\vartheta_{2i+2}$ integrand factors from d-simplices. The sign $\pm$ from the measure (resp. integrand) is expressed by the orange (resp. green) arrow. For instance, the arrow is directed from ϑ_{2i} to ϑ_{2i+1} if we have a $+$ sign on the measure, otherwise directed in the opposite direction. (a): If v is away from the orientation reversing wall, we can see that all the signs from the measure share the same sign. We can also check that signs from the integrand have the same sign. In such a situation, we have $\sigma(\delta\lambda) = -1$. (b): If v is placed on the orientation reversing wall (red thick line), we have to flip the direction of all arrows on one side of the wall. The total number of flipped arrows is odd; odd number of orange arrows and even number of green arrows. Thus, the value of the integral in (b) has the opposite sign from that of (a). Hence, we have $\sigma(\delta\lambda) = +1$, when the two Grassmann variables attached on one side of the wall have different colors. (b'): On the other hand, we have $\sigma(\delta\lambda) = -1$, when the two Grassmann variables attached on one side of the wall have the same color.

Now we can see that $\sigma(\delta\lambda)$ becomes -1 exactly when λ is nonzero on S_M dual of $w_2 + w_1^2$.

First, let us recall that the set of all $(d-2)$-simplices of the barycentric subdivision gives the representative of the dual of w_2. Thus, we can express the linear term of $\sigma(\delta\lambda)$ as

$$(-1)^{\int_{w_2}\lambda}(-1)^{\sum_{e\in M}\chi(e)\lambda(e)}. \tag{4.15}$$

Here, $(-1)^{\sum_{e\in M}\chi(e)\lambda(e)}=1$ if λ is nonzero away from the orientation reversing wall. When λ is nonzero on the wall, $(-1)^{\sum_{e\in M}\chi(e)\lambda(e)}=1$ (resp. -1) if the two Grassmann variables on one side of the wall have the same (resp. different) color. We can express such a linear term as $(-1)^{\int_{w_1^2}\lambda}$. To see this, first we observe that the choice of the assignment of Grassmann variables on the wall corresponds to choosing the slight deformation of the wall, such that the deformation intersects transversally with the wall at $(d-2)$-simplices. Concretely, we deform the wall on each $(d-1)$-simplices of the wall to the side where θ (black dot) is contained, see Fig. 4.1 (b). Now we can see that $(-1)^{\sum_{e\in M}\chi(e)\lambda(e)}=-1$ when $\lambda=1$ at the intersection of these two walls, otherwise 1. Here, both walls before and after deformation give a representative of the dual of w_1, and thus the intersection of two walls gives a representative of the dual of w_1^2. Hence, we have $(-1)^{\sum_{e\in M}\chi(e)\lambda(e)}=(-1)^{\int_{w_1^2}\lambda}$, proving (4.8).

Summarizing, we have constructed a Pin$^-$ theory in the form of $\sigma(M,\alpha)(-1)^{\sum_E\alpha}$, with the quadratic property (5.16) and an 't Hooft anomaly $(-1)^{\int \mathrm{Sq}^2\alpha}$. We will simply express the Pin$^-$ structure as η with $\delta\eta=w_2+w_1^2$, and write the theory as $\sigma(M,\alpha)(-1)^{\int_M\eta\cup\alpha}$. If we instead have a Pin$^+$ structure, we have $\partial E=S_M$ with S_M a representative of w_2, i.e., a set of all $(d-2)$-simplices of M. In that case, we instead have $\delta\eta=w_2$ and the anomaly is given by $(-1)^{\int \mathrm{Sq}^2\alpha+w_1^2\cup\alpha}$.

WZW-like expressions of the Grassmann integral

We can also see 't Hooft anomaly of $\sigma(M,\alpha)$ more directly. To do this, we introduce an expression of $\sigma(M,\alpha)$ convenient for our purpose. Let us assume that the spacetime manifold M equipped with the background gauge field $\alpha\in Z^{d-1}(M,\mathbb{Z}_2)$ is null-bordant, i.e., M is a boundary of some $(d+1)$-dimensional manifold K and α is extended to K. Then, we can consider the Wess-Zumino-Witten (WZW) like expression of the Grassmann integral

$$\sigma'(M,\alpha)=(-1)^{\int_K \mathrm{Sq}^2\alpha}(-1)^{\sum_{S_K}\alpha}, \tag{4.16}$$

where S_K represents the dual of $w_2+w_1^2$, that is, a set of all $(d-1)$-simplices of K plus extra $(d-1)$-simplices that represent the dual of w_1^2 in K. Due to the Wu relation [39], $\mathrm{Sq}^2(\alpha)+(w_2+w_1^2)\cup\alpha$ is exact for an arbitrary $(d+1)$-dimensional manifold. Hence, the above expression does not depend on the extending manifold K. We can explicitly check that (4.16) satisfies the properties of the Grassmann integral (4.7), (4.8). First, let us check the quadratic property,

$$\begin{aligned}\sigma'(\alpha)\sigma'(\alpha')&=(-1)^{\int_K(\alpha\cup_{d-3}\alpha'+\alpha'\cup_{d-3}\alpha)}\sigma'(\alpha+\alpha')\\&=(-1)^{\int_M\alpha\cup_{d-2}\alpha'}\sigma(\alpha+\alpha').\end{aligned} \tag{4.17}$$

Next, when $\alpha=\delta\lambda$ for some $\lambda\in C^{d-2}(K,\mathbb{Z}_2)$, we have

$$\begin{aligned}\sigma'(\delta\lambda)&=(-1)^{\int_K \mathrm{Sq}^2\delta\lambda}(-1)^{\sum_{S_K}\delta\lambda}\\&=(-1)^{\int_M\lambda\cup_{d-3}\delta\lambda+\lambda\cup_{d-4}\lambda}(-1)^{\sum_{S_M}\lambda},\end{aligned} \tag{4.18}$$

where we used $\partial S_K = S_M$, namely the boundary of S_K again gives the dual of $w_2 + w_1^2$ on M. This will be demonstrated in Sec. 5.1.2. Then, the above WZW definition $\sigma'(M,\alpha)$ is almost identified as $\sigma(M,\alpha)$. Concretely, we can immediately see that the combination $\overline{\sigma}(\alpha)\sigma'(\alpha)$ is a linear function of α with $\sigma(\delta\lambda) = 1$ for coboundaries. So, $\overline{\sigma}(\alpha)\sigma'(\alpha)$ is a linear function on cohomology, and then

$$\sigma(\alpha) = \sigma'(\alpha)\epsilon(\alpha), \tag{4.19}$$

where $\epsilon : H^{d-1}(M,\mathbb{Z}_2) \to \mathbb{Z}_2$ is some linear function that takes value in ± 1. The additional term $\epsilon(\alpha)$ is regarded as a gauge-invariant counterterm which does not affect on the response to gauge transformation or re-triangulation, so we can identify $\sigma(M,\alpha)$ as the WZW expression $\sigma'(M,\alpha)$ for a practical purpose.

Based on the WZW expression, we immediately know the effect of re-triangulation as follows. Suppose we have two configurations of α, orientation reversing walls and triangulations on $M \times \{0\}$ and $M \times \{1\}$ interpolated by $K = M \times [0,1]$. Then, according to the WZW expression for $\overline{\sigma}(M \times \{0\})\sigma(M \times \{1\})$, up to gauge invariant counterterms $\sigma(M \times \{0\})$ is given by

$$\sigma(M \times \{0\}) = (-1)^{\int_K \mathrm{Sq}^2(\alpha)}(-1)^{\sum_{S_K} \alpha} \cdot \sigma(M \times \{1\}), \tag{4.20}$$

where $K = M \times [0,1]$, and α on $M \times \{0\}$, $M \times \{1\}$ is extended to K. This expression directly shows that the effect of gauge transformation and re-triangulation of $\sigma(M,\alpha)$ is controlled by the bulk response action

$$(-1)^{\int_K \mathrm{Sq}^2 \alpha}(-1)^{\sum_{S_K} \alpha}. \tag{4.21}$$

Chapter 5

Gu-Wen SPT phases and their boundaries

The content of this chapter is based on the author's works [22, 23].

In this chapter, we construct lattice path integrals for fermionic SPT phases based on the prescription given in Chapter 4, and study the property of their boundary states. In particular, we focus on the simplest class of intrinsically fermionic phases called the Gu-Wen SPT phases. As mentioned in Sec. 2.4, the d space-time dimensional Gu-Wen SPT phase with the global symmetry G has a complex fermion assigned on the junction of G in codimension $d-1$. Here, we limit ourselves to the case that the bosonic symmetry group is decomposed as $G_0 \times O(d)$. Then, the G_0 connection $g_0 : M \to BG_0$ together with w_1 defines a connection of $G_0 \times \mathbb{Z}_2^T$, $g : M \to B(G_0 \times \mathbb{Z}_2^T)$, where $\mathbb{Z}_2^T$ is the $\mathbb{Z}_2$ subgroup of $O(d)$ generated by the orientation reversing element. From now, we will simply write $G := G_0 \times \mathbb{Z}_2^T$.

Concretely, the Gu-Wen SPT phase is described by a pair of cohomological data

$$(m_{d-1}, x_d) \in Z^{d-1}(BG, \mathbb{Z}_2) \times C^d_\rho(BG, U(1)), \tag{5.1}$$

where the subscript ρ means the twisted cochain with the action of anti-unitary symmetry on $U(1)$ by complex conjugation, see Appendix D for detail. The data m_{d-1} controls the decoration of fermionic degrees of freedom on the junction of G-defects. The above data m_{d-1}, x_d are not independent, but related by the constraint called the Gu-Wen equation. The Gu-Wen equation depends on what spacetime structure we want to consider. For example, in the case of Pin$^-$ structure, we must have

$$\delta_\rho x_d = \frac{1}{2}\mathrm{Sq}^2(m_{d-1}) \qquad \mod 1, \tag{5.2}$$

where the periodicity of $U(1)$ is taken mod 1, $U(1) = \mathbb{R}/\mathbb{Z}$. In the case of Pin$^+$ structure, we instead have

$$\delta_\rho x_d = \frac{1}{2}(\mathrm{Sq}^2(m_{d-1}) + \overline{w}_1^2 \cup m_{d-1}) \qquad \mod 1. \tag{5.3}$$

Here, we define $\overline{w}_1 \in Z^1(BG, \mathbb{Z}_2)$ such that $w_1 = g^*\overline{w}_1$, as a map that sends $\mathbb{Z}_2^T$ odd element of G to 1, otherwise 0. Then, the lattice path integral for the Gu-Wen SPT phases can be described in a similar fashion to the case of bosonic SPT phases reviewed in Sec. 2.1.

A well-understood class of d-dimensional bosonic SPT phases is classified by the ρ-twisted cohomology group $H^d_\rho(BG, U(1))$ [47], in the presence of the time-reversal symmetry. For a given $\omega \in Z^d_\rho(BG, U(1))$, the action of the SPT phase on an unoriented d-manifold M is given by a certain product of weights $g^*\omega$ on each d-simplex of M with the G-gauge field $g : M \to BG$, which is constructed as follows.

First, let us consider the case that the d-simplex t is away from the orientation reversing wall. In this case, we simply define the weight as $g^*\omega^{s(t)}$, where $s(t) = +1$ if t is a $+$ simplex, and $s(t) = -1$ if t is a $-$ simplex, which is identical to the definition of the oriented case. However, when the d-simplex t overlaps with the orientation reversing wall, the definition of the weight described above should be modified, since the choice of the sign $s(t)$ has an ambiguity. To resolve such ambiguity, we first assign $+1$ to every vertex of t on one side of the wall, and assign -1 on the other side. Then, we define $s(t)$ as the sign given by comparing the ordering on t and the orientation of M on the side of vertices labeled by $+1$. Let us denote ϵ as the number ± 1 assigned on the vertex of the smallest ordering in t. Then, we define the weight on t as $g^*\omega^{\epsilon\cdot s(t)}$.

We note that this definition is independent of the choice of assigning ± 1 to one side of the wall, since flipping the sign of ± 1 on vertices changes the sign of ϵ and $s(t)$ simultaneously, which leaves $g^*\omega^{\epsilon\cdot s(t)}$ invariant. Then, let us write the action as the product of weights for all d-simplices in M. We simply denote the action as $\int_M g^*\omega$. One can see that the action defined in such a manner is invariant under G-gauge transformation [41]. If we take a general cochain $x \in C^d_\rho(BG, U(1))$ which is not necessarily a cocycle, we can see that $\int_M g^*x$ is no longer invariant under the gauge transformation, whose variation is controlled by a response action in one more dimension $\int g^*(\delta_\rho x)$.

Now, we are ready to consider the fermionic case. For a given $g : M \to BG$ where M is a Pin$^-$ d-dimensional manifold, The Gu-Wen SPT phase based on Pin$^-$ structure is constructed by using the Grassmann integral constructed in Chapter 4 as

$$\sigma(g^*m_{d-1})(-1)^{\int_M \eta\cup g^*m_{d-1}} \exp(2\pi i \int_M g^*x_d), \tag{5.4}$$

where $\delta\eta = w_2 + w_1^2$ specifies the chosen Pin$^-$ structure. We note that this expression is free of 't Hooft anomalies, since the $\sigma(g^*m_{d-1})(-1)^{\int_M \eta\cup g^*m_{d-1}}$ carries the 't Hooft anomaly characterized by $(-1)^{\int \mathrm{Sq}^2(g^*m_{d-1})}$, which is canceled by the bosonic term $\exp(2\pi i \int_M g^*x_d)$ due to the Gu-Wen equation (5.2).

The same construction also works for the case of Pin$^+$ Gu-Wen phases, where the structure η is modified as $\delta\eta = w_2$. Then, the bosonic term $\exp(2\pi i \int_M g^*x_d)$ based on the Gu-Wen equation (5.3) again cancels the anomaly of the Grassmann integral in the expression (5.2).

5.1 Gapped boundary of Gu-Wen SPT phase

In the rest of this chapter, we study the boundary states of Gu-Wen SPT phases. Since the boundary of SPT phases have an 't Hooft anomaly for the global symmetry, the spectrum at long distances becomes nontrivial. If the anomaly is the perturbative one, the anomaly must be saturated by gapless degrees of freedom. However, for a general non-perturbative anomaly including the Gu-Wen anomaly, this is not always the case; the anomaly can be matched by gapped topological ordered state without spontaneous symmetry breaking.

We demonstrate that Gu-Wen Pin G-SPT phases admit a symmetry-preserving gapped boundary, by writing down the explicit d-dimensional action on the boundary of $(d+1)$-dimensional Gu-Wen Pin G-SPT phase specified by the Gu-Wen data (n_d, y_{d+1}). To construct the gapped boundary, we prepare a symmetry extension by a (0-form) symmetry $\widetilde{K}$ [48],

$$0 \to \widetilde{K} \to \widetilde{H} \xrightarrow{\widetilde{p}} G \to 0, \tag{5.5}$$

such that n_d trivializes as an element of $H^d(B\widetilde{H}, \mathbb{Z}_2)$; $[\widetilde{p}^* n_d] = 0 \in H^d(B\widetilde{H}, \mathbb{Z}_2)$. When G is finite, such an extension can be prepared by generalizing the argument of [49].

We now take $\widetilde{m}_{d-1} \in C^{d-1}(B\widetilde{H}, \mathbb{Z}_2)$ such that $\widetilde{p}^* n_d = \delta \widetilde{m}_{d-1}$. In the Pin$^-$ case, we see that $z_{d+1} = \widetilde{p}^* y_{d+1} - \mathrm{Sq}^2(\widetilde{m}_{d-1})$ is a (ρ-twisted) cocycle, where the Steenrod square for a cochain that is not closed is given by

$$\mathrm{Sq}^2(\widetilde{m}_{d-1}) = \widetilde{m}_{d-1} \cup_{d-3} \widetilde{m}_{d-1} + \delta \widetilde{m}_{d-1} \cup_{d-2} \widetilde{m}_{d-1}, \tag{5.6}$$

as reviewed in Appendix B. Therefore, for the Pin$^-$ case the bulk Gu-Wen data pull back to

$$(\delta \widetilde{m}_{d-1}, \mathrm{Sq}^2(\widetilde{m}_{d-1}) + z_{d+1}). \tag{5.7}$$

In the Pin$^+$ case, we instead define the ρ-twisted cocycle $z_{d+1} = \widetilde{p}^* y_{d+1} - \mathrm{Sq}^2(\widetilde{m}_{d-1}) - (\widetilde{p}^* \overline{w}_1)^2 \cup \widetilde{m}_{d+1}$. Then, one can see that the Gu-Wen data pull back to

$$(\delta \widetilde{m}_{d-1}, \mathrm{Sq}^2(\widetilde{m}_{d-1}) + (\widetilde{p}^* \overline{w}_1)^2 \cup \widetilde{m}_{d-1} + z_{d+1}). \tag{5.8}$$

Without loss of generality we can assume that $z_{d+1} = \delta_\rho x_d$ for some $x_d \in C^d_\rho(BH, U(1))$, by a further extension of the symmetry

$$0 \to K \to H \xrightarrow{p} \widetilde{H} \to 0. \tag{5.9}$$

Again, such an extension for twisted cocycle can be prepared by generalizing the argument of [49]. We set $m_{d-1} = p^* \widetilde{m}_{d-1}$. We now expect that the action on the boundary is given by the K-gauge theory,

$$Z_{\text{boundary gauge}} \propto \sum_{p(h)=g} \sigma(h^* m_{d-1}) (-1)^{\int_M \eta \cup h^* m_{d-1}} \exp(2\pi i \int_M h^* x_d), \tag{5.10}$$

with $h : M \to BH$. But to make sense of this expression we have to extend the definition of the Gu-Wen Grassmann integral $\sigma(\alpha_{d-1})$ to the case when $\alpha_{d-1} \in C^{d-1}(M, \mathbb{Z}_2)$ is not necessarily closed. By extending the Grassmann integral to the case of a bulk-boundary system, we will see that the extended Gu-Wen integral nicely couples to the bulk in a gauge invariant fashion.

5.1.1 Bulk-boundary Gu-Wen Grassmann integral for the pin case

When we naively use the above definition (3.32) when α is not closed: $\delta\alpha = \beta$, the resulting expression is problematic since $u(t)$ can become Grassmann-odd. Following [23], we avoid this conundrum by coupling with the Gu-Wen integral $\sigma(N,\beta)$ in $(d+1)$ dimensional bulk N such that $\partial N = M$, making all components in the path integral Grassmann-even.

Now let us write down the boundary Gu-Wen integral coupled with bulk; we denote the entire integral by $\sigma(\alpha;\beta)$. We assign Grassmann variables $\theta_e, \overline{\theta}_e$ on each $(d-1)$-simplex e of M, and $\theta_f, \overline{\theta}_f$ on each d-simplex f of $N \setminus M$. We define the Gu-Wen integral as

$$\sigma(\alpha;\beta) = \int \prod_{f|\beta(f)=1} d\theta_f d\overline{\theta}_f \int \prod_{e|\alpha(e)=1} d\theta_e d\overline{\theta}_e \prod_t u(t) \prod_{f|\text{wall}} (\pm i)^{\beta(f)} \prod_{e|\text{wall}} (\pm i)^{\alpha(e)}, \tag{5.11}$$

where we assume that the orientation reversing wall in N intersects M transversally at $(d-1)$-simplices, which are regarded as making up the wall in M. $u(t)$ is a monomial of Grassmann variables defined on a $(d+1)$-simplex of N. $u(t)[\beta]$ is defined in the same fashion as in the case without boundary if t is away from the boundary, but modified when t shares a d-simplex with the boundary. For simplicity, we assign an ordering on vertices of such $t = (01\dots d+1)$, so that the d-simplex shared with M becomes $f_0 = (12\dots d+1)$; the vertex 0 is contained in $N \setminus M$. For instance, we can take a barycentric subdivision on N, and assign 0 to vertices associated with $(d+1)$-simplices. We further define the sign of d-simplices on M, such that f_0 and t have the same sign.

Then, $u(t)$ neighboring with M is defined by replacing the position of ϑ_{f_0} in $u(t)[\beta]$ with the boundary action on f_0, $u(f_0)[\alpha] = \prod_{e\in f_0} \vartheta_e^{\alpha(e)}$. We then have: On a $+$ simplex,

$$u(t) = u(f_0)[\alpha] \cdot \prod_{f\in\partial t, f\neq f_0} \vartheta_f^{\beta(f)}. \tag{5.12}$$

On a $-$ simplex,

$$u(t) = \prod_{f\in\partial t, f\neq f_0} \vartheta_f^{\beta(f)} \cdot u(f_0)[\alpha]. \tag{5.13}$$

One can check that $u(t)$ defined above becomes Grassmann-even. Then, using the exactly same logic as Sec. 4, one can obtain the quadratic property of $\sigma(\alpha;\beta)$ as

$$\sigma(\alpha+\alpha';\beta+\beta') = \sigma(\alpha;\beta)\sigma(\alpha';\beta')(-1)^{\int_M (\alpha\cup_{d-2}\alpha' + \alpha\cup_{d-1}\delta\alpha') + \int_N \beta\cup_{d-1}\beta'}. \tag{5.14}$$

5.1.2 Effect of gauge transformations

In this section, we study the effect of gauge transformation of the $(d-1)$-form symmetry $\beta \mapsto \beta + \delta\lambda$ on N, which also transforms α on M via $\alpha \to \alpha + \lambda$. When N is closed, we have seen in Sec. 4 that the $(d-1)$-form symmetry has an 't Hooft anomaly, and becomes gauge invariant after coupling with the $(d+2)$-dimensional response action $(-1)^{\int_{\widetilde{N}} \mathrm{Sq}^2 \beta}$ on a $(d+2)$-dimensional manifold $\widetilde{N}$ with $\partial\widetilde{N} = N$ and considering it as a bulk-boundary system. However, when N has a boundary $M = \partial N$, the effect of gauge transformations of our theory becomes more subtle. In that case, the variation of the action is again controlled by a response action in one more dimension, but the way to couple the theory with the response action is modified as represented in Fig. 5.1, due to the presence of the boundary.

That is, a d-dimensional manifold M is bounded by $\widetilde{M}$ with $\partial\widetilde{M} = M$ where a $(d+1)$-dimensional response action canceling the gauge variation on M is supported. Further, we have a $(d+2)$-dimensional manifold $\widetilde{N}$ with $\partial\widetilde{N} = N \sqcup \widetilde{M}$ where a $(d+2)$-dimensional response action canceling the variation on N is supported. After all, M looks like a "corner" in the resulting geometry of the bulk-boundary system.

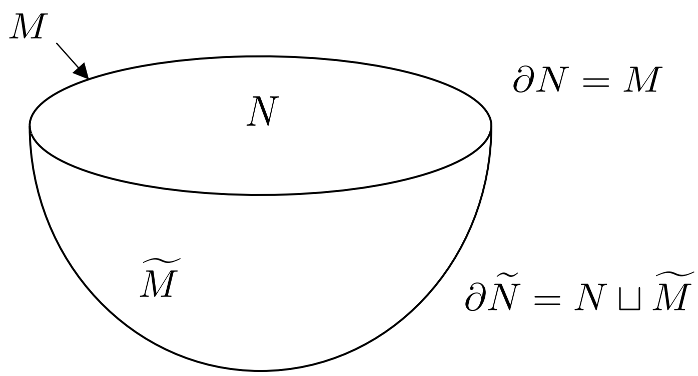

Figure 5.1: The geometry for the "anomaly inflow" of our bulk-boundary Grassmann integral. The gauge variation of the bulk-boundary Grassmann integral supported on M and N is accounted for by the response action on $\widetilde{M}$ and $\widetilde{N}$, respectively.

Then, we will see that the gauge variation of $\sigma(\alpha;\beta)$ under the gauge transformation $\alpha \mapsto \alpha+\lambda, \beta \mapsto \beta+\delta\lambda$ is precisely controlled by the response action on $\widetilde{M}$ and $\widetilde{N}$ expressed as

$$(-1)^{\int_{\widetilde{M}} \mathrm{Sq}^2 \alpha + (w_2+w_1^2)\cup\alpha}(-1)^{\int_{\widetilde{N}} \mathrm{Sq}^2 \beta + (w_2+w_1^2)\cup\beta}, \tag{5.15}$$

where we assume that α and β with $\delta\alpha = \beta$ extends to $\widetilde{M}$ and $\widetilde{N}$, respectively. Analogously to what we did in Sec. 4, this can be again shown by using a couple of formulae

- The quadratic property obtained in (5.14),

$$\sigma(\alpha+\alpha';\beta+\beta') = \sigma(\alpha;\beta)\sigma(\alpha';\beta')(-1)^{\int_M (\alpha\cup_{d-2}\alpha' + \alpha\cup_{d-1}\delta\alpha') + \int_N \beta\cup_{d-1}\beta'}. \tag{5.16}$$

- When β is a coboundary $\beta = \delta\lambda$, the bulk-boundary Grassmann integral is explicitly computed as

$$\sigma(\lambda;\delta\lambda) = (-1)^{\int_N \mathrm{Sq}^2 \lambda}(-1)^{\sum_{e\in S}\lambda(e)}, \tag{5.17}$$

  where S denotes the set of $(d-1)$-simplices of N that represents $w_2+w_1^2$ on N/M, and further satisfies that ∂S gives the representative of $w_2+w_1^2$ on the boundary M. The definition of S will be explained in detail later.

Let us demonstrate (5.17). The quadratic part of $\sigma(\lambda;\delta\lambda)$ is determined by noting that the quadratic property

$$\sigma(\lambda+\lambda';\delta\lambda+\delta\lambda') = \sigma(\lambda;\delta\lambda)\sigma(\lambda';\delta\lambda')(-1)^{\int_M \lambda\cup_{d-2}\lambda'}(-1)^{\int_N \lambda\cup_{d-2}\delta\lambda' + \delta\lambda'\cup_{d-2}\lambda}, \tag{5.18}$$

is solved by $(-1)^{\mathrm{Sq}^2\lambda}$ up to a linear term. So, $\sigma(\lambda;\delta\lambda)$ can be expressed as

$$\sigma(\lambda;\delta\lambda) = (-1)^{\int_N \mathrm{Sq}^2 \lambda}(-1)^{\sum_{e\in S}\lambda(e)}, \tag{5.19}$$

with S some set of $(d-1)$-simplices e of N. The linear term is fixed by computing $\sigma(\lambda;\delta\lambda)$ explicitly in the simplest case; $\lambda(e)=1$ on a single $(d-1)$-simplex, otherwise 0. If we take a barycentric subdivision on N, we can see that $\sigma(\lambda;\delta\lambda)=-1$ when λ is nonzero on the dual of $w_2+w_1^2$ described in Sec. 4, at least if λ is nonzero away from the boundary of N. When λ is nonzero on the boundary, it requires more careful treatment. In this situation, by arranging the sign of f_0 chosen to be identical to t, we can see that $\sigma(\lambda;\delta\lambda)=-1$ when λ is nonzero in M away from the orientation reversing wall. In the case that $\lambda(e)$ is nonzero for a $(d-1)$-simplex e on the orientation reversing wall of M, there is a d-simplex f of N on the orientation reversing wall that bounds e. By arranging the color of the Grassmann variables on e to be the same as those on f for each side of the orientation reversing wall, we can see that $\sigma(\lambda;\delta\lambda)=-1$ also holds on the wall of M.

Summarizing, the $(d-1)$-chain S in (5.19) consists of all $(d-1)$-simplices of N that represents the dual of w_2, plus the extra $(d-1)$-simplices that represent w_1^2 of N as the intersection of the orientation reversing wall with itself. Hence we express the linear term as (5.17). An important property of S is that the $(d-2)$-cycle ∂S gives a representative for the dual of $w_2+w_1^2$ on M. To see this, let us further set $\lambda=\delta\chi$ for $\sigma(\lambda;\delta\lambda)$. Then, $\sigma(\delta\chi;0)$ reduces to the ordinary Grassmann integral $\sigma(\delta\chi)$ supported on M, which has the form of (4.8)

$$\sigma(\delta\chi)=(-1)^{\int_M \mathrm{Sq}^2\chi}(-1)^{\int_{w_2+w_1^2}\chi}. \tag{5.20}$$

Meanwhile, using $\mathrm{Sq}^2(\delta\chi)=\delta(\mathrm{Sq}^2\chi)$, we can see that

$$\sigma(\delta\chi;0)=(-1)^{\int_M \mathrm{Sq}^2\chi}(-1)^{\sum_{v\in\partial S}\chi(v)}. \tag{5.21}$$

These expressions should be identical, which shows that ∂S constitutes the dual of $w_2+w_1^2$ on M.

Based on (5.16), (5.17), the gauge transformation of $\sigma(\alpha;\beta)$ is expressed as

$$\begin{aligned}\sigma(\alpha+\lambda;\beta+\delta\lambda)&=\sigma(\alpha;\beta)\sigma(\lambda;\delta\lambda)(-1)^{\int_M(\alpha\cup_{d-2}\lambda+\alpha\cup_{d-1}\delta\lambda)}(-1)^{\int_N\beta\cup_{d-1}\delta\lambda}\\&=\sigma(\alpha;\beta)(-1)^{\int_M(\alpha\cup_{d-2}\lambda+\alpha\cup_{d-1}\delta\lambda)}(-1)^{\int_N(\beta\cup_{d-1}\delta\lambda+\mathrm{Sq}^2\lambda)}(-1)^{\sum_{e\in S}\lambda(e)}.\end{aligned} \tag{5.22}$$

Firstly, let us observe that the quadratic part of the gauge variation (5.22) corresponds to the gauge variation of the bulk response action

$$(-1)^{\int_{\widetilde{M}}\mathrm{Sq}^2\alpha}(-1)^{\int_{\tilde{N}}\mathrm{Sq}^2\beta}, \tag{5.23}$$

which transforms as

$$\begin{aligned}&(-1)^{\int_{\widetilde{M}}\mathrm{Sq}^2\alpha}(-1)^{\int_{\widetilde{M}}\lambda\cup_{d-3}\alpha+\alpha\cup_{d-3}\lambda+\delta\lambda\cup_{d-2}\alpha+\lambda\cup_{d-3}\lambda+\delta\alpha\cup_{d-2}\lambda+\delta\lambda\cup_{d-2}\lambda}\times\\&(-1)^{\int_{\tilde{N}}\mathrm{Sq}^2\beta}(-1)^{\int_{\tilde{N}}\beta\cup_{d-2}\delta\lambda+\delta\lambda\cup_{d-2}\beta+\delta\lambda\cup_{d-2}\delta\lambda}\\=&(-1)^{\int_{\widetilde{M}}\mathrm{Sq}^2\alpha}(-1)^{\int_{\widetilde{M}}\delta(\alpha\cup_{d-2}\lambda+\alpha\cup_{d-1}\delta\lambda)+\beta\cup_{d-1}\delta\lambda+\lambda\cup_{d-3}\lambda+\delta\lambda\cup_{d-2}\lambda}\times\\&(-1)^{\int_{\tilde{N}}\mathrm{Sq}^2\beta}(-1)^{\int_{\tilde{N}}\delta(\beta\cup_{d-1}\delta\lambda+\lambda\cup_{d-3}\lambda+\delta\lambda\cup_{d-2}\lambda)}\\=&(-1)^{\int_{\widetilde{M}}\mathrm{Sq}^2\alpha}(-1)^{\int_{\tilde{N}}\mathrm{Sq}^2\beta}(-1)^{\int_M(\alpha\cup_{d-2}\lambda+\alpha\cup_{d-1}\delta\lambda)}(-1)^{\int_N(\beta\cup_{d-1}\delta\lambda+\mathrm{Sq}^2\lambda)}.\end{aligned} \tag{5.24}$$

We can see that the effect of the gauge transformation is identical to the variation of $\sigma(\alpha;\beta)$ up to the linear term.

Secondly, the linear term in (5.22) can be canceled by adding the term that encodes the Pin^- structure of the spacetime. To see this, we prepare a Pin^- structure on M represented by $E_M \in C_{d-1}(M,\mathbb{Z}_2)$ with $\partial E_M = \partial S$, where ∂S represents $w_2+w_1^2$ on M. Then, we can see that $S' = E_M + S$ is closed $S' \in Z^{d-1}(N,\mathbb{Z}_2)$, and further matches the representative of the obstruction class $w_2+w_1^2$ as an element of $Z_{d-1}(N;\partial N,\mathbb{Z}_2)$. Thus, the Pin^- structure on N is prepared as the trivialization of S' prepared by $\partial E_N = S'$. After all, we consider the combination

$$\sigma(\alpha;\beta)(-1)^{\sum_{e\in E_M}\alpha(e)}(-1)^{\sum_{f\in E_N}\beta(f)}, \tag{5.25}$$

which precisely cancels the linear term in (5.22), and carries the 't Hooft anomaly

$$(-1)^{\int_{\widetilde{M}}\mathrm{Sq}^2\alpha}(-1)^{\int_{\tilde{N}}\mathrm{Sq}^2\beta}. \tag{5.26}$$

For convenience, let us simply denote the Pin^- structure on both bulk and boundary as η, and write the action (5.25) as

$$\sigma(\alpha;\beta)(-1)^{\int_M \eta\cup\alpha}(-1)^{\int_N \eta\cup\beta}. \tag{5.27}$$

If we instead have a Pin^+ structure, E_M and E_N are the trivialization of w_2 on the boundary and bulk, i.e., S is taken as a set of all $(d-2)$-simplices of N that represents w_2 and then define $\partial E_M = \partial S, \partial E_N = S + E_M$. Then, the 't Hooft anomaly of the theory (5.25) is given by

$$(-1)^{\int_{\widetilde{M}}\mathrm{Sq}^2\alpha + w_1^2\cup\alpha}(-1)^{\int_{\tilde{N}}\mathrm{Sq}^2\beta+w_1^2\cup\beta}. \tag{5.28}$$

5.1.3 WZW-like expressions of the bulk-boundary Grassmann integral

Analogously to what we did in Sec. 4, we also have a useful expression for the bulk-boundary Grassmann integral $\sigma(\alpha;\beta)$, which allows us to observe the 't Hooft anomaly directly. Assuming the geometry shown in Fig. 5.1 with α and β extended to $\widetilde{M}$ and $\widetilde{N}$ respectively, we can express $\sigma(\alpha;\beta)$ up to gauge invariant counterterm as

$$\sigma(\alpha;\beta) = (-1)^{\int_{\widetilde{M}}\mathrm{Sq}^2\alpha}(-1)^{\int_{\tilde{N}}\mathrm{Sq}^2\beta}(-1)^{\sum_{S_{\widetilde{M}}}\alpha}(-1)^{\sum_{S_{\tilde{N}}}\beta}, \tag{5.29}$$

where $S_{\widetilde{M}}$ represents the dual of $w_2+w_1^2$ on $\widetilde{M}$, and further $\partial S_{\widetilde{M}} = S_M$ represents the dual of $w_2+w_1^2$ on M. Similarly, $S_{\tilde{N}}$ represents the dual of $w_2+w_1^2$ on $\widetilde{N}$, and further $\partial S_{\tilde{N}} = S_N + S_{\widetilde{M}}$ represents the dual of $w_2+w_1^2$ on $N \sqcup \widetilde{M}$. [1]

[1] To be precise, we have $\partial S_{\tilde{N}} = S_N + S_{\widetilde{M}} + s$, where $s \in Z_{d-1}(M,\mathbb{Z}_2)$ is the set of all $(d-1)$-simplices of M, since N and $\widetilde{M}$ shares the boundary M. The effect of s can be canceled by adding the gauge invariant counterterm $(-1)^{\sum_s \alpha}$, and then this effect doesn't affect on the analysis in this section.

Firstly, one can check that the quadratic property (5.16) is satisfied for (5.29), since

$$\begin{aligned}\int_{\widetilde{M}} \mathrm{Sq}^2(\alpha+\alpha') + \int_{\widetilde{N}} \mathrm{Sq}^2(\beta+\beta') &= \int_{\widetilde{M}} (\mathrm{Sq}^2\alpha + \mathrm{Sq}^2\alpha') + \int_{\widetilde{M}} (\mathrm{Sq}^2\beta + \mathrm{Sq}^2\beta') \\ &\quad + \int_{\widetilde{M}} \delta(\alpha\cup_{d-2}\alpha' + \alpha\cup_{d-1}\delta\alpha') + \delta\alpha\cup_{d-1}\delta\alpha' \\ &\quad + \int_{\widetilde{N}} \delta(\beta\cup_{d-1}\beta') \\ &= \int_{\widetilde{M}} (\mathrm{Sq}^2\alpha + \mathrm{Sq}^2\alpha') + \int_{\widetilde{M}} (\mathrm{Sq}^2\beta + \mathrm{Sq}^2\beta') \\ &\quad + \int_M \alpha\cup_{d-2}\alpha' + \alpha\cup_{d-1}\delta\alpha' + \int_N \beta\cup_{d-1}\beta'.\end{aligned} \tag{5.30}$$

Secondly, when the gauge fields are expressed as $\alpha = \delta\chi, \beta = 0$, it reduces to the Grassmann integral supported on M, since we have

$$\sigma(\delta\chi; 0) = (-1)^{\int_M \mathrm{Sq}^2\chi}(-1)^{\sum_{S_M}\chi}, \tag{5.31}$$

which reproduces (4.8).

Finally, for the case that $\alpha = \lambda, \beta = \delta\lambda$, we have

$$\sigma(\lambda; \delta\lambda) = (-1)^{\int_N \mathrm{Sq}^2\lambda}(-1)^{\sum_{S_N}\lambda}. \tag{5.32}$$

which reproduces (5.17). Thus, the WZW-like expression (5.29) corresponds to the bulk-boundary Grassmann integral up to a gauge invariant counterterm (which is a linear function of $H^d(N;\partial N,\mathbb{Z}_2)\times H^{d-1}(M,\mathbb{Z}_2)$). This expression of $\sigma(\alpha;\beta)$ directly shows that

$$\sigma(\alpha;\beta)(-1)^{\sum_{e\in E_M}\alpha(e)}(-1)^{\sum_{f\in E_N}\beta(f)} \tag{5.33}$$

has the 't Hooft anomaly characterized by the response action

$$(-1)^{\int_{\widetilde{M}}\mathrm{Sq}^2\alpha}(-1)^{\int_{\widetilde{N}}\mathrm{Sq}^2\beta}. \tag{5.34}$$

5.1.4 Gapped boundary for the Gu-Wen pin phase

After all these preparations, it is a simple matter to show that the boundary gauge theory (5.10) correctly couples to the bulk Gu-Wen Pin SPT phase. Indeed, the partition function of the coupled system has the action

$$\sigma(h^*m_{d-1}; g^*n_d)(-1)^{\int_M \eta\cup h^*m_{d-1}}(-1)^{\int_N \eta\cup g^*n_d}\exp\left(-2\pi i\int_M h^*x_d + 2\pi i\int_N g^*y_{d+1}\right) \tag{5.35}$$

for both Pin^- and Pin^+ case, where we take $\alpha = h^*m_{d-1}$ and $\beta = g^*n_d$. For the Pin^- case, the last bosonic term in (5.35) has the variation under the gauge transformation controlled by

$$(-1)^{\int_{\widetilde{M}}\mathrm{Sq}^2(h^*m_{d-1})}(-1)^{\int_{\widetilde{N}}\mathrm{Sq}^2(g^*n_d)}, \tag{5.36}$$

due to the Gu-Wen equation. The anomaly is precisely canceled by the fermionic terms in the form of (5.27). For the Pin^+ case, the bosonic term instead carries the anomaly

$$(-1)^{\int_{\widetilde{M}} \mathrm{Sq}^2(h^* m_{d-1})+w_1^2\cup h^* m_{d-1}}(-1)^{\int_{\widetilde{N}} \mathrm{Sq}^2(g^* n_d)+w_1^2\cup h^* n_d}, \tag{5.37}$$

which is again canceled by the fermionic term. [2]

[2]Technically, the orientation reversing wall used to define the Grassmann integral is defined on chains of the triangulation, while w_1 that appears in the bosonic side is a cochain. In fact, there exists a map f_∞ which turns a cochain into a chain, $f_\infty : Z^k(M,\mathbb{Z}_2) \to Z_{d-k}(M,\mathbb{Z}_2)$. The map satisfies for a given $\beta \in Z^k(M,\mathbb{Z}_2)$ that $\int \alpha \cup \beta = \int_{f_\infty \beta} \alpha$ for any $\alpha \in C^{d-k}(M,\mathbb{Z}_2)$. Then, the orientation reversing wall can be taken to be $f_\infty w_1$ using $w_1 \in Z^1(M,\mathbb{Z}_2)$, and we conjecture that this choice of the orientation reversing wall makes the 't Hooft anomaly of the combined action (5.4) precisely canceled with each other in the Pin^+ case. See Appendix D of [50] for the detailed description of the f_∞ map.

Chapter 6

(1+1)d topological superconductor

The content of this chapter is based on the author's works [22, 24].

In this chapter, we construct a lattice path integral for a field theory that describes a (1+1)-dimensional topological superconductor called a Kitaev wire. As reviewed in Sec. 2.2, in the presence of a time-reversal symmetry with $T^2 = 1$, the Kitaev wire generates the SPT phase classified by $\mathbb{Z}_8$ [10]. The SPT classification corresponds to Pin$^-$ cobordism group $\Omega^2_{\mathrm{pin}^-} = \mathbb{Z}_8$, and its generator is known as an invertible topological field theory whose partition function on a closed manifold becomes the Arf-Brown-Kervaire (ABK) invariant [46]. We also construct a Hilbert space of the above theory for the ABK invariant, built on the Fock space of complex fermions.

Then, we consider the following problem; if we are given a wave function of a (1+1)-dimensional fermionic gapped invertible phase, find a way to diagnose the SPT classification of the wave function. While we cannot characterize topological phases via local order parameters, the quantities constructed from the ground states by acting with a non-local operation, can sometimes detect topological classifications. We loosely call these quantities defined via non-local operations as non-local order parameters.

In the presence of time-reversal symmetry, it was proposed [51, 52] that one can obtain a quantized non-local parameter for SPT phases defined via the operation called "partial time-reversal" on the reduced density matrix of the ground state. In particular, the non-local order parameter for the Kitaev wire with time-reversal symmetry was computed in [52], and it was found in [52] that its phase distinguishes the $\mathbb{Z}_8$ classification of topological superconductors. In this chapter, we study the connection between the proposed non-local order parameters and the underlying field theory constructed in Sec. 6.1. We clarify the nature of the non-local order parameters as the quantized ABK invariant, by computing the non-local order parameter on the state prepared by the path integral for the ABK invariant.

this chapter is organized as follows. In Sec. 6.1, we obtain a path integral whose partition function gives the ABK invariant. In Sec. 6.2, we construct the Hilbert space for the theory and then evaluate the non-local order parameter that diagnoses the $\mathbb{Z}_8$ classification.

6.1 Path integral: Arf and ABK invariant

Here, we construct the $\mathbb{Z}_8$-valued ABK invariant on a two-dimensional Pin^- manifold, by fermionizing a bosonic theory by coupling with the Grassmann integral on lattice, generalizing the fermionization procedure reviewed in Chapter 3 to the Pin^- case.

The weight for the bosonic theory on a two dimensional triangulated manifold M is assigned in the same manner as the case of the Arf invariant for oriented spin manifolds [21], described as follows. For a given configuration $\alpha \in C^1(M, \mathbb{Z}_2)$, we assign weight $1/2$ to each 1-simplex e, and also assign weight 2 to each 2-simplex f when $\delta\alpha = 0$ at f, otherwise 0. Let us denote the product of the whole weight as $\widetilde{Z}(\alpha)$. Then, we consider a Pin^- theory obtained by gauging the $\mathbb{Z}_2$ symmetry,

$$\begin{aligned} Z[M,\eta] &= \sum_{\alpha\in Z^1(M,\mathbb{Z}_2)} \sigma(M,\alpha)(-1)^{\int_M \eta\cup\alpha}\widetilde{Z}(\alpha) \\ &= 2^{|F|-|E|}\cdot \sum_{\alpha\in Z^1(M,\mathbb{Z}_2)} \sigma(M,\alpha)(-1)^{\int_M \eta\cup\alpha} \\ &= 2^{\chi(M)-1}\cdot \sum_{[\alpha]\in H^1(M,\mathbb{Z}_2)} \sigma(M,\alpha)(-1)^{\int_M \eta\cup\alpha} \\ &= \sqrt{2}^{\chi(M)}\mathrm{ABK}[M,\eta], \end{aligned} \tag{6.1}$$

where $|F|$, $|E|$ denotes the number of 2-simplices, 1-simplices in M, respectively. $\chi(M)$ denotes the Euler number of M, and $\mathrm{ABK}[M,\eta]$ is the ABK invariant,

$$\mathrm{ABK}[M,\eta] = \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_2)|}} \sum_{[\alpha]\in H^1(M,\mathbb{Z}_2)} i^{Q_\eta[\alpha]}. \tag{6.2}$$

Here, $i^{Q_\eta[\alpha]} = \sigma(M,\alpha)(-1)^{\int_M \eta\cup\alpha}$ is a $\mathbb{Z}_4$-valued quadratic function that satisfies

$$Q_\eta[\alpha] + Q_\eta[\alpha'] = Q_\eta[\alpha+\alpha'] + 2\int_M \alpha\cup\alpha'. \tag{6.3}$$

The ABK invariant determines the Pin^- bordism class of two dimensional manifolds $\Omega_2^{\mathrm{pin}^-}(\mathrm{pt}) = \mathbb{Z}_8$, which is generated by $\mathbb{RP}^2$ [46]. To compute the partition function on $\mathbb{RP}^2$, let α be a nontrivial 1-cocycle that generates $H^1(\mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2$. Then, using the quadratic property for $\alpha = \alpha'$ in (6.3), one can see that $Q_\eta[\alpha]$ takes value in ± 1, since $Q_\eta[0] = 0$ and $\int_M \alpha\cup\alpha' = 1$. $Q_\eta[\alpha] = \pm 1$ corresponds to two possible choices of Pin^- structure on $\mathbb{RP}^2$. Then, the ABK invariant is computed as an 8th root of unity,

$$\mathrm{ABK}[\mathbb{RP}^2,\eta] = \frac{1\pm i}{\sqrt{2}} = e^{\pm 2\pi i/8}. \tag{6.4}$$

6.2 Diagnostic for $\mathbb{Z}_8$ classification

In the rest of this chapter, we formulate the non-local operations in the Hilbert space of our field theory that is utilized to diagnose the $\mathbb{Z}_8$ classification of the SPT phase, and then study the connection of the non-local order parameter for the $\mathbb{Z}_8$ classification proposed in [51] with the ABK invariant. This section is based on the author's work [24].

6.2.1 The Hilbert space of the ABK theory

First, we illustrate the Hilbert space for the theory given in (6.1). When the spacetime manifold M has a boundary, the Hilbert space on ∂M is constructed on the Fock space of n complex fermions, where n is the number of boundary 1-simplices. For simplicity, let us assume that M is an oriented disk D^2. Then, the wave function for the prepared Hilbert space is given by evaluating the path integral (6.1), which is expressed as

$$|\psi\rangle = \sum_{\alpha\in Z^1(M,\mathbb{Z}_2)} \widetilde{Z}(\alpha)\sigma(M,\alpha;\mathrm{ord}(1,\cdots,n))(-1)^{\int_M \eta\cup\alpha}(c_1^\dagger)^{\alpha(e_1)}\cdots(c_n^\dagger)^{\alpha(e_n)}\,|0\rangle\,, \tag{6.5}$$

where $c_j/c_j^\dagger$ denotes a complex fermion annihilation/creation operator at a boundary 1-simplex e_j. This is regarded as an analogue of matrix product state (MPS) representation of the wave function [53, 54] applied for fermionic systems. A similar representation of the wave function of a fermionic wave function is found in [55, 56], which developed MPS-like representation of fermionic states in one spatial dimension. $\sigma(M,\alpha;\mathrm{ord}(1,\cdots,n))$ evaluates the Grassmann integral on an open surface M, which is defined via the following relation

$$\int_M \prod_{e|\alpha(e)=1} d\theta_e d\overline{\theta}_e \prod_t u(t) = \sigma(M,\alpha;\mathrm{ord}(1,\cdots,n))\vartheta_1^{\alpha(e_1)}\cdots\vartheta_n^{\alpha(e_n)}. \tag{6.6}$$

Here, ϑ_j represents θ_j or $\overline{\theta}_j$ depending on an assignment of Grassmann variables on boundaries. η in (6.5) is the spin structure of M, given in a similar fashion to the description in Sec. 5.1.2 about the spin structure of a manifold with a boundary. That is, the term $(-1)^{\int_M \eta\cup\alpha}$ is expressed in the form of

$$(-1)^{\int_M \eta\cup\alpha} = (-1)^{\int_{E_M}\alpha}. \tag{6.7}$$

Here, ∂E_M gives a set of all 0-simplices of the barycentric subdivision S of M as an element of $C_0(M,\partial M,\mathbb{Z}_2) := C_0(M,\mathbb{Z}_2)/C_0(\partial M,\mathbb{Z}_2)$, i.e., ∂E_M is identical to S in the interior of M.

Finally, $\widetilde{Z}(\alpha)$ in (6.5) denotes the bosonic weight. While it was defined for closed manifolds in (6.1) as $\widetilde{Z}(\alpha) = 2^{|F|-|E|}$ using the number of faces $|F|$ and edges $|E|$ of the manifold, we will modify the definition of the bosonic weight on the boundary, in order for the wave function (6.5) to have unit norm. In (6.5), we define

$$\widetilde{Z}(\alpha) = 2^{|F|-|E|+|E_b|/2-1/2}, \tag{6.8}$$

where $|E_b|$ denotes the number of boundary 1-simplices. Based on the definition of $\widetilde{Z}(\alpha)$, the state $|\psi\rangle$ is shown to be correctly normalized to have the unit norm, as demonstrated in Appendix E.2.

6.2.2 Partial time-reversal

Now we describe the quantized non-local order parameter for SPT phases proposed in [51] in the case of the SPT state (6.5). Firstly, we prepare the reduced density matrix ρ_I of the state defined on an interval I in the ring $S^1 = \partial M$. Then, let us take a bipartition of I as

$I = I_1 \sqcup I_2$. Roughly speaking, the order parameter is defined via the process of taking the "time-reversal" of the density matrix, restricted to the interval I_1. With a proper definition of the time-reversing operation in the partial region I_1 in I, the order parameter is given by

$$\mathrm{tr}_I(\rho_I \rho_I^{\mathcal{T}_1}), \tag{6.9}$$

where $\rho_I^{\mathcal{T}_1}$ denotes the density matrix acted by "partial time-reversal". The definition of partial time-reversal is transparently expressed in the coherent state basis. That is, we introduce n Grassmann variables $\xi_1, \dots, \xi_n$ and denote a state like $|\{\xi_i\}\rangle = \prod_j \exp(-\xi_j c_j^\dagger)|0\rangle$. The density matrix is rewritten in the coherent state basis as

$$\rho_I = \int d[\overline{\xi}, \xi] d[\overline{\chi}, \chi] \,|\{\xi_j\}\rangle\, \rho_I(\{\overline{\xi}_j\}; \{\chi_j\}) \,\langle\{\overline{\chi}_j\}|\,, \tag{6.10}$$

where $d[\overline{\xi}, \xi] = \prod_j d\overline{\xi}_j d\xi_j e^{-\sum_j \overline{\xi}_j \xi_j}$, and $\rho_I(\{\overline{\xi}_j\}; \{\chi_j\}) = \langle\{\overline{\xi}_j\}|\, \rho_I \,|\{\chi_j\}\rangle$. Then, the operation $\rho_I^{\mathcal{T}_1}$ is defined as

$$\rho_I^{\mathcal{T}_1} := \int d[\overline{\xi}, \xi] d[\overline{\chi}, \chi] \,|\{i\overline{\chi}_j\}_{j\in I_1}, \{\xi_j\}_{j\in I_2}\rangle\, \rho_I(\{\overline{\xi}_j\}; \{\chi_j\}) \,\langle\{i\xi_j\}_{j\in I_1}, \{\overline{\chi}_j\}_{j\in I_2}|\,. \tag{6.11}$$

This operation on I_1 is called partial time-reversal in [51], since it acts on Grassmann variables in I_1 in the same fashion as the time-reversal symmetry with $T^2 = 1$.

Here, let us comment on an intuition of how the quantity (6.9) diagnoses the $\mathbb{Z}_8$ classification of the topological superconductor. Roughly speaking, the reduced density matrix ρ_I is regarded as a path integral on a spacetime D^2 with a boundary $S^1 = I \sqcup \overline{I}$. Then, we take two copies of ρ_I's and glue the two disks on the boundaries by taking the trace. During that process, the partial time-reversal reverses the orientation of the partial interval $I_1 \sqcup \overline{I}_1$ for the boundary of one of the disks, by exchanging the bra and ket in that region. This works as introducing a cross-cap when we glue two disks, see Fig. 6.1. After all, the resulting spacetime yields an unoriented spacetime with a single cross-cap, which turns out $\mathbb{RP}^2$. Since the $\mathbb{RP}^2$ generates the Pin$^-$ bordism group $\Omega_2^{\mathrm{pin}^-} = \mathbb{Z}_8$, we expect that (6.9) gives a partition function of Pin$^-$ invertible theory whose phase is quantized as the 8th root of unity, fully diagnosing the classification.

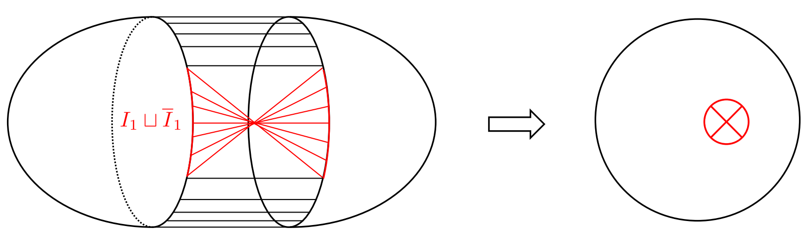

Figure 6.1: The geometry for $\mathrm{tr}_I(\rho_I \rho_I^{\mathcal{T}_1})$. The disks in the left figure represent the reduced density matrix ρ_I. Since the partial time-reversal exchanges the bra and ket partially, it amounts to introducing the cross-cap when gluing two disks. The resulting spacetime is a sphere with a single cross-cap, which turns out to be $\mathbb{RP}^2$.

The above intuition actually turns out to be the case. In Appendix E.1, we explicitly compute the quantity (6.9) for the SPT state (6.5) given via the path integral, and show that

$$\mathrm{tr}_I(\rho_I \rho_I^{\mathcal{T}_1}) = \frac{1}{2\sqrt{2}} \mathrm{ABK}[\mathbb{RP}^2, \eta], \tag{6.12}$$

where the ABK invariant is defined as (6.1), and η is a specific Pin^- structure on the resulting spacetime $\mathbb{RP}^2$. This gives an understanding of the $\mathbb{Z}_8$ quantization of the non-local order parameter (6.9) in terms of the ABK invariant.

Chapter 7

Beyond Gu-Wen: Review on $(2+1)$d $\mathbb{Z}_2$ SPT phase

In Chapter 5, we studied a subclass of fermionic SPT phases called the Gu-Wen phases. While the Gu-Wen phases cover a large class of SPT phases, the classification leaves out "beyond Gu-Wen" phases, whose classification was developed in [57, 58]. The simplest beyond Gu-Wen phase is the (1+1)-dimensional topological superconductor (Kitaev wire) [18]. In the absence of global symmetries except for fermion parity, the Kitaev wire generates the $\mathbb{Z}_2$ classification of SPT phases. If we take a time-reversal symmetry with $T^2 = 1$ into account, it instead generates the $\mathbb{Z}_8$ classification, as we have seen in Chapter 6.

In (2+1) dimensions, a seminal example for the beyond Gu-Wen phase is a fermionic SPT phase with unitary $\mathbb{Z}_2$ symmetry, classified by the cobordism group $\Omega^3_{\text{spin}}(B\mathbb{Z}_2) = \mathbb{Z}_8 \times \mathbb{Z}$. In that case, the Gu-Wen phase generates the $\mathbb{Z}_4$ subgroup of the $\mathbb{Z}_8$ torsion part, as reviewed in Sec. 2.4. The generator of $\mathbb{Z}_8$ is outside of the Gu-Wen phases, and particularly important because this SPT phase also describes the inflow of the 't Hooft anomaly for a single Majorana fermion in (1+1) dimensions [59, 60].

This $\mathbb{Z}_8$ generator is understood by decorating a Kitaev wire on the codimension-1 $\mathbb{Z}_2$ symmetry defects, represented by the $n_1 \in H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$ layer shown in (2.14). In particular, there is a known construction of an exactly solvable Hamiltonian model on a lattice for the $\mathbb{Z}_8$ generator, based on the description of the Kitaev wire located on the $\mathbb{Z}_2$ defects [61]. Such a construction of a SPT phase via decorating the symmetry defect with lower-dimensional phases is loosely called the "decorated domain wall" construction. Concretely, the wave function for the model is given by first decorating the Kitaev wire on the domain wall of the $\mathbb{Z}_2$ symmetry, and then fluctuating the domains to respect the $\mathbb{Z}_2$ symmetry. In order to conserve fermion parity under fluctuation of the Kitaev wire, one requires a specific choice of directions on edges on the lattice called "Kasteleyn direction", which is understood as encoding a spin structure on the spatial manifold, regarded as the trivialization of the 2nd Stiefel-Whitney class w_2 [62]. In this chapter, we review the construction of the wave function for the (2+1)-dimensional SPT phase with the $\mathbb{Z}_2$ symmetry on a lattice, following [61].

Brief review on the Kitaev wire

Firstly, we briefly recall the wave function of the Kitaev wire which will be decorated on the $\mathbb{Z}_2$ domain wall. Let us consider an open chain of complex fermions with the length L. The Hilbert space is given by a Fock space of L complex fermions $c_1, c_2, \dots c_L$. To describe the state realized by the Kitaev wire, it is convenient to introduce a pair of Majorana fermions a_i, b_i on each site i.

$$\begin{aligned} a_i &= c_i^\dagger + c_i, \\ b_i &= i(c_i^\dagger - c_i). \end{aligned} \tag{7.1}$$

Then the wave function of the Kitaev wire is characterized by the following condition,

$$ib_i a_{i+1} = 1 \qquad \text{for } 1 \le i \le L-1. \tag{7.2}$$

These conditions specify the two-dimensional Hilbert space, and these two states are characterized by the eigenvalue $ia_1 b_L = \pm 1$. That is, the Hilbert space of the Kitaev wire is effectively described by the two-dimensional Fock space constructed via the pair of Majorana fermions a_1, b_L on the boundary. We emphasize that the Kitaev wire is an invertible phase, and the two-fold degeneracy of the Hilbert space originates from the presence of s a single unpaired Majorana fermion on each end of the boundary. This unpaired Majorana mode (a_1 or b_L) is sometimes called "Majorana zero mode", which is identified as the nontrivial edge state of the Kitaev wire. Meanwhile, the trivial invertible phase is realized by the condition

$$ia_i b_i = 1 \qquad \text{for } 1 \le i \le L. \tag{7.3}$$

This is regarded as a trivial atomic insulator with $c_i^\dagger c_i = 1$ on each site. The Kitaev wire (7.2) is regarded as shifting each pair of Majorana fermions by a "half translation" from the trivial phase. In the presence of the time-reversal symmetry $T : a_i \to a_i, b_i \to -b_i$, the Kitaev wire generates the $\mathbb{Z}_8$ classification of the (1+1)-dimensional topological superconductor discussed in Chapter 6 [10]. Actually, the non-local order parameter (6.9) is computed in [51] and shown to be quantized as the 8th of unity, which implies that the topological action is given by the Arf-Brown-Kervaire invariant of the spacetime.

Wave function the (2+1)d $\mathbb{Z}_2$ SPT phase

Now let us review the description of the (2+1)-dimensional $\mathbb{Z}_2$ SPT phase proposed by Tarantino and Fidkowski [61]. While their lattice model was originally built on a honeycomb lattice in [61], we obtain the wave function for a graph on any two-dimensional oriented manifold equipped with a triangulation. The wave function is based on an assignment of direction on edges of a graph called Kasteleyn direction. Here Kasteleyn direction means that, the number of clockwise-directed edges bounding a face of the graph must be odd for any face of a graph.

We consider a trivalent directed graph Γ on a two-dimensional oriented spin manifold M given as follows. We consider a triangulated M with a branching structure. The simplical complex for this triangulation is denoted as $\mathcal{T}$. We have local ordering on each 2-simplex of $\mathcal{T}$ according to the branching structure (see Appendix B for the definition of branching structure). Each 2-simplex can then be either a $+$ simplex or a $-$ simplex, depending on whether the ordering agrees with the orientation or not. Then, the trivalent graph Γ is obtained by filling each 2-simplex of $\mathcal{T}$ with a pattern described in Fig. 7.1. The edges of Γ are Kasteleyn directed, and they are assigned in the following steps [57];

1. We start with directing edges of the graph Γ, as described in Fig. 7.1, for $+$ and $-$ simplices of $\mathcal{T}$. At this stage, some faces of Γ are not necessarily Kasteleyn.

2. Each non-triangular face of Γ is in 1-1 correspondence to a 0-simplex of $\mathcal{T}$. We denote as S the set of 0-simplices that correspond to non-Kasteleyn faces of Γ. $S \in Z_0(M, \mathbb{Z}_2)$ is known to represent the dual of the second Stiefel-Whitney class w_2 [63, 64]. Then, we specify the spin structure of M by a choice of a set of 1-simplices $E \in C_1(M, \mathbb{Z}_2)$ of $\mathcal{T}$, with $\partial E = S$.

3. We reverse the directions on the edge of Γ which cross 1-simplices of E, which makes all faces of Γ Kasteleyn.

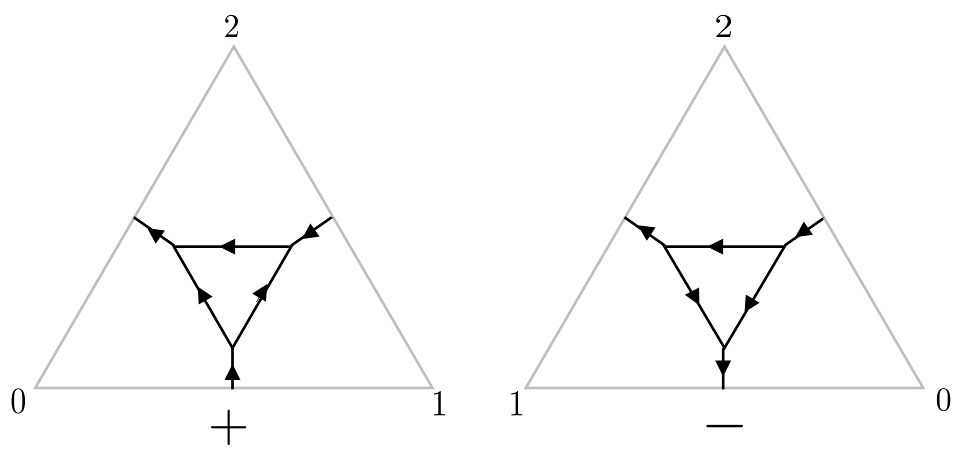

Figure 7.1: Directions on edges of Γ.

After these steps, we introduced the Kasteleyn direction on edges of Γ based on the spin structure of M. We refer to triangular faces in Γ as triangles. Let $t(v), t(w)$ be triangles in Γ that contain vertices v, w respectively. If an edge $\langle vw \rangle$ satisfies $t(v) = t(w)$, we refer to $\langle vw \rangle$ as a long edge. If we have $t(v) \neq t(w)$, we refer to $\langle vw \rangle$ as a short edge. Then, the degrees of freedom in the model are given as follows:

- A qubit located on each face of Γ except for triangles, operated by Pauli operators τ^x, τ^y, τ^z. Dually, we can also think of τ qubits as living on 0-simplices of $\mathcal{T}$.

- A complex fermion on each short edge e of Γ, created and annihilated by $c_e^\dagger, c_e$ respectively. Dually, we can also think of complex fermions as living on 1-simplices of $\mathcal{T}$.

The qubits are charged under the unitary $\mathbb{Z}_2$ symmetry, $U_{\mathbb{Z}_2} : |1\rangle \mapsto |0\rangle, |0\rangle \mapsto |1\rangle$, while the fermions are invariant under the symmetry. The wave function for this model is given by decorating the Kitaev wire on the codimension-1 domain wall of qubits. To introduce the Kitaev wire decoration, it is convenient to decompose each complex fermion into a pair of Majorana fermions. Let $e = \langle \overrightarrow{vv'} \rangle$ be a short edge oriented from v to v'. Then, each complex fermion on e is represented by a pair of Majorana fermions

$$\begin{aligned} a_v &= c_e^\dagger + c_e, \\ b_{v'} &= i(c_e^\dagger - c_e), \end{aligned} \tag{7.4}$$

located on v and v', respectively. The wave function is then given by pairing Majorana fermions on vertices of Γ, according to the dimer covering of the edges of Γ. That is, the Majorana fermions are paired along the non-overlapping dimers that each occupy two neighbouring vertices, and completely cover the vertices of the graph.

As a technical detail, we locate a fictitious qubit on each triangle of Γ, whose τ^z is fixed according to the majority rule: if the triangle is contained in a 2-simplex $\langle v_0 v_1 v_2 \rangle$, it is $|1\rangle$ or $|0\rangle$ depending on whether the majority of three qubits on v_1, v_2, v_3 have $|1\rangle$ or $|0\rangle$. Then, away from the domain wall of qubits, we pair up Majorana fermions along short edges $\langle \overrightarrow{vw} \rangle$ by a pairing term $i\gamma_v\gamma_w$ (γ is a or b). On the domain wall, we pair up Majorana fermions along long edges $\langle \overrightarrow{vw} \rangle$ by a pairing term $i\gamma_v\gamma_w$. These pairing rules amount to decorating the Kitaev wire on the codimension-1 domain wall of qubits; see Fig. 7.2. The wave function of the model is given by the equal superposition of all possible configurations of qubits, associated with the Majorana pairings discussed above.

Since the way of pairing up Majorana fermions along the $\mathbb{Z}_2$ domain wall differs from the case without the domain wall by a half translation, the given wave function is regarded as locating the Kitaev wire on the $\mathbb{Z}_2$ domain wall. The wave function constructed in such a way generates the $\mathbb{Z}_8$ classification of the (2+1)-dimensional $\mathbb{Z}_2$ SPT phase. The field theoretical description for the decorated domain wall construction of this phase is given in Sec. 8.2.2.

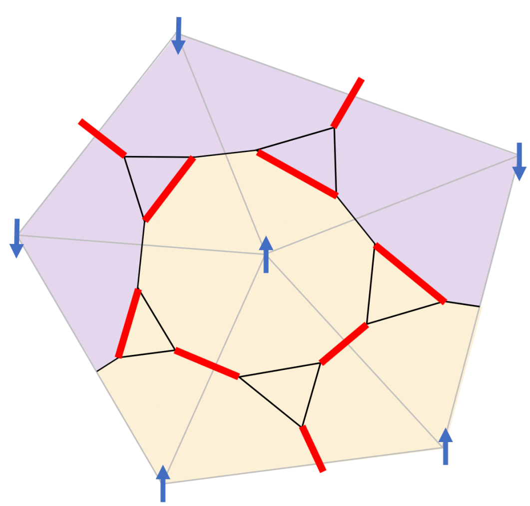

Figure 7.2: Majorana pairings on the two-dimensional graph Γ.

Kasteleyn direction and conservation of fermion parity

Here let us comment on why we need the Kasteleyn direction for the construction. In short, the Kasteleyn property of Γ is required in order to have the symmetric state under the fermion parity $(-1)^F$.

To see this, let us consider a wave function with a specific pairing of Majorana fermions, which corresponds to a dimer covering $\mathcal{D}_i$ on Γ. By flipping some qubits for this wave function, we finally obtain a different dimer covering $\mathcal{D}_f$. These two dimer coverings are related by sliding a sequence of dimers along a closed path C of Γ. Concretely, suppose edges $\langle v_1 v_2\rangle$, $\langle v_3 v_4\rangle$, $\ldots \langle v_{2n-1} v_{2n}\rangle$ form dimers in $\mathcal{D}_i$. Then, the dimers are rearranged to $\langle v_2 v_3\rangle$, $\langle v_4 v_5\rangle$, $\ldots \langle v_{2n} v_1\rangle$ in $\mathcal{D}_f$. We can easily show that the two wave functions for $\mathcal{D}_i$ and $\mathcal{D}_i$ have the same fermion parity, *iff* the path C is Kasteleyn directed. If we work on the reduced Fock space of these $2n$ Majorana fermions on C, the fermion parity for $\mathcal{D}_i$ is

$$i^n \gamma_1 \gamma_2 \ldots \gamma_{2n-1} \gamma_{2n} = s_{1,2} s_{3,4} \ldots s_{2n-1,2n}, \tag{7.5}$$

where $s_{i,j} = 1$ if the direction for $\langle v_i v_j\rangle$ is $\langle \overrightarrow{v_i v_j}\rangle$, and $s_{i,j} = -1$ for the opposite direction. The fermion parity for $\mathcal{D}_f$ is

$$i^n \gamma_1 \gamma_2 \ldots \gamma_{2n-1} \gamma_{2n} = -i^n \gamma_2 \ldots \gamma_{2n-1} \gamma_{2n} \gamma_1 = -s_{2,3} s_{4,5} \ldots s_{2n,1}. \tag{7.6}$$

These two expressions are identical iff C is Kasteleyn directed.

Chapter 8

Beyond Gu-Wen: (3+1)d topological superconductor with $T^2 = (-1)^F$

The content of this chapter is based on the author's works [25].

In Chapter 7, we reviewed the construction of (2+1)-dimensional $\mathbb{Z}_2$ SPT phase generating the $\mathbb{Z}_8$ classification, which lies outside of the Gu-Wen phases. An even more interesting example of beyond Gu-Wen phases is found in (3+1) dimensions. For instance, (3+1)-dimensional SPT phases based on the time-reversal symmetry with $T^2 = (-1)^F$ is classified by the Pin$^+$ cobordism group $\Omega^4_{\mathrm{Pin}^+} = \mathbb{Z}_{16}$. According to the description in Sec. 2.4, only the $\mathbb{Z}_4$ subgroup of the $\mathbb{Z}_{16}$ corresponds to the Gu-Wen phase. The $2 \in \mathbb{Z}_{16}$ phase is the simplest phase beyond Gu-Wen in the $\mathbb{Z}_{16}$ classification, and this $2 \in \mathbb{Z}_{16}$ phase is again understood via the decoration of the Kitaev wire on the codimension-2 junction of the T symmetry defects. The $\mathbb{Z}_{16}$ generator $1 \in \mathbb{Z}_{16}$ phase is more intricate, which has to do with the decoration of a $p + ip$ superconductor on the T defect. This $1 \in \mathbb{Z}_{16}$ phase is beyond the scope of this thesis.

In this chapter, as a generalization of the $\mathbb{Z}_2$ SPT phase in (2+1) dimensions reviewed in Chapter 7, we will describe (3+1)-dimensional T SPT phases generating the $\mathbb{Z}_8$ subgroup of the $\mathbb{Z}_{16}$ classification, in terms of the Kitaev wire decorations. We will provide a wave function for that phase, and our model is obtained by a version of decorated domain wall construction [65–67], where the T symmetry defect carries a two dimensional wave function of the fluctuating Kitaev wires, see Fig. 8.1. This is regarded as the decorated domain wall construction repeated twice, which after all describes the decoration of the Kitaev wire on the codimension-2 junction of T symmetry defects.

The wave function is constructed on a graph in the three-dimensional spatial manifold. By deliberately assigning the directions on edges of the three dimensional graph, we always have a two dimensional graph on the T symmetry defect whose edges are completely Kasteleyn directed, allowing us to fluctuate Kitaev wires on the wall in a fashion respecting fermion parity. We will see such a specific assignment of directions on edges is again made possible by a spin structure on the spatial manifold, i.e., a trivialization of the 2nd Stiefel-Whitney class w_2.

The iterated decoration of the topological phases on the symmetry defect described in Fig. 8.1 deserves understanding more in detail. In particular, the wave function supported on the two-dimensional T symmetry defect of the (3+1)-dimensional T SPT phase itself

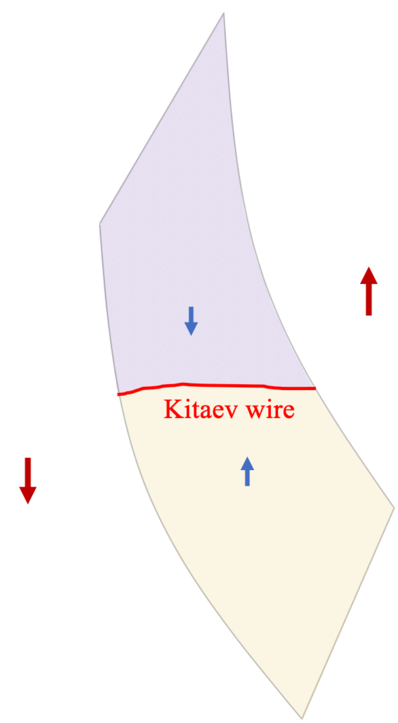

Figure 8.1: A schematic illustration for our wave function in three spatial dimension. On a two dimensional T symmetry defect, we have a two-dimensional wave function of fluctuating Kitaev wires located on the symmetry defects induced on the two-dimensional surface. The symmetry defects are realized as a domain wall separating two distinct vacua of the symmetry broken phase, and the domains are characterized by the polarized qubits (red and blue arrows) charged by the time-reversal symmetry.

looks like a wave function of the (2+1)-dimensional SPT phase, where the Kitaev wire is located on the symmetry defect induced on the (2+1)-dimensional locus of the symmetry defect. So, the naturally arising question for the situation is what global symmetry the (2+1)-dimensional theory on the T symmetry defect possesses. Then, we are interested in what kind of (2+1)-dimensional SPT phase the T symmetry defects support, and how the classification of the SPT phases on the symmetry defects are related to that of the (3+1)-dimensional T SPT phases. We expect that the (2+1)-dimensional phase on the domain wall inherits the classification of (3+1)-dimensional SPT phases, and then the study on the domain wall allows us to find the classification of our (3+1)-dimensional SPT phase.

To understand these questions, we make contact with the recent field theoretical argument on global symmetry of theories supported on the symmetry defect [68, 69]. These works study the global symmetry of the localized theory on the symmetry domain wall in the spontaneously symmetry broken phase, assuming the Lorentz symmetry of the theory. In particular it was argued in [68] that there is the unitary $\mathbb{Z}_2$ symmetry induced on the (2+1)-dimensional domain wall of the T symmetry, for a (3+1)-dimensional system with T symmetry such that $T^2 = (-1)^F$. More formally, for a (3+1)-dimensional spacetime manifold with Pin$^+$ structure that corresponds to the time-reversal symmetry, we can find a spacetime structure induced on the (2+1)-dimensional defect of the T symmetry, which is dual to the first Stiefel-Whitney class w_1. The induced structure turns out to become the spin structure with the $\mathbb{Z}_2$ gauge field, which corresponds to the fermionic system with a unitary $\mathbb{Z}_2$ symmetry. Hence, we expect that one observes a (2+1)-dimensional SPT phase based on unitary $\mathbb{Z}_2$ symmetry, on the T symmetry domain wall in the spontaneously broken phase of our model for the (3+1)-dimensional T SPT phase.

Moreover, the classification of the (3+1)-dimensional T SPT phase turns out to be closely related to that of the (2+1)-dimensional $\mathbb{Z}_2$ SPT phase on the T domain wall. In fact, there is a linear relation between the classification of the (2+1)-dimensional SPT phase on the T symmetry defect and that of (3+1)-dimensional T SPT phases [68, 69]. This relationship allows us to determine the classification of the (3+1)-dimensional SPT phase, from a given theory on the symmetry defect. This linear map between SPT classifications is nicely formulated in terms of the classification scheme of SPT phases based on cobordism group, mathematically known as the Smith map of cobordism groups [12].

In our wave function for the (3+1)-dimensional T SPT phase, we find that the T domain wall supports a nontrivial SPT phase based on the $\mathbb{Z}_2$ symmetry, after fixing the configuration of the wall by breaking the T symmetry. The classification of the (2+1)-dimensional fermionic $\mathbb{Z}_2$ SPT phase is given by cobordism group $\Omega^3_{\text{spin}}(B\mathbb{Z}_2) = \mathbb{Z}_8 \times \mathbb{Z}$. The domain wall is effectively described by the wave function given in Chapter 7 for an SPT phase generating the $\mathbb{Z}_8$ classification. Then, the classification of the (3+1)-dimensional T SPT phase is totally encoded in the SPT phase on the domain wall, which allows us to find that our model spans the $\mathbb{Z}_8$ subgroup of the $\mathbb{Z}_{16}$ classification, based on the field theoretical argument made in [68, 69].

this chapter is organized as follows. In Sec. 8.1, we provide the wave function for (3+1)-dimensional T SPT phase given by decorating the Kitaev wire. In Sec. 8.2, we review the field theoretical perspective that is helpful to understand our construction of the wave function. In Sec. 8.3, we claim that the wave function realizes the $2 \in \mathbb{Z}_{16}$ phase in classification of the T SPT phase, by making a contact with the field theoretical argument.

8.1 Construction of the wave function

Let us provide our wave function for a (3+1)-dimensional time-reversal SPT phase with $T^2 = (-1)^F$, which is provided by a sort of decorated domain wall construction of the Kitaev wire. We consider a trivalent graph Γ on a three-dimensional oriented spin manifold M given as follows.

We first endow M with a triangulation. In addition, we take the barycentric subdivision for the triangulation of M. Namely, each 3-simplex in the initial triangulation of M is subdivided into $4! = 24$ simplices, whose vertices are barycenters of the subsets of vertices in the 3-simplex. We further assign a local ordering to vertices of the barycentric subdivision, such that a vertex on the barycenter of each i-simplex is labeled as "i." The obtained simplical complex after taking barycentric subdivision is denoted as $\mathcal{T}$. Each 3-simplex can then be either a + simplex or a − simplex, depending on whether the ordering agrees with the orientation or not.

The trivalent graph Γ is given by connecting patterns illustrated in Fig. 8.2 on each 3-

simplex of $\mathcal{T}$. For later convenience, we illustrate the following way to obtain Γ step by step:

1. First, we consider a simplical complex $\mathcal{T}'$ obtained by further subdividing each 3-simplex of $\mathcal{T}$ into 12 simplices as described in Fig. 8.3. The set of 2-simplices of $\mathcal{T}'$ is denoted as $\widetilde{\Lambda}$. Then, we take the dual lattice Λ of $\widetilde{\Lambda}$ described in Fig. 8.4.

2. All vertices of Λ have degree 4. For each vertex of Λ, we first resolve the vertex into a pair of trivalent vertices, and then change one of the trivalent vertices (which is not a vertex of a triangle) into three trivalent vertices, see Fig. 8.5. The obtained trivalent graph is Γ in Fig. 8.2.

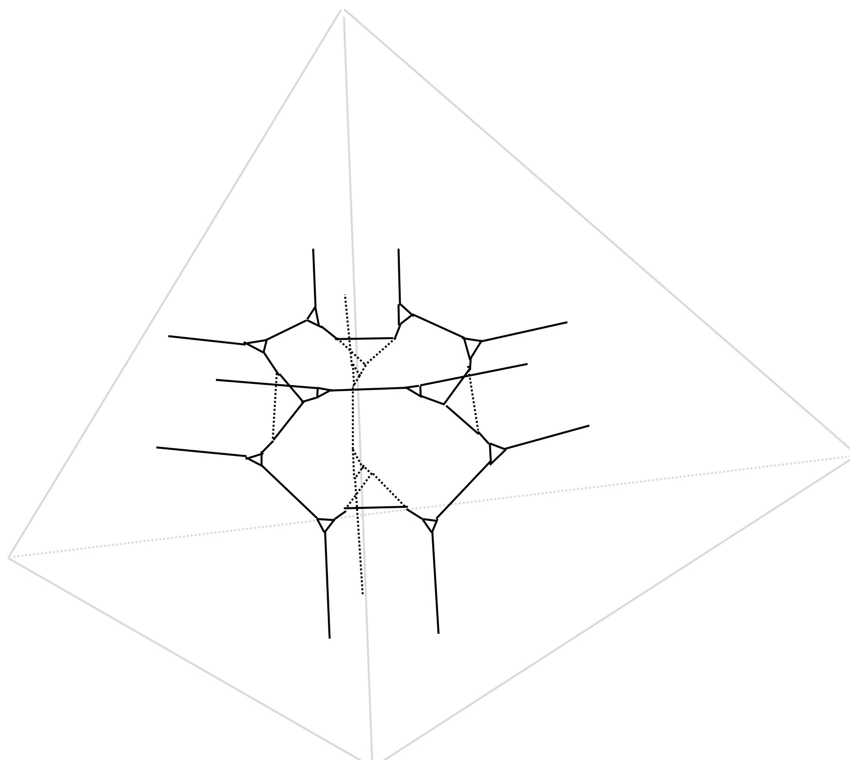

Figure 8.2: Γ on a 3-simplex of $\mathcal{T}$.

Following the notations in Sec. 7, we refer to triangular faces in Γ as triangles. Let $t(v), t(w)$ be triangles in Γ which contain vertices v, w respectively. If an edge $\langle vw\rangle$ satisfies $t(v) = t(w)$, we refer to $\langle vw\rangle$ as a long edge. If we have $t(v) \neq t(w)$, we refer to $\langle vw\rangle$ as a short edge. Then, the degrees of freedom in our model are given as follows:

- A qubit located on each vertex of $\mathcal{T}'$, which is operated by Pauli operators $\sigma^x, \sigma^y, \sigma^z$. We sometimes call these qubits "σ qubits". Dually, we can also think of σ qubits as living on three-dimensional cells of Γ.

- A qubit located on each 1-simplex of $\mathcal{T}'$, except for 1-simplices connecting a barycenter of a 2-simplex and a 3-simplex of $\mathcal{T}$. This qubit is operated by Pauli operators τ^x, τ^y, τ^z,

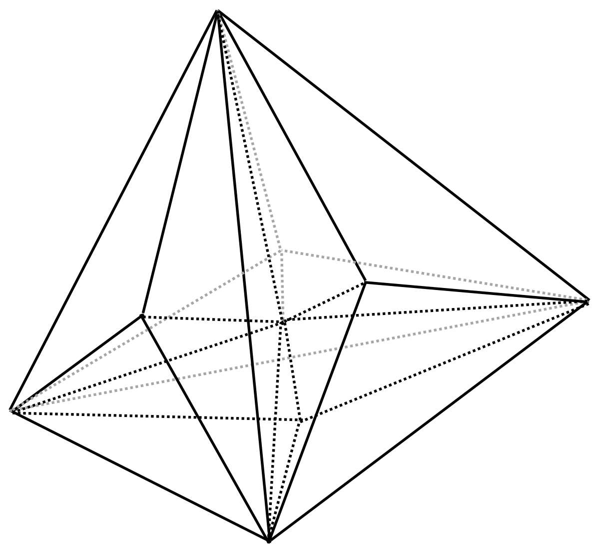

Figure 8.3: Subdividing each 3-simplex of $\mathcal{T}$ into 12 simplices yields $\mathcal{T}'$.

and we sometimes call these qubits "τ qubits". Dually, we can also think of τ qubits as living on faces of Γ except for triangles.

- A pair of complex fermions located on each short edge e, created and annihilated by $c_e^{s\dagger}, c_e^s$ ($s = \uparrow, \downarrow$) respectively.

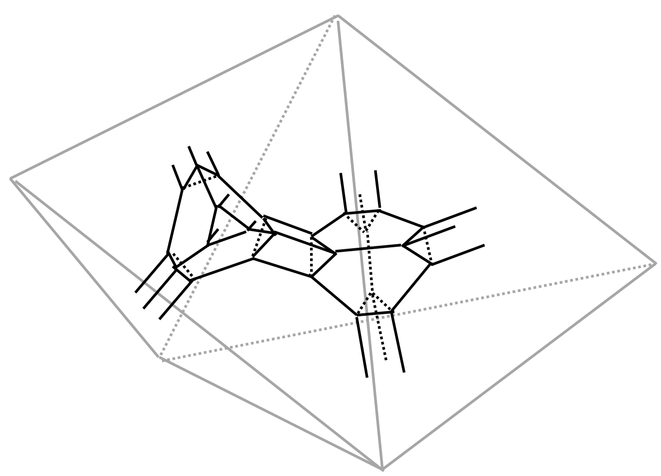

Figure 8.4: Λ is obtained by connecting truncated tetrahedra on 3-simplices of $\mathcal{T}$, by triangular prisms.

Both σ and τ qubits are charged under time-reversal as the Pauli x,

$$T : |1\rangle \mapsto |0\rangle \, , |0\rangle \mapsto |1\rangle \, . \tag{8.1}$$

Since $T^2 = (-1)^F$, fermions are also acted upon by time-reversal in a nontrivial fashion. Before discussing the symmetry property of fermions, let us outline how we perform the

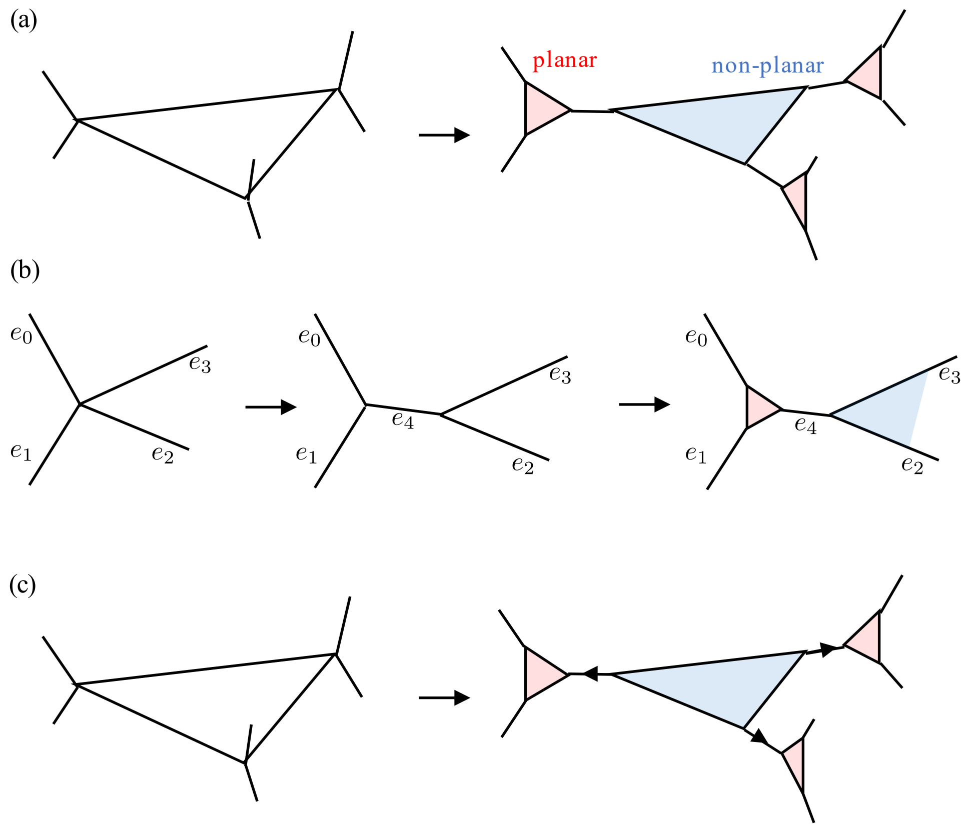

Figure 8.5: The figure shows the process of obtaining Γ from Λ. (a), (b): The left figure represents a triangular face of Λ. Γ is given by resolving each degree 4 vertex into trivalent vertices and then replacing one vertex with a triangle. A planar (resp. non-planar) triangle is represented as a red (resp. blue) triangle. (c): When we direct edges of Γ, the process of (a) is associated with directing newly added short edges.

domain wall decoration. Since σ qubits are located on three-dimensional cells of Γ, their configuration specifies a two-dimensional domain wall on Γ, which forms a graph supported on a two-dimensional surface, as we will define later in Sec. 8.1.2. Then, the configuration of τ qubits on the two-dimensional domain wall further gives us a one-dimensional domain wall, where we put the Kitaev wire; see Fig. 8.6.

8.1.1 Directions of edges and discrete spin structure

Analogously to the Tarantino-Fidkowski type wave function in Sec. 7, we need Kasteleyn directions on edges of Γ restricted to the two-dimensional domain wall of σ qubits, in order to ensure the conservation of the fermion parity under domain wall fluctuations. Let us assume that we have obtained a graph K on the two-dimensional manifold K that represents the T domain wall, whose edges will be directed in a Kasteleyn fashion. A caveat is that the assignment of Kasteleyn direction on K depends on how we choose the section of the normal bundle NK of the two-dimensional domain wall K; for instance, we can choose the section of NK directed from the side of $|1\rangle$ domain of σ qubits to that of $|0\rangle$ domain. Then, the

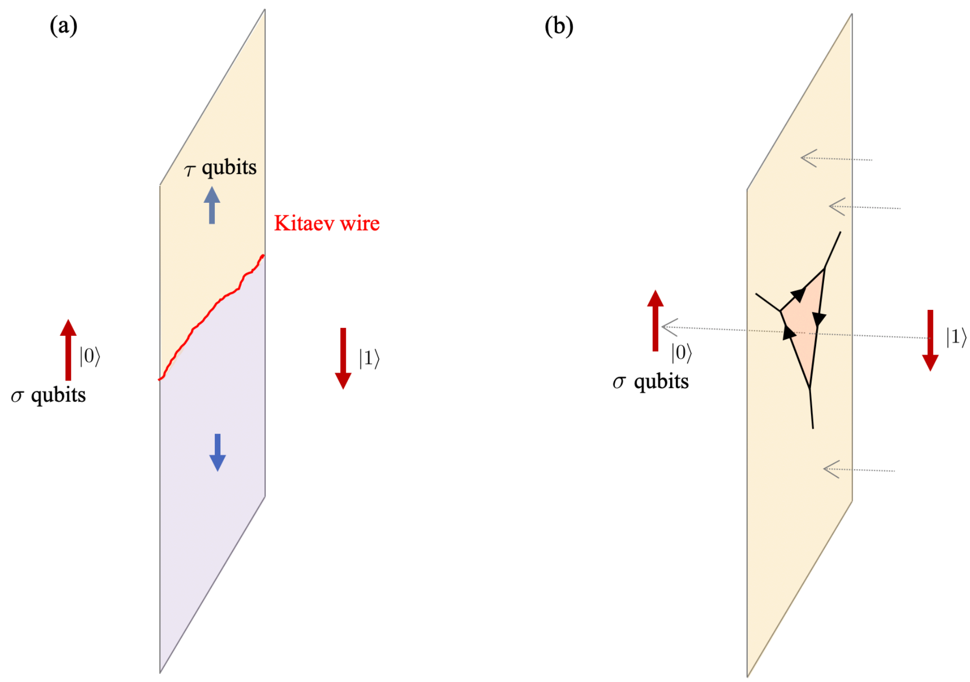

Figure 8.6: (a): We decorate the domain wall of τ qubits with the Kitaev wire placed on the two-dimensional domain wall of σ qubits. (b): The section NK of the normal bundle of the two-dimensional domain wall is directed from the side of the $|1\rangle$ domain of σ qubits to that of the $|0\rangle$ domain (gray arrows). Then, on each planar triangle on K (pink triangle), we assign directions on edges bounding the triangle clockwise around the axis parallel to the section.

Kasteleyn property is defined by the number of clockwise directed edges on a closed path of K, around the axis parallel to the section of NK (see Fig. 8.9 (b)). We note that such defined Kasteleyn direction on K is not necessarily invariant under time-reversal, since the section of NK is reversed by time-reversal, thereby transforming the definition of the Kasteleyn property on K; clockwise edges now become anticlockwise.

The above observation implies that the time-reversal symmetry also acts on the directions of edges. In this subsection, we will first introduce directions that are invariant under time-reversal, and then discuss non-invariant directions. For later convenience, we classify triangles (i.e., triangular faces) in Γ into two types: a "non-planar" triangle which originates from a triangular face of Λ, and a "planar" triangle obtained by replacing a trivalent vertex in Fig. 8.5 (a).

Invariant directions on edges

Here, we introduce directions on edges of Γ that are invariant under time-reversal, determined independently of the configuration of qubits. These invariant directions are assigned on edges of Γ except for edges bounding a planar triangle. Now we provide invariant directions step by step;

1. We start by assigning directions on edges of the graph Λ (Fig. 8.4), as described in

Fig. 8.7 (resp. Fig. 8.8) for + simplices (resp. − simplices) of $\mathcal{T}$.

2. Next, we modify the directions on edges of Λ, according to the combinatorial spin structure on M. To define the spin structure of M, we first prepare the representative of the dual of the second Stiefel-Whitney class w_2 on the simplical complex $\mathcal{T}'$ (see Fig. 8.3). It is represented by a 1-cycle $S \in Z_1(M, \mathbb{Z}_2)$,

$$S = \sum_{e \in \mathcal{T}'} e - \sum_{\Delta_+} (\langle v_{013} v_{0123} \rangle + \langle v_{123} v_{0123} \rangle) - \sum_{\Delta_-} (\langle v_{012} v_{0123} \rangle + \langle v_{023} v_{0123} \rangle), \qquad (8.2)$$

   where the first sum runs over all 1-simplices of $\mathcal{T}'$. The second sum is over 1-simplices of $\mathcal{T}'$ contained in a + simplex $\Delta_+ = \langle 0123 \rangle$ of $\mathcal{T}$. Here, the vertices of $\mathcal{T}'$ which is a barycenter of a simplex $\Delta \in \mathcal{T}$ are written as v_Δ. Similarly, the third sum is over − simplices of $\mathcal{T}$. The validity of the expression (8.2) is proven in Appendix C. The spin structure is specified by a trivialization $\partial E = S$ of S. Here, $E \in C_2(M, \mathbb{Z}_2)$ is a subcomplex of $\widetilde{\Lambda}$.

   Then, we reverse the directions of edges of Λ that cross 2-simplices of E.

3. Finally, we complete the assignment of directions on Γ, by generating Γ from Λ associated with directions to newly added short edges, as described in Fig. 8.5 (c).

Non-invariant directions on edges

Here, we introduce directions on yet undirected edges of Γ that are acted upon by time-reversal in a nontrivial fashion. As we will see in Sec. 8.1.2, there will be a two-dimensional graph K supported on the two-dimensional domain wall of σ qubits. We assign directions on edges bounding planar triangles, *iff* the planar triangle is contained in the two-dimensional domain wall K (see Fig. 8.6 (b)). We do not assign directions if the planar triangles are away from K.

On the two-dimensional domain wall K, we choose the section of the normal bundle NK, such that the section of NK is directed from the side of $|1\rangle$ domain of σ qubits to that of $|0\rangle$ domain. Then, on each planar triangle on K, we assign directions on edges bounding the triangle clockwise around the axis parallel to the section; see Fig. 8.6 (b). These directions are reversed by the time-reversal action, since the chosen section of NK is flipped by time-reversal.

8.1.2 Kasteleyn direction on the 2d domain wall

Here, we define the two-dimensional graph K on the two-dimensional domain wall of σ qubits, on which the Kasteleyn direction will be induced.

As a technical detail, we fix each σ qubit on the barycenter of a 2-simplex of $\mathcal{T}$ according to the majority rule: if the σ qubit is located on the barycenter of a 2-simplex $\langle v_0 v_1 v_2 \rangle$, it is $|1\rangle$ or $|0\rangle$ depending on whether the majority of three σ qubits on vertices v_0, v_1, v_2 have $|1\rangle$ or $|0\rangle$. Each σ qubit on the barycenter of a 3-simplex $\langle v_0 v_1 v_2 v_3 \rangle$ of $\mathcal{T}$ is also determined by the majority rule: it is $|1\rangle$ or $|0\rangle$ depending on whether the majority of four σ qubits on vertices

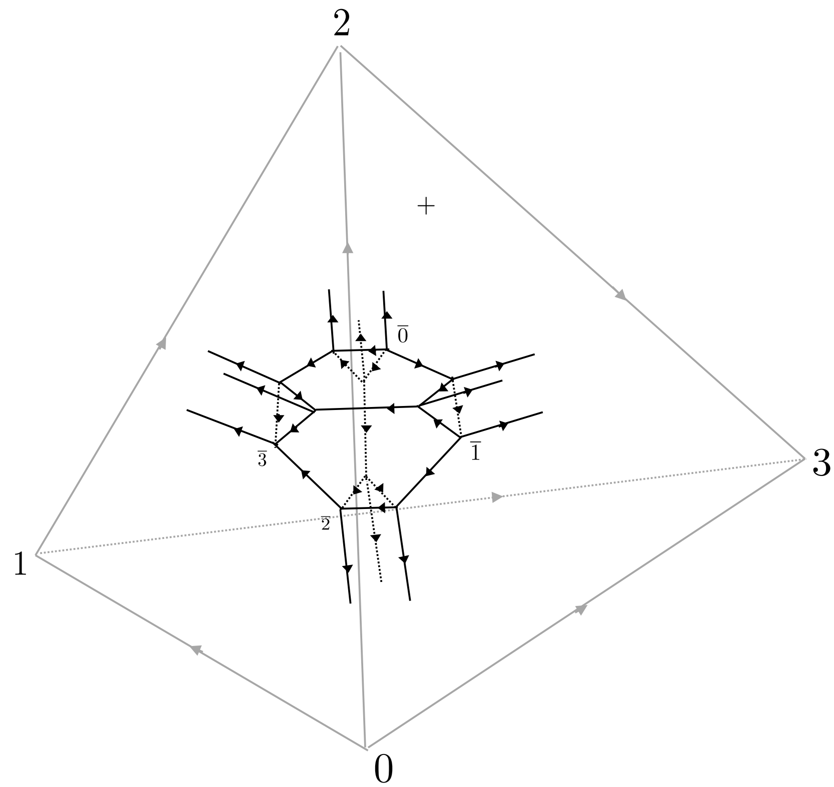

Figure 8.7: Initial assignment of directions on edges of Λ in a $+$ simplex.

v_0, v_1, v_2, v_3 have $|1\rangle$ or $|0\rangle$, if the numbers of $|1\rangle$ differs from that of $|0\rangle$. If the number of $|1\rangle$ and $|0\rangle$ on v_0, v_1, v_2, v_3 are both 2, we leave the σ qubit on $\langle v_0 v_1 v_2 v_3\rangle$ undetermined.

Then, a two-dimensional graph K′ is defined according to the configuration of σ qubits, as described in Fig. 8.9. After some efforts, we can see that the two-dimensional graph K′ is "almost Kasteleyn directed" when seen from the side of the $|1\rangle$ domain of σ qubits, except for non-planar triangles contained in K′ in Fig. 8.9 (b).

To prepare a graph whose edges are completely Kasteleyn directed, we gather four faces of K′ in Fig. 8.9 (b) into a single face, as described in Fig. 8.10. We denote the obtained graph as K, which is completely Kasteleyn directed.

8.1.3 Wave function: decorated 1d domain wall on the 2d domain wall

Here, we precisely describe the Kitaev wire decoration on the domain wall of τ qubits on K, which was schematically illustrated in Fig. 8.6 (a). The decoration is based on the Kasteleyn direction on K introduced in the previous subsection.

In our model, we have two complex fermions on each short edge of Γ. Analogously to the (2+1)-dimensional case in Sec. 7, we will represent each complex fermion on a short edge $\langle vv'\rangle$ in terms of a pair of Majorana fermions placed on vertices v, v', whose assignment depends on the direction of $\langle vv'\rangle$. Let $e = \langle\overrightarrow{vv'}\rangle$ be a short edge oriented from v to v'. Then,

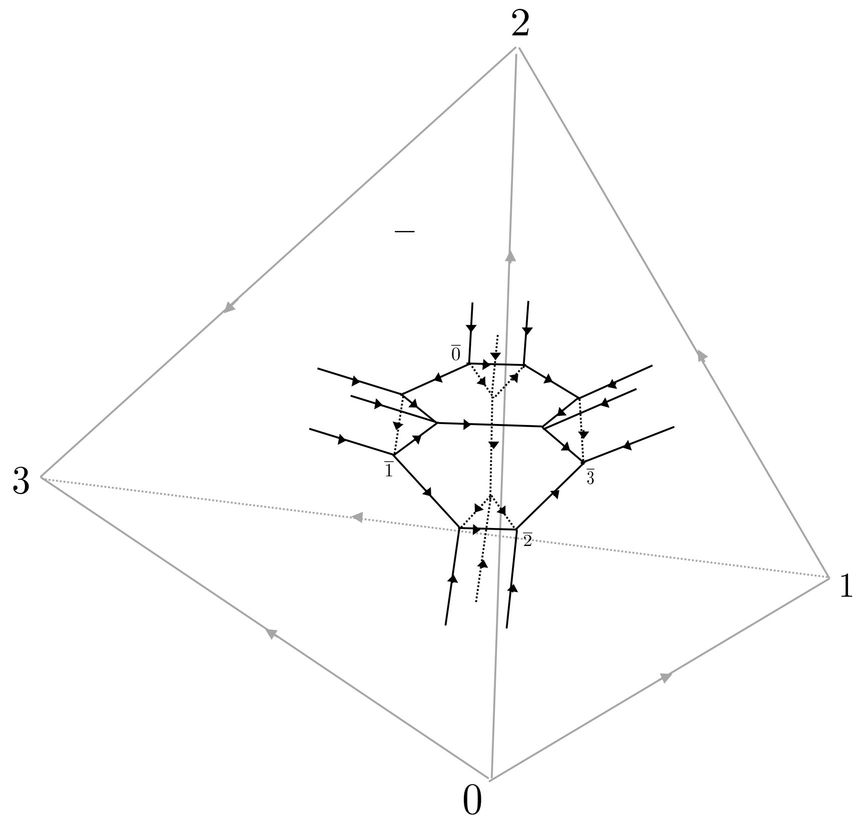

Figure 8.8: Initial assignment of directions on edges of Λ in a $-$ simplex.

each complex fermion on e is represented by a pair of Majorana fermions

$$\begin{aligned} a_v^s &= c_e^{s\dagger} + c_e^s, \\ b_{v'}^s &= i(c_e^{s\dagger} - c_e^s), \end{aligned} \tag{8.3}$$

located on v and v', respectively. Then, we introduce the symmetry property of fermions. Since $T^2 = (-1)^F$, the fermion on $\langle\overrightarrow{vv'}\rangle$ is a Kramers doublet under T,

$$T : \begin{cases} a_v^\uparrow \to a_v^\downarrow \\ a_v^\downarrow \to -a_v^\uparrow, \end{cases} \quad \begin{cases} b_{v'}^\uparrow \to -b_{v'}^\downarrow \\ b_{v'}^\downarrow \to b_{v'}^\uparrow. \end{cases} \tag{8.4}$$

The wave function of our model is given by pairing up Majorana fermions on vertices, according to a dimer configuration on Γ. Similar to the (2+1)-dimensional case in Sec. 7, away from the two-dimensional domain wall K, we pair up Majorana fermions that share a short edge $\langle\overrightarrow{vv'}\rangle$, by a pairing term $ia_v^\uparrow b_{v'}^\uparrow + ia_v^\downarrow b_{v'}^\downarrow$. Furthermore, the τ qubits on the face of Γ are fixed away from K, depending on the domain of σ qubits: the τ qubits are $|1\rangle$ (resp. $|0\rangle$) if contained in the domain of $|1\rangle$ (resp. $|0\rangle$).

Next, we consider the domain wall of τ qubits on K. As a technical detail, we recall that the two-dimensional graph K was obtained by gathering four faces in K$'$ into a single face, which was described in Fig. 8.10. Since we have one τ qubit on each face of K$'$ except for triangles, the newly obtained single face of K in Fig. 8.10 (a) contains two τ qubits. To

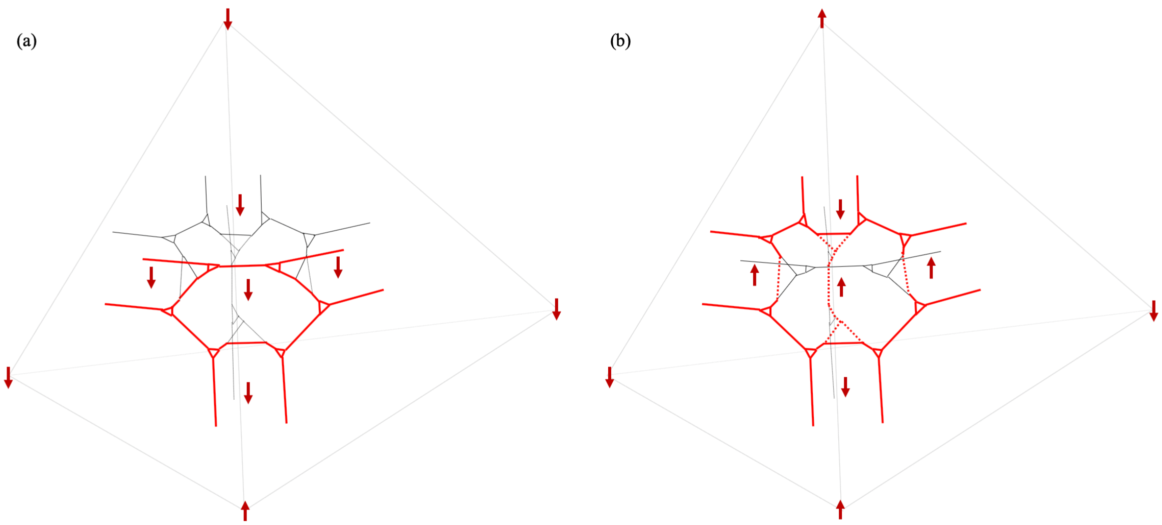

Figure 8.9: Two possible patterns of the two-dimensional graph K$'$ in a single 3-simplex of $\mathcal{T}$, where σ qubits are drawn as red arrows and K$'$ is represented as a red graph. For (a), it is also possible to have the situation in which all σ qubits are flipped. For (b), two non-planar triangles are contained in K$'$, which are not necessarily Kasteleyn directed.

consider the Kitaev wire decoration on K instead of K$'$, we have to make sure that two τ qubits share the same state, i.e., $|00\rangle$ or $|11\rangle$, as described in Fig. 8.10 (b).

Then, away from the domain wall of τ qubits on K, we also pair up Majorana fermions that share a short edge $\langle\overrightarrow{vv'}\rangle$, by $ia_v^\uparrow b_{v'}^\uparrow + ia_v^\downarrow b_{v'}^\downarrow$. These pairings away from the Kitaev wire decoration are invariant under T, which is consistent with the fact that the directions of short edges are unchanged by time-reversal, according to Sec. 8.1.1.

Kitaev wire on the 1d domain wall

Now we explain the way to put the Kitaev wire on the one-dimensional domain wall of τ qubits on K. Analogously to the (2+1)-dimensional case in Sec. 7, this is done by pairing Majorana fermions along the long edges on the one-dimensional domain wall. To do this, it is convenient to label the planar triangle on K in a "bipartite" fashion. For a 1-simplex e of $\mathcal{T}$ crossing the two-dimensional domain wall, we find a pair of planar triangles contained in a single 3-simplex of $\mathcal{T}$, which is located in the nearest position of e, as described in Fig. 8.11. Then, we label the pair of planar triangles by "A" and "B", such that the "A" triangle is located in the clockwise direction of the "B" triangle, when seen from the side of $|1\rangle$ domain of σ qubits.

Then, on the one-dimensional domain wall τ qubits on K, we pick out Majorana modes $\gamma_v^{s_v}$ (γ is a or b) from each vertex v, and we pair them along the long edges bounding a planar triangle $\langle\overrightarrow{vw}\rangle$ as $i\gamma_v^{s_v}\gamma_w^{s_w}$, so that they form the Kitaev wire. s_v and s_w are determined on each vertex according to the following rule,

- If the planar triangle is labeled by "A", $s=\uparrow$ when γ is a, and $s=\downarrow$ when γ is b.
- If the planar triangle is labeled by "B", $s=\downarrow$ when γ is a, and $s=\uparrow$ when γ is b.

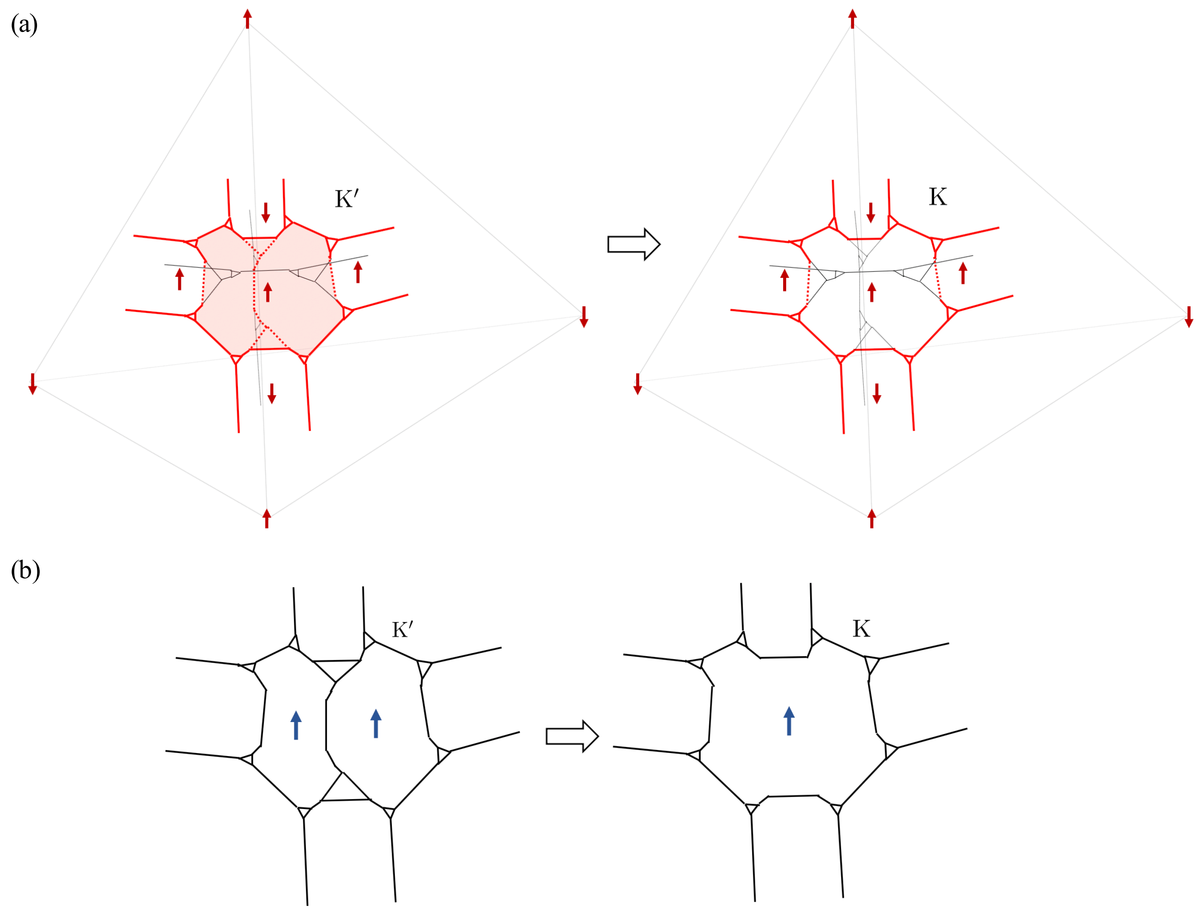

Figure 8.10: (a): The process of obtaining K from K′. This is done by gathering the four pink faces of K′ including two non-planar triangles into a single face. (b): Two τ qubits on the identified faces of K′ should share the same state.

After pairing up these Majorana fermions, we are left with one unpaired Majorana on each vertex of planar triangles, and two on each vertex of non-planar triangles, on the one-dimensional domain wall. Then, we pair up yet unpaired Majorana fermions on short edges $\langle\overrightarrow{vw}\rangle$, as $i\gamma_v^{\bar{s}_v}\gamma_w^{\bar{s}_w}$. Here, we will choose the pairing such that $\bar{s}_v = \bar{s}_w$.

Finally, we have one unpaired Majorana fermion on each vertex of non-planar triangles. We pair them up along long edges bounding a non-planar triangle $\langle\overrightarrow{vw}\rangle$ as $i\gamma_v^{s_v}\gamma_w^{s_w}$. Here, we can see that s_v is flipped from s_w, $s_v = -s_w$ (here, $-s$ denotes the opposite spin to s).

Wave function

The wave function of our model is given by the equal superposition of all possible configurations of σ qubits and τ qubits, associated with the Kitaev wire decoration discussed above.

Let us demonstrate the time-reversal invariance of this wave function. To see this, we first note that the pairing of Majorana fermions away from the Kitaev wire decoration $ia_v^{\uparrow}b_{v'}^{\uparrow}+ia_v^{\downarrow}b_{v'}^{\downarrow}$ is invariant under time-reversal. On the Kitaev wire decoration, according to the pairing

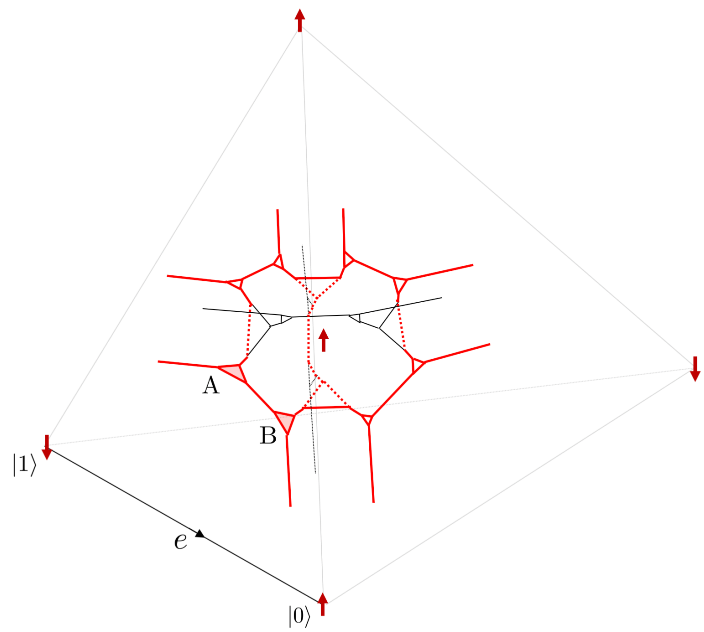

Figure 8.11: Labeling all planar triangles on K with either "A" or "B". For a 1-simplex e of $\mathcal{T}$ crossing the 2d domain wall, we find a pair of planar triangles contained in a single 3-simplex of $\mathcal{T}$, which is located in the nearest position of e. Then, we label the pair of planar triangles by "A" and "B", such that the "A" triangle is located in the clockwise direction of the "B" triangle, when seen from the side of $|1\rangle$ domain of σ qubits.

rule, the spins of paired Majorana fermions s_v, s_w flip their signs under time-reversal, which is consistent with the transformation law of fermions. This is because the labels of planar triangles ("A" or "B") are changed under time-reversal, thereby the spins of paired Majorana fermions are also flipped. We can also check that the pairings of Majorana fermions are consistent with the induced Kasteleyn direction on K under time-reversal. On one hand, on short edges and long edges bounding a non-planar triangle on K, the sign of the pairing $i\gamma_v^{s_v}\gamma_w^{s_w}$ is invariant under time-reversal, which is consistent with the invariance of the direction on $\langle vw\rangle$. On the other hand, on long edges of planar triangles of K, the pairing $i\gamma_v^{s_v}\gamma_w^{s_w}$ flips its sign under time-reversal. It is also consistent with the Kasteleyn direction on K, which flips the directions on long edges of planar triangles.

Thanks to the Kasteleyn directions induced on the two-dimensional domain walls, the obtained wave function also preserves the fermion parity $(-1)^F$, analogously to the (2+1)-dimensional case in Sec. 7.

8.2 Phases on the domain wall: the Smith map

To see what T SPT phase the above wave function realizes, we consider the T symmetry domain wall of the phase and study the relation between the (2+1)-dimensional phases on the domain wall and our (3+1)-dimensional SPT phase. For that purpose, here we review the field theoretical argument by [68] about the phases on the domain wall.

Let us consider a (3+1)-dimensional Lorentz invariant QFT with time-reversal symmetry such that $T^2 = (-1)^F$. Firstly, we are interested in a global symmetry of the theory supported on the T symmetry defect. It turns out that the global symmetry on the defect can be understood geometrically in terms of the induced spacetime structure on the defect, which will be discussed later in this section. Meanwhile, there is a useful perspective on the symmetry of the T defect when the T symmetry is spontaneously broken.

8.2.1 The symmetry broken phase

Let us start with the phase with the broken T symmetry. In that case, the T defect is realized as a T domain wall separating two distinct vacua of the symmetry broken phase. We consider an infinite system of our (3+1)-dimensional QFT, and make up a T domain wall of the theory by breaking the T symmetry, dividing the system into the left and right domain. We are interested in a global symmetry supported on the T domain wall in this setup. Since we assume the Lorentz invariance of our theory, we can find a global symmetry induced on the domain wall with help of the CPT symmetry [68, 69]. [1] Concretely, when the T domain wall is located in a reflection symmetric fashion, the combined transformation of T and $CP_\perp T$ fixes the domains, thus acts solely on the domain wall. Here, $P_\perp$ denotes a spatial reflection fixing the configuration of the domain wall, see Fig. 8.12. Since the CPT is anti-unitary, the combined transformation $T \cdot (CP_\perp T)$ turns out to behave as a unitary $\mathbb{Z}_2$ symmetry on the domain wall. The theory on the T domain wall is based on this induced $\mathbb{Z}_2$ symmetry.

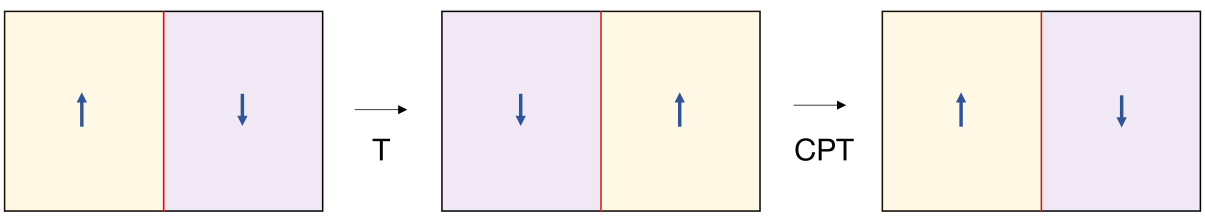

Figure 8.12: The illustration for the T domain wall. The T domain wall separates the two distinct vacua in the T broken phase. The T symmetry acts on the figure by changing two vacua (i.e., yellow $\leftrightarrow$ purple). Since the CPT commutes with T (up to fermion parity), the CPT leaves the vacua of the T-broken phase invariant, and acts as the parity that reflects the figure across the domain wall. T alone cannot be a symmetry on the domain wall since it flips the domain, but $T \cdot (CP_\perp T)$ works as the symmetry on the wall, since $CP_\perp T$ reflects back the configuration of domains.

More generally, if one has a $\mathbb{Z}_2$ symmetry which may be unitary or anti-unitary, we can find an induced symmetry operation that acts solely on the domain wall by combining with the CPT symmetry. We can find the algebra of the combined symmetry operation by the following properties of the CPT symmetry [68];

[1] In this thesis, the operation P actually means a spatial reflection that reverses one of spatial directions rather than the parity that reverses all directions. For example, [40] writes this operation as R as an abbreviation of the reflection.

1. Any unitary internal global symmetry U commutes with the CPT symmetry,

$$U \cdot (CPT) = CPT \cdot U. \tag{8.5}$$

2. Any time-reversal symmetry T commutes with the CPT symmetry up to fermion parity,

$$T \cdot (CPT) = CPT \cdot T \cdot (-1)^F. \tag{8.6}$$

3. The CPT symmetry squares to 1,

$$(CPT)^2 = 1. \tag{8.7}$$

For example, on the domain wall of the T symmetry with $T^2 = (-1)^F$, we have a unitary $\mathbb{Z}_2$ symmetry $T \cdot (CP_\perp T)$ since

$$T \cdot (CP_\perp T) \cdot T \cdot (CP_\perp T) = T^2 \cdot (CP_\perp T)^2 \cdot (-1)^F = 1. \tag{8.8}$$

If we start with a unitary $\mathbb{Z}_2$ symmetry U, its domain wall has a time-reversal symmetry with $T^2 = 1$, since

$$U \cdot (CP_\perp T) \cdot U \cdot (CP_\perp T) = U^2 \cdot (CP_\perp T)^2 = 1. \tag{8.9}$$

If we start with a T symmetry with $T^2 = 1$, its domain wall has a unitary $\mathbb{Z}_4^F$ symmetry U with $U^2 = (-1)^F$, since

$$T \cdot (CP_\perp T) \cdot T \cdot (CP_\perp T) = T^2 \cdot (CP_\perp T)^2 \cdot (-1)^F = (-1)^F. \tag{8.10}$$

Finally, if we start with a unitary $\mathbb{Z}_4^F$ symmetry with $U^2 = (-1)^F$, its domain wall has a time-reversal symmetry T with $T^2 = (-1)^F$, since

$$U \cdot (CP_\perp T) \cdot U \cdot (CP_\perp T) = U^2 \cdot (CP_\perp T)^2 = (-1)^F. \tag{8.11}$$

After all, we find a hierarchy of the global symmetry on the domain wall with the mod 4 periodicity,

$$\cdots \Rightarrow \left\{T^2 = (-1)^F\right\} \Rightarrow \left\{U^2 = 1\right\} \Rightarrow \left\{T^2 = 1\right\} \Rightarrow \left\{U^2 = (-1)^F\right\} \Rightarrow \cdots, \tag{8.12}$$

where the induced symmetry on the domain wall is written on the right.

8.2.2 Induced spacetime structure and the Smith map

The global symmetry on the domain wall can also be understood geometrically. For example, let us consider a $(d+1)$-dimensional QFT for fermions with a $\mathbb{Z}_2$ symmetry. In that case, we have a $\mathbb{Z}_2$ symmetry defect in a $(d+1)$-dimensional spacetime manifold with equipped with a spin structure. The global symmetry for theories localized on the defect can be read by the spacetime structure of a d-dimensional $\mathbb{Z}_2$ defect. Actually, we can see that the d-dimensional $\mathbb{Z}_2$ defect has a Pin^- structure. To see this, let us denote M as a $(d+1)$-dimensional spin manifold and K as a d-dimensional submanifold of M that represents the $\mathbb{Z}_2$ defect. We have

$w_1(TM) = w_2(TM) = 0$ according to the spin structure on M. On the symmetry defect K, tangent bundle TM is expressed as

$$TM|_K \cong TK \oplus NK, \tag{8.13}$$

where NK represents the normal bundle of K. Due to $w_1(TM) = w_1(TK) + w_1(NK) = 0$, we have $w_1(TK) = w_1(NK)$. Then, we can see that $w_2(TK) + w_1^2(TK) = 0$, since

$$w_2(TM) = w_2(TK) + w_1(TK)w_1(NK) = w_2(TK) + w_1^2(TK) = 0, \tag{8.14}$$

due to the Whitney sum formula $w_n(E \oplus F) = \sum_i w_i(E)w_{n-i}(F)$ for real vector bundles E, F with finite rank. This implies that K has a Pin$^-$ structure. In general, let L be a real line bundle on M associated to the principal $\mathbb{Z}_2$ bundle. Then, in the presence of the $\mathbb{Z}_2$ gauge field and the associated line bundle L, the spacetime structures on M that correspond to the mod 4 periodicity are known to be expressed in terms of spin structure on $TM \oplus L^{\oplus m}$ (called twisted spin structure),

$$\begin{aligned} \{U^2 = 1\} &\Rightarrow \text{spin structure on } TM \oplus L^{\oplus 4k}, \\ \{T^2 = 1\} &\Rightarrow \text{spin structure on } TM \oplus L^{\oplus 4k+1}, \\ \{U^2 = (-1)^F\} &\Rightarrow \text{spin structure on } TM \oplus L^{\oplus 4k+2}, \\ \{T^2 = (-1)^F\} &\Rightarrow \text{spin structure on } TM \oplus L^{\oplus 4k+3}, \end{aligned} \tag{8.15}$$

with $k \in \mathbb{Z}$. In particular, Pin$^-$, Spin$^{\mathbb{Z}_4}$, Pin$^+$ structures are all given in terms of the twisted spin structure. Since the spin structure on $TM \oplus L^{\oplus m}$ depends on m only mod 4, the expression of symmetries by twisted spin structure explains the mod 4 periodicity of the symmetry on the $\mathbb{Z}_2$ defect (8.12). That is, when M has a spin structure on $TM \oplus L^{\oplus m}$, the $\mathbb{Z}_2$ defect K has the induced spin structure on $TM \oplus L^{\oplus m+1}$, since $TM = TK \oplus NK$ and we have [70]

$$L|_K \cong NK \tag{8.16}$$

on the symmetry defect. Thus, we have a mod 4 periodicity for the induced spacetime structure on the $\mathbb{Z}_2$ defect,

$$\cdots \Rightarrow \{\text{Pin}^+\} \Rightarrow \{\text{Spin} \times \mathbb{Z}_2\} \Rightarrow \{\text{Pin}^-\} \Rightarrow \{\text{Spin}^{\mathbb{Z}_4}\} \Rightarrow \cdots, \tag{8.17}$$

which corresponds to (8.12).

Then, picking up the $\mathbb{Z}_2$ defect turns out to define a map between bordism group of manifolds. Namely, we have a sequence of linear maps of bordism groups

$$\cdots \Omega_d^{\text{spin}}(B\mathbb{Z}_2) \to \Omega_{d-1}^{\text{pin}^-} \to \Omega_{d-2}^{\text{spin}^{\mathbb{Z}_4}} \to \Omega_{d-3}^{\text{pin}^+} \to \cdots, \tag{8.18}$$

which are called the Smith maps of bordism groups. Dually, we also have a sequence of the Smith maps for the cobordism groups running in the opposite way

$$\cdots \Omega^{d-3}_{\text{pin}^+} \to \Omega^{d-2}_{\text{spin}^{\mathbb{Z}_4}} \to \Omega^{d-1}_{\text{pin}^-} \to \Omega^{d}_{\text{spin}}(B\mathbb{Z}_2) \to \cdots. \tag{8.19}$$

In general, the Smith map for the cobordism groups describes the gapped phase carried by the T symmetry defect. That is, when the theory is trivially gapped away from the symmetry defect, the classification of the $(d+1)$-dimensional SPT phase ν_{d+1} is totally encoded in that of the d-dimensional symmetry defect ν_d, as $\nu_{d+1} = f(\nu_d)$. In that case, one can construct an action for the $(d+1)$-dimensional SPT phase in terms of the d-dimensional action localized on the symmetry defect. For example, let us consider the Smith map for the (2+1)-dimensional $\mathbb{Z}_2$ SPT phase,

$$f_2 : \Omega^2_{\mathrm{pin}^-} \to \Omega^3_{\mathrm{spin}}(B\mathbb{Z}_2), \tag{8.20}$$

which defines a map $\mathbb{Z}_8 \to \mathbb{Z}_8 \oplus \mathbb{Z}$ expressed as

$$f_2 : \nu \to (\nu, 0). \tag{8.21}$$

In this case, the $\mathbb{Z}_8$ generator of the (2+1)-dimensional $\mathbb{Z}_2$ SPT phase $\nu = 1 \in \mathbb{Z}_8$ can be described as the (1+1)-dimensional action on a $\mathbb{Z}_2$ defect that constitutes a Pin$^-$ surface. Namely, the action can be given by evaluating the ABK invariant illustrated in Sec. 6.1 on the $\mathbb{Z}_2$ defect, which is schematically expressed as "$a \cup \mathrm{ABK}$" with a a background $\mathbb{Z}_2$ gauge field. Here, the symbol for cup product just means that we evaluate the ABK invariant on the Poincaré dual of the $\mathbb{Z}_2$ gauge field. A theory constructed in such a manner has the partition function quantized as 8th root of unity, when evaluated on $\mathbb{RP}^3$ equipped with a nontrivial $\mathbb{Z}_2$ gauge field a. Actually, the Poincaré dual of $a \in H^1(\mathbb{RP}^3, \mathbb{Z}_2)$ becomes $\mathbb{RP}^2$ equipped with a Pin$^-$ structure, with the ABK invariant 8th root of unity. This shows that the (2+1)-dimensional action "$a \cup \mathrm{ABK}$" works as the generator for the $\mathbb{Z}_8$ part of $\Omega^3_{\mathrm{spin}}(B\mathbb{Z}_2)$.

In addition, the T symmetry defect of a (3+1)-dimensional SPT phase with $T^2 = (-1)^F$ is described by the Smith map

$$f_3 : \Omega^3_{\mathrm{spin}}(B\mathbb{Z}_2) \to \Omega^4_{\mathrm{pin}^+}, \tag{8.22}$$

which gives a homomorphism $\mathbb{Z}_8 \oplus \mathbb{Z} \to \mathbb{Z}_{16}$. The form of this map is given by

$$f_3 : (\nu, k) \to 2\nu - k \mod 16. \tag{8.23}$$

In this case, the action for the $\mathbb{Z}_8$ subgroup of the $\mathbb{Z}_{16}$ classification in (3+1) dimensions can be prepared by a (2+1)-dimensional action on the T defect. On the T defect, the restriction of w_1 on the T defect further defines an induced $\mathbb{Z}_2$ gauge field on the (2+1)-dimensional spacetime. Using the induced $\mathbb{Z}_2$ gauge field, one can define the action in the form of "$a \cup \mathrm{ABK}$" on the T defect, which defines the action for the $\nu = 2$ of the $\mathbb{Z}_{16}$ classification. Schematically, the (3+1)-dimensional action can be expressed as "$w_1 \cup (w_1 \cup \mathrm{ABK})$", where the symbol for the cup product means the evaluation on the Poincaré dual of w_1.

A theory constructed in such a manner has the partition function quantized as 8th root of unity, when evaluated on $\mathbb{RP}^4$ equipped with a Pin$^+$ structure. Actually, the Poincaré dual of $w_1 \in H^1(\mathbb{RP}^4, \mathbb{Z}_2)$ becomes $\mathbb{RP}^3$ equipped with a nontrivial $\mathbb{Z}_2$ gauge field, with the action "$a \cup \mathrm{ABK}$" 8th root of unity. This shows that the (3+1)-dimensional action "$w_1 \cup (w_1 \cup \mathrm{ABK})$" works as the generator for the $\mathbb{Z}_8$ part of $\Omega^4_{\mathrm{pin}^+}$, since $\mathbb{RP}^4$ generates the Pin$^+$ bordism group $\Omega_4^{\mathrm{pin}^+} = \mathbb{Z}_{16}$.

In contrast, the generator of the $\Omega^4_{\mathrm{pin}^+} = \mathbb{Z}_{16}$ classification cannot be expressed by an action isolated on the T defect. In this case, according to the Smith map (8.23) we have odd k copies of $p + ip$ superconductors represented by the gravitational Chern-Simons theory on the T defect,

$$k \cdot \int_{K=\partial W} \mathrm{CS}_{\mathrm{grav}} = \frac{k}{192\pi} \int_W \mathrm{Tr}(R \wedge R). \tag{8.24}$$

which means that $c_- = k/2$ on the T defect. In particular, when the T defect is realized as the T domain wall in the symmetry broken phase, it implies that the left and right domain separated by the T domain wall must carry different gravitational response actions, $c_{-,L} - c_{-,R} = k/2$, where $c_{-,L}, c_{-,R}$ denote the framing anomaly in the left and right domain, respectively. In addition, we have $c_{-,L} = -c_{-,R}$, since the vacua for the left domain is obtained by acting the time-reversal on the right domain, which reverses the gravitational response c_-. Combining these facts, we can see that both the left and right domain have $c_- = 1/4 \bmod 1/2$ when k is odd. This nontrivial quantization of c_- is one consequence for the (3+1)-dimensional T SPT phase with the odd index in $\mathbb{Z}_{16}$. See also the author's work [71] and [72].

8.3 (3+1)d T SPT phase via decorated domain wall

Here, we claim that an invertible gapped phase realized by the wave function with the T symmetry constructed in Sec. 8.1 generates the $\mathbb{Z}_8$ subgroup in the $\mathbb{Z}_{16}$ classification, by comparing with the field theoretical argument in Sec. 8.2. This can be seen by studying the (2+1)-dimensional phase localized on the $\mathbb{Z}_2$ defect, which is realized as a domain wall of σ qubits on the 3-dimensional cells. For a given configuration of σ qubits, we get a wave function for the fluctuating Kitaev wires on the two-dimensional T domain wall K, as described in Fig. 8.6. This two-dimensional wave function has the same form as a wave function constructed by Tarantino-Fidkowski in Chapter 7 for the $\mathbb{Z}_2$ SPT phase. In particular, we find a unitary $\mathbb{Z}_2$ symmetry of the wave function on the T domain wall, generated by flipping τ qubits on the domain wall, $U_{\mathbb{Z}_2} = \prod_{\mathrm{K}} \tau^x$.

Since τ qubits are charged under the T symmetry, the configuration of τ^z on the $\mathbb{Z}_2$ domain wall is thought to work as the placeholder for the section of orientation line bundle L restricted to the T domain wall. Hence, we identify the τ qubits on the $\mathbb{Z}_2$ domain wall as the section of the induced $\mathbb{Z}_2$ gauge field on the domain wall, i.e., $U_{\mathbb{Z}_2}$ is regarded as generating the induced $\mathbb{Z}_2$ symmetry on the wall, and the one-dimensional domain wall of τ qubits is identified as the domain wall of the induced $\mathbb{Z}_2$ symmetry. Since the T domain wall carries the (2+1)-dimensional $\mathbb{Z}_2$ SPT phase generating the $\mathbb{Z}_8$ classification, we regard our wave function as the image of $(1, 0) \in \Omega^3_{\mathrm{spin}}(B\mathbb{Z}_2)$ via the Smith map f_3 in (8.22).

Chapter 9

Conclusion

In conclusion, we studied local definitions of fermionic SPT phases and their boundaries. We proposed a way to construct a path integral for fermionic topological phases with time-reversal symmetry, by generalizing the fermionization proposed in [21] to the unoriented Pin^+ or Pin^- case. By utilizing the fermionization, we provided a path integral for Gu-Wen phases in generic spacetime dimensions and (1+1)-dimensional topological superconductor.

We also studied an operation called partial time-reversal on a ground state wave function of (1+1)-dimensional topological superconductor with T symmetry such that $T^2 = 1$. We have shown that the partial time-reversal allows us to diagnose the $\mathbb{Z}_8$ classification of a given ground state. We further obtained a symmetry-preserving gapped boundary of Gu-Wen phases with finite group symmetry based on the path integral of the bulk-boundary system. Finally, we discussed a wave function of the (3+1)-dimensional beyond Gu-Wen phase with $T^2 = (-1)^F$ based on the decorated domain wall construction. In the T symmetry broken phase, we have seen how the degrees of freedom on the T domain wall inherits the classification of the original (3+1)-dimensional SPT phase by utilizing the Smith map of cobordism groups.

We close this thesis with some possible future prospects. The bosonization and fermionization for lattice systems discussed in Chapter 3 and Chapter 4 would be practically useful for the development of fermionic simulation algorithms, since one can apply bosonic numerical methods to the dual Hamiltonian instead of directly dealing with the fermionic Hamiltonian. For instance, the path integral description of fermionic phases enables us to represent the wave function of a fermionic system in terms of tensor network representation for the bosonic dual system. In addition, a Hamiltonian version of the bosonization that corresponds to the Gaiotto-Kapustin's path integral description is proposed in [73]. For instance, these bosonization techniques should be helpful for finding the ground state of two-dimensional fermionic system by utilizing the bosonic dual system by e.g., density matrix renormalization group (DMRG) [74] on a cylinder. Recently, the bosonization is also utilized to simulate a fermionic system on qubit system on a digital superconducting quantum processor [75]. It would be interesting to realize the bosonic dual of a fermionic topological phase studied in this thesis on a qubit system.

For the partial transpose introduced in Chapter 6 to diagnose the $\mathbb{Z}_8$ classification of the (1+1)-dimensional topological superconductor, there are experimental proposals [76, 77] to measure moments of the partially transposed density matrix with ion traps and cold atoms.

It is interesting to establish experimental protocols to measure the non-local order parameter discussed in Chapter 6. We have shown in Chapter 8 that a wave function for a (3+1)-dimensional topological superconductor with $T^2 = (-1)^F$ can be described on a system of qubits and complex fermions. In [25], we constructed a commuting projector Hamiltonian realizing the wave function, by summing over local Hamiltonians that fluctuates the decorated domain wall. The existence of a commuting projector Hamiltonian for an SPT phase is important, since the many-body localized phases exhibiting SPT phases in generic finite energy density eigenstates can exist only when the SPT phase in question has a representation in terms of a commuting projector Hamiltonian [78]. In addition, it is promising to give an efficient tensor network representation useful for numerical implementations, based on out wave function of a (3+1)-dimensional topological superconductor.

In this thesis, we did not mention the generator of $\mathbb{Z}_{16}$ for the (3+1)-dimensional topological superconductor with $T^2 = (-1)^F$. In [50], the author and his collaborators obtained a path integral for $3 \in \mathbb{Z}_{16}$ phase that generates the $\mathbb{Z}_{16}$ classification. This is also done by fermionizing a specific (3+1)-dimensional bosonic path integral following the prescription of Chapter 4 in this thesis. The construction of the bosonic path integral is given by a generalized Crane-Yetter state sum starting with a super-modular tensor category that describes $SO(3)_3$ Chern-Simons theory. It should be interesting to study the symmetry-preserving gapped boundary of this theory by utilizing the bulk-boundary Grassmann integral, which is expected to be the $SO(3)_3$ Chern-Simons theory with T symmetry.

Though we have constructed the symmetry-preserving gapped boundary for the SPT phases with finite discrete symmetry, it is intriguing whether a generic SPT phases with continuous symmetry admits a symmetry-preserving gapped boundary. Hopefully, a large class of symmetry-preserving gapped boundary is obtained by fermionizing a generalized Crane-Yetter model equipped with the continuous symmetry.

It is also very interesting to study the SPT phases and their boundary with crystalline symmetries. In that case, there is a simple way to reduce the (3+1)-dimensional SPT phase to the lower dimensional system with internal symmetries [79]. For example, consider a unitary reflection symmetry across the (2+1)-dimensional plane which protects the the (3+1)-dimensional SPT phase. Then, we can operate the unitary circuit respecting the reflection symmetry away from the reflection plane, which can disentangle the SPT phase away from the reflection plane. After all, we obtain the reduced (2+1)-dimensional SPT phase supported on the reflection plane, where the reflection symmetry acts internally. Based on this logic, hopefully one can obtain a lattice model for a (3+1)-dimensional SPT phase with the spatial reflection symmetry such that $R^2 = 1$ which is classified by $\mathbb{Z}_{16}$.

Appendix A

Homology and cohomology

In this appendix, we review the concepts of simplical homology and cohomology used in this thesis.

A.1 chains and cochains

Let M be a triangulated d-dimensional manifold. For $0 \leq k \leq d$, there is a set of "k-chains" denoted as $C_k(M, X)$ with an abelian group X. $C_k(M, X)$ is a vector space with coefficient X, with one basis element on each k-simplex of M. Hence, $c \in C_k(M, X)$ is regarded as a subset of the k-simplices of the triangulation, and written as

$$c = \sum_{\langle 0 \dots k \rangle \in M} c_{\langle 0 \dots k \rangle} \left| 0 \dots k \right\rangle, \qquad c_{\langle 0 \dots k \rangle} \in X, \tag{A.1}$$

where $\langle 0 \dots k \rangle$ denotes a k-simplex. Then, there is a linear operator $\partial : C_k(M, X) \to C_{k-1}(M, X)$ called a boundary operator, defined as

$$\partial \left| 0 \dots k \right\rangle = \sum_{i=0}^{k} (-1)^i \left| 0 \dots \hat{i} \dots k \right\rangle, \tag{A.2}$$

where $\hat{i}$ means that we exclude i from the collection of vertices. Since each $(k-1)$-simplex $\langle 0 \dots \hat{i} \dots k \rangle$ lives on a boundary of a k-simplex $\langle 0 \dots k \rangle$, ∂c actually represents the boundary of a k-simplex c. We say c is a closed chain if $\partial c = 0$. The set of closed k-chains is written as $Z_k(M, X)$. In addition, the set of k-chains written as $c = \partial b$ with some $b \in C_{k+1}(M, X)$ is written as $B_k(M, X)$. Since we have $\partial\partial c = 0$ for any c, $B_k(M, X)$ is a subset of $Z_k(M, X)$, $B_k(M, X) \subset Z_k(M, X)$, meaning that a boundary of some chain is always closed.

Now let us illustrate k-cochains of M denoted as $C^k(M, X)$. This is the dual space of $C_k(M, X)$ spanned by linear maps $C_k(M, X) \to X$, that is, $C^k(M, X) = \mathrm{Hom}(C_k(M, X), X)$. A k-cochain $\gamma \in C^k(M, X)$ can be represented as

$$\gamma = \sum_{\langle 0 \dots k \rangle \in M} \gamma_{\langle 0 \dots k \rangle} \left\langle 0 \dots k \right|, \qquad \gamma_{\langle 0 \dots k \rangle} \in X. \tag{A.3}$$

Then, the coboundary map $\delta : C^k(M,X) \to C^{k+1}(M,X)$ is defined as a dual map of $\partial : C_{k+1}(M,X) \to C_k(M,X)$. That is, for a given $\gamma \in \mathrm{Hom}(C_k(M,X),X)$, $\delta(\gamma) := \gamma\partial \in \mathrm{Hom}(C_{k+1}(M,X),X)$. Specifically, the coboundary acts on a k-cochain as

$$\delta\gamma = \sum_{\langle 0\ldots k+1\rangle \in M} \gamma(\partial\,|0\ldots k+1\rangle)\,\langle 0\ldots k+1|\,. \tag{A.4}$$

Then, we again say that γ is closed if $\delta\gamma = 0$, and the set of closed k-cochains is denoted as $Z^k(M,X)$. In addition, the set of k-cochains written as $\gamma = \delta\beta$ with some $\beta \in C^{k-1}(M,X)$ is denoted as $B^k(M,X)$. By the definition of δ, we have $\delta\delta\gamma = 0$ for any γ, and then $B^k(M,X) \subset Z^k(M,X)$.

Now, the homology group of M is defined as $H_k(M,X) := Z_k(M,X)/B_k(M,X)$, and the cohomology group is $H^k(M,X) := Z^k(M,X)/B^k(M,X)$.

A.2 Poincaré duality

We can also consider the dual cellulation $M^\vee$ of M. In general, the dual cellulation $M^\vee$ of the triangulation is not a triangulation. Each k-simplex $\langle v_0 \ldots v_k\rangle$ of M is the dual of a $(d-k)$-cell $P_{\langle v_0\ldots v_k\rangle}$ of $M^\vee$, as shown in Fig. A.1 for two dimensions. Though we do not introduce the construction of the dual cellulation in detail, $M^\vee$ can be obtained by firstly taking barycentric subdivision of M and then gathering some subdivided d-simplices into a single d-cell.

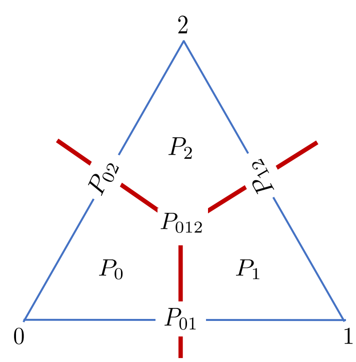

Figure A.1: The dual cellulation $M^\vee$ for the original triangulation M.

Then, the k-chains for the dual cellulation is again defined as a formal linear combination of the k-cells of $M^\vee$ in the form of

$$c = \sum_{P_{0\ldots d-k} \in M^\vee} c_{P_{0\ldots d-k}}\,|P_{0\ldots d-k}\rangle\,, \qquad c_{P_{0\ldots d-k}} \in X. \tag{A.5}$$

The set of $(d-k)$-chains of $M^\vee$ is written as $C_{d-k}(M^\vee,X)$. Then, the boundary operator ∂ can be defined via the coboundary δ of the original triangulation M,

$$\partial\,|P_{0\ldots d-k}\rangle = \sum_{\langle \tilde r_0\ldots\tilde r_{d-k+1}\rangle} \left((\delta\,\langle 0\ldots d-k|)\,|\tilde r_0\ldots\tilde r_{d-k+1}\rangle\right)|P_{\langle \tilde r_0\ldots\tilde r_{d-k+1}\rangle}\rangle\,. \tag{A.6}$$

That is, the boundary is given by the set of the dual of a $(d-k+1)$-cells whose boundaries contain the $(d-k)$-simplex $\langle 0 \dots d-k \rangle$. Based on this boundary operator, we can naturally define $Z_{d-k}(M^\vee, X)$, $B_{d-k}(M^\vee, X)$. By definition, we again have $\partial\partial c = 0$ for any k-chain c.

Then, k-cochains are defined as the linear map $C_k(M^\vee, X) \to X$, and we write $C^k(M^\vee, X) := \mathrm{Hom}(M^\vee, X)$. The coboundary operator is defined via the boundary of M,

$$\delta \langle P_{0\dots d-k}| = \sum_{\langle \tilde{r}_0 \dots \tilde{r}_{d-k+1} \rangle} (\langle 0 \dots d-k|\, \partial(|\tilde{r}_0 \dots \tilde{r}_{d-k+1}\rangle)) \langle P_{\langle \tilde{r}_0 \dots \tilde{r}_{d-k+1} \rangle}| \,. \tag{A.7}$$

Then we can define $Z^k(M^\vee, X)$, $B^k(M^\vee, X)$ using the coboundary. Note that $C_{d-k}(M^\vee, X)$ is identical to $C^k(M, X)$, and the boundary (resp. coboundary) of $C_{d-k}(M^\vee, X)$ has the same action as the coboundary (resp. boundary) of $C^k(M, X)$. This means that we have $Z^k(M^\vee, X) \cong Z_{d-k}(M, X)$, $Z_k(M^\vee, X) \cong Z^{d-k}(M, X)$, and the same statement holds for $B^k(M^\vee, X)$ and $B_k(M^\vee, X)$. This is called the Poincaré duality between M and $M^\vee$.

Appendix B

Cup product and higher cup product

A branching structure on a triangulation is a local ordering of vertices, which can be specified by an arrow on each 1-simplex $\langle ij\rangle$, such there are no closed loops on any 2-simplices. This defines a total ordering of vertices on every single d-simplex $\langle 0\dots d\rangle$. In this appendix, we review cochain-level product operation called higher cup product, whose definitions are based on branching structure of the triangulation. Here we limit ourselves to the $\mathbb{Z}_2$-valued cochains, setting $X = \mathbb{Z}_2$ in Appendix A. See also [80] for a nice reference on higher cup product.

Let M be a triangulated d-dimensional manifold. Firstly, the cup product gives the product of cochains

$$-\cup- : C^k(M,\mathbb{Z}_2)\times C^l(M,\mathbb{Z}_2)\to C^{k+l}(M,\mathbb{Z}_2), \tag{B.1}$$

whose explicit form is written as

$$(\alpha\cup\beta)(0,\dots,k+l) = \alpha(0,\dots,k)\beta(k,\dots,k+l). \tag{B.2}$$

Note that this definition of cup product depends on the branching structure on the triangulation, where the ordering of vertices on each $(k+l)$-simplex is specified as $0\to 1\to\cdots\to k+l$. The cup product satisfies the Leibniz rule at the cochain level,

$$\delta(\alpha\cup\beta) = \delta\alpha\cup\beta+\alpha\cup\delta\beta. \tag{B.3}$$

According to the Leibniz rule, one can show that the cup product defines the product of cohomologies $H^k(M,\mathbb{Z}_2)\times H^l(M,\mathbb{Z}_2)\to H^{k+l}(M,\mathbb{Z}_2)$. Actually, for given $\alpha\in Z^k(M,\mathbb{Z}_2)$, $\beta\in Z^l(M,\mathbb{Z}_2)$, the shift of these cocycles by coboundaries is evaluated as

$$(\alpha+\delta A)\cup(\beta+\delta B) = \alpha\cup\beta+\delta(\alpha\cup B + A\cup\beta + A\cup\delta B), \tag{B.4}$$

so this also shifts $\alpha\cup\beta$ by a coboundary, thus defines a map between cohomologies. Such a product operation defined on cohomologies is called a cohomology operation.

It is known that the cup product has the geometrical interpretation in the picture of the Poincaré dual. That is, for given cochains $\alpha\in C^k(M,\mathbb{Z}_2)$ and $\beta\in C^l(M,\mathbb{Z}_2)$, the cup product $\alpha\cup\beta$ is the Poincaré dual to the intersection of Poincaré duals $\alpha^\vee\cap\beta^\vee$, for $\alpha^\vee\in C_{d-k}(M^\vee,\mathbb{Z}_2), \beta^\vee\in C_{d-l}(M^\vee,\mathbb{Z}_2)$. This can be understood as follows. Let us consider

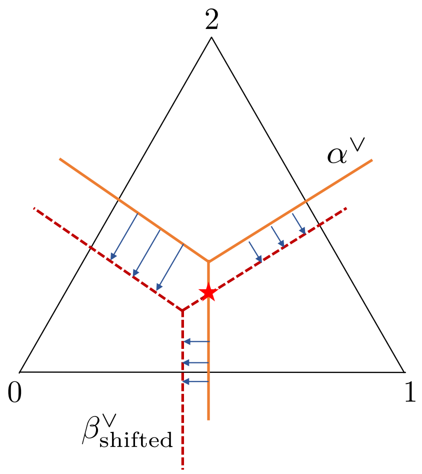

Figure B.1: The cup product $(\alpha \cup \beta)(012) = \alpha(01)\beta(12)$ can be understood as the intersection of the dual chains in the Poincaré dual picture.

a shifted version of the Poincaré dual $\beta^\vee_{\mathrm{shift}}$, where the shifting vector is determined by the branching structure. See Fig. B.1 for two dimensions. Then, the intersection of $\alpha^\vee$ and $\beta^\vee_{\mathrm{shifted}}$ corresponds to the dual of $\alpha \cup \beta$.

As a generalization of the cup product, the higher cup product $\cup_i$ gives

$$- \cup_i - : C^k(M, \mathbb{Z}_2) \times C^l(M, \mathbb{Z}_2) \to C^{k+l-i}(M, \mathbb{Z}_2), \tag{B.5}$$

whose explicit form is written as

$$(\alpha \cup_i \beta)(0, \dots, k+l-i) = \sum_{0 \leq j_0 < \cdots < j_i \leq k+l-i} \alpha(0 \to j_0, j_1 \to j_2, \dots)\beta(j_0 \to j_1, j_1 \to j_3, \dots). \tag{B.6}$$

Here, the notation $i \to j$ denotes all vertices from i to j, $\{i, i+1, \dots, i+j\}$. In particular, $\cup_0$ is identified as the cup product $\cup$ defined in (B.2). The higher cup product is subject to the generalized Leibniz rule,

$$\delta(\alpha \cup_i \beta) = (\delta\alpha) \cup_i \beta + \alpha \cup_i (\delta\beta) + \alpha \cup_{i-1} \beta + \beta \cup_{i-1} \alpha, \tag{B.7}$$

which is regarded as that the non-commutative property of $\cup_{i-1}$ is controlled by the $\cup_i$ product. According to the above Leibniz rule, for closed α and β, one can see that $\alpha \cup_i \beta$ is not necessarily closed, $\delta(\alpha \cup_i \beta) = \alpha \cup_{i-1} \beta + \beta \cup_{i-1} \alpha$ for $\alpha \in Z^k(M, \mathbb{Z}_2), \beta \in Z^l(M, \mathbb{Z}_2)$. Hence, the product $\cup_i$ doesn't give a cohomology operation for $i > 0$.

However, it turns out that the map

$$\begin{aligned} &\mathrm{Sq}^{d-i}(\alpha) : Z^k(M, \mathbb{Z}_2) \to Z^{k+d-i}(M, \mathbb{Z}_2) \\ &\mathrm{Sq}^{d-i}(\alpha) := \alpha \cup_{i+k-d} \alpha \end{aligned} \tag{B.8}$$

does give a cohomology operation. Actually, one can check that $\mathrm{Sq}^{d-i}(\alpha + \delta A) = \mathrm{Sq}^{d-i}(\alpha) + \delta(\alpha \cup_{i+k-d} A + A \cup_{i+k-d} \alpha + A \cup_{i+k-d-1} A + A \cup_{i+k-d} \delta A)$ by using the generalized Leibniz rule. This shows that Sq^{d-i} defines a map $H^k(M, \mathbb{Z}_2) \to H^{k+d-i}(M, \mathbb{Z}_2)$.

For convenience, we extend the definition of the Sq^{d-i} operation to non-closed cochains. For a given $\lambda \in C^k(M, \mathbb{Z}_2)$, we define

$$\mathrm{Sq}^{d-i}(\lambda) : C^k(M, \mathbb{Z}_2) \to C^{k+d-i}(M, \mathbb{Z}_2) \tag{B.9}$$

$$\mathrm{Sq}^{d-i}(\lambda) := \lambda \cup_{i+k-d} \lambda + \delta\lambda \cup_{i+k-d+1} \lambda. \tag{B.10}$$

One can immediately check that Sq^{d-i} commutes with the coboundary,

$$\delta\, \mathrm{Sq}^{d-i}(\lambda) = \mathrm{Sq}^{d-i}(\delta\lambda). \tag{B.11}$$

Appendix C

A useful formula for Stiefel-Whitney homology classes

In this appendix, we prove the expression for the dual of the representative of w_2 used in (8.2). First we recall the theorem in [73],

Theorem. In a three-dimensional manifold M with triangulation and branching structure, the homology class of the dual of w_2 is represented by a 1-chain $S' \in C_1(M, \mathbb{Z}_2)$,

$$S' = \sum_e e - \sum_{\Delta_+ = \langle 0123 \rangle} \langle 02 \rangle - \sum_{\Delta_- = \langle 0123 \rangle} \langle 13 \rangle, \tag{C.1}$$

where the first sum is over all 1-simplices of the triangulation, and Δ_+ (resp. Δ_-) denotes a $+$ (resp. $-$) 3-simplex.

We show that the above 1-chain S' is homologically equivalent to S in (8.2). To do this, we consider a branching structure of $\mathcal{T}'$ defined as follows. First, we assign a local ordering to vertices of $\mathcal{T}$, such that the vertex on the barycenter of a i-simplex is labeled as i. Then, while respecting the ordering on vertices of $\mathcal{T}$, we further assign a local ordering on vertices of $\mathcal{T}'$, such that a barycenter of a j-simplex of $\mathcal{T}$ has a larger ordering than that of an i-simplex if $j > i$. Then, we have an induced branching structure on $\mathcal{T}'$. Based on this branching structure, after some efforts we can write S' in (C.1) as

$$\begin{aligned} S' = \sum_{e \in \mathcal{T}'} e &- \sum_{\Delta_+} (\langle v_1 v_{0123} \rangle + \langle v_3 v_{0123} \rangle + \langle v_0 v_{013} \rangle + \langle v_2 v_{123} \rangle) \\ &- \sum_{\Delta_-} (\langle v_1 v_{0123} \rangle + \langle v_3 v_{0123} \rangle + \langle v_0 v_{012} \rangle + \langle v_2 v_{023} \rangle), \end{aligned} \tag{C.2}$$

where the convention is the same as the expression in (8.2). Up to a boundary of a 2-chain, the above S' is written as

$$\begin{aligned} S' = \sum_{e \in \mathcal{T}'} e &- \sum_{\Delta_+} (\langle v_{013} v_{0123} \rangle + \langle v_{123} v_{0123} \rangle + \langle v_1 v_3 \rangle + \langle v_0 v_2 \rangle) \\ &- \sum_{\Delta_-} (\langle v_{012} v_{0123} \rangle + \langle v_{023} v_{0123} \rangle + \langle v_1 v_3 \rangle + \langle v_0 v_2 \rangle). \end{aligned} \tag{C.3}$$

Since the contributions of $\langle v_1 v_3 \rangle, \langle v_0 v_2 \rangle$ cancel out on $+$ and $-$ simplices, we finally get (8.2).

Appendix D

Group cohomology

In this appendix, we review group cohomology utilized to define topological terms for SPT phases.

D.1 Classifying space and group cohomology

There exists a space BG called a classifying space, which has the property that a flat G-gauge field on a d-dimensional manifold M is specified by a map $g : M \to BG$. The classifying space BG is constructed as follows. Firstly, let EG be a simplical complex whose k-simplices are given by a collection of $(k+1)$ group elements $\{|g_0, \ldots, g_k\rangle \, | g_i \in G\}$ glued along $(k-1)$-simplices $[g_0, \ldots, \hat{g}_i, \ldots, g_k]$ in an obvious way, where $\hat{g}_i$ means skipping over g_i in the collection of group elements. We define a group action on EG as

$$g : EG \to EG, \quad |g_0, \ldots g_k\rangle \to |gg_0, \ldots gg_k\rangle \, . \tag{D.1}$$

Now we define BG as the quotient space by the group action $BG := EG/G$. A k-simplex of BG can be expressed in the form of

$$|g_1|g_2| \ldots |g_k\rangle := G \, |e, g_1, g_1 g_2, \ldots, g_1 \ldots g_k\rangle \, , \tag{D.2}$$

where $e \in G$ is the unit element of G. The boundary of a k-simplex of BG is given by

$$\partial \, |g_1|g_2| \ldots |g_k\rangle = |g_2| \ldots |g_k\rangle + (-1)^k \, |g_1| \ldots g_{k-1}\rangle + \sum_{i=1}^{k-1} (-1)^i \, |g_1| \ldots |g_i g_{i+1}| \ldots |g_k\rangle \, . \tag{D.3}$$

This defines $Z_k(BG, X)$, $B_k(BG, X)$ and $H_k(BG, X)$. Then, the cochain $\omega \in C^k(BG, X)$ is a linear map $C_k(BG, X) \to X$,

$$\omega = \sum_{g_1, g_2, \cdots, g_k} \omega(g_1, g_2, \ldots, g_k) \, \langle g_1|g_2| \ldots |g_k| \, . \tag{D.4}$$

The coboundary is expressed as

$$\delta\omega = \sum_{g_1, \cdots, g_{k+1}} \omega(\partial \, |g_1| \ldots |g_{k+1}\rangle) \, \langle g_1| \ldots |g_{k+1}| \, . \tag{D.5}$$

Concretely, each coefficient of $\delta\omega$ is written as

$$(\delta\omega)(g_1,\ldots,g_{k+1}) = \omega(g_2,\ldots,g_{k+1}) + (-1)^k\omega(g_1,\ldots g_k) + \sum_{i=1}^{k-1}(-1)^i\omega(g_1,\ldots,g_ig_{i+1},\ldots,g_{k+1}). \tag{D.6}$$

This defines $Z^k(BG,X)$, $B^k(BG,X)$ and the group cohomology $H^k(BG,X)$. In particular, we can simply regard the element of $Z^k(BG,X)$ as a function $\omega : G^k \to X$ with the property $\delta\omega = 0$ given by (D.6).

D.2 Twisted group cohomology

When there is a group action ρ of G on X, we can define the twisted version of group cohomology. $\omega \in Z^k_\rho(BG,X)$ is a function $\omega : G^k \to X$ with the property $\delta_\rho\omega = 0$, where the twisted coboundary δ_ρ is defined as

$$(\delta_\rho\omega)(g_1,\ldots,g_{k+1}) = \rho_{g_1}\omega(g_2,\ldots,g_{k+1}) + (-1)^k\omega(g_1,\ldots g_k) + \sum_{i=1}^{k-1}(-1)^i\omega(g_1,\ldots,g_ig_{i+1},\ldots,g_{k+1}). \tag{D.7}$$

In particular, when G contains the anti-unitary symmetry, it acts on $X = U(1)$ as the complex conjugation $\rho : e^{i\theta} \to e^{-i\theta}$. One can see that $\delta_\rho\delta_\rho\omega = 0$ for any $\omega \in C^k_\rho(BG,U(1))$, and it defines $Z^k_\rho(BG,U(1))$, $B^k_\rho(BG,U(1))$ and the twisted group cohomology $H^k_\rho(BG,U(1))$.

Appendix E

Evaluation of partial time-reversal

E.1 Computation of $\mathrm{tr}_I(\rho_I \rho_I^{\mathcal{T}_1})$

Here, we perform the explicit computation of the quantity (6.9)

$$\mathrm{tr}_I(\rho_I \rho_I^{\mathcal{T}_1}), \tag{E.1}$$

for a wave function prepared by the theory for the ABK invariant (6.5). We start with constructing the reduced density matrix. We first prepare the state on the boundary circle of a disk $M = D^2$ and its conjugation in the form of (6.5)

$$|\psi\rangle = \sum_{\alpha \in Z^1(M,\mathbb{Z}_2)} \widetilde{Z}(M,\alpha)\sigma(M,\alpha;\mathrm{ord}(-n,\cdots,n))(-1)^{\int_M \eta\cup\alpha}(c^\dagger_{-n})^{\alpha(e_{-n})}\cdots(c^\dagger_n)^{\alpha(e_n)}|0\rangle, \tag{E.2}$$

$$\langle\psi| = \sum_{\alpha \in Z^1(\overline{M},\mathbb{Z}_2)} \widetilde{Z}(\overline{M},\alpha)\sigma(\overline{M},\alpha;\mathrm{ord}(\overline{n},\cdots,-\overline{n}))(-1)^{\int_{\overline{M}} \eta\cup\alpha}\langle 0| c_{\overline{n}}^{\alpha(e_{\overline{n}})}\cdots c_{-\overline{n}}^{\alpha(e_{-\overline{n}})}, \tag{E.3}$$

where we let the number of boundary 1-simplices $2n$ here, and label 1-simplices in ∂M as $e_{-n},\dots,e_{-1},e_1,\dots,e_n$, for later convenience. $\overline{M}$ is given by reversing the orientation of M, and we denote 1-simplices in $\partial\overline{M}$ as $e_{-\overline{n}},\dots,e_{-\overline{1}},e_{\overline{1}},\dots e_{\overline{n}}$. Starting from the density matrix $\rho = |\psi\rangle\langle\psi|$, we take the reduced density matrix ρ_I for the interval $I = \sum_{1\le|j|\le l} e_j$, see Fig. E.1. For simplicity, we set l, n as even. Then, ρ_I is expressed as

$$\begin{aligned}\rho_I =& \frac{1}{\sqrt{2}} \sum_{\alpha\in Z^1(N,\mathbb{Z}_2)} \widetilde{Z}(N,\alpha)\sigma(M,\alpha|_M;\mathrm{ord}(-n,\cdots,n))\sigma(\overline{M},\alpha|_{\overline{M}};\mathrm{ord}(\overline{n},\cdots,-\overline{n}))\\ &\times(-1)^{\sum_{E_M+E_{\overline{M}}}\alpha}\times(c^\dagger_{-l})^{\alpha(e_{-l})}\cdots(c^\dagger_l)^{\alpha(e_l)}|0\rangle\langle 0| c_l^{\alpha(e_{\overline{l}})}\cdots c_{-l}^{\alpha(e_{-\overline{l}})},\end{aligned} \tag{E.4}$$

where N is given by gluing M and $\overline{M}$ along the complement of I on ∂M. $E_M, E_{\overline{M}}$ denotes a dual of η introduced in (6.7).

(E.4) reduces to the form of the path integral on N. To see this, first we associate the product of Grassmann integrals on M, $\overline{M}$ with that of N, by the following relation

$$\begin{aligned}&\sigma(M,\alpha|_M;\mathrm{ord}(-n,\cdots,n))\sigma(\overline{M},\alpha|_{\overline{M}};\mathrm{ord}(\overline{n},\cdots,-\overline{n}))\\ &=\prod_{l+1\le j\le n}^{\mathrm{odd}}(-1)^{\alpha(e_j)}\prod_{l+1\le j\le n}^{\mathrm{even}}(-1)^{\alpha(e_{-j})}\sigma(N,\alpha;\mathrm{ord}(-l,\cdots,l,\overline{l},\cdots,-\overline{l})),\end{aligned} \tag{E.5}$$

which can be shown by an explicit computation of the Grassmann integral. Then, (E.4) is rewritten in the form of the path integral on N (see Fig. E.1),

$$\begin{aligned}\rho_I =&\frac{1}{\sqrt{2}} \sum_{\alpha\in Z^1(N,\mathbb{Z}_2)} \widetilde{Z}(N,\alpha)\sigma(N,\alpha;\mathrm{ord}(-l,\cdots,l,\bar{l},\cdots,-\bar{l}))(-1)^{\sum_{E_N}\alpha} \\ &\times (c^{\dagger}_{-l})^{\alpha(e_{-l})}\cdots(c^{\dagger}_{l})^{\alpha(e_l)}\left|0\right\rangle\left\langle 0\right| c_l^{\alpha(e_{\bar{l}})}\cdots c_{-l}^{\alpha(e_{-\bar{l}})},\end{aligned} \tag{E.6}$$

where we define E_N as

$$E_N := E_M + E_{\overline{M}} + \sum_{l+1\le j\le n}^{\text{odd}} e_j + \sum_{l+1\le j\le n}^{\text{even}} e_{-j}. \tag{E.7}$$

One can check that ∂E_N correctly gives the dual of w_2 on N, when restricted to the interior of N. Thus, E_N actually works as a dual of η on N.

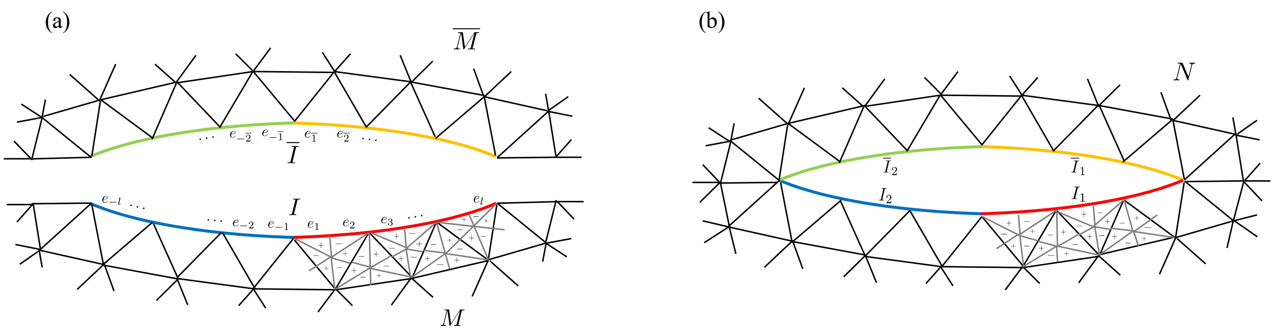

Figure E.1: (a): $|\psi\rangle$ and $\langle\psi|$ are prepared by the path integral on M and $\overline{M}$ respectively. (b): Taking the partial trace amounts to gluing M and $\overline{M}$, and the resulting surface is denoted as N.

To compute the partial time-reversal of ρ_I, we express ρ_I using the coherent state basis,

$$\begin{aligned}\rho_I =&\frac{1}{\sqrt{2}} \sum_{\alpha\in Z^1(N,\mathbb{Z}_2)} \widetilde{Z}(N,\alpha)\sigma(N,\alpha;\mathrm{ord}(-l,\cdots,l,\bar{l},\cdots,-\bar{l}))(-1)^{\sum_{E_N}\alpha} \\ &\times \int \overleftarrow{d}\,\xi_{-l}^{\alpha(e_{-l})}\cdots\int\overleftarrow{d}\,\xi_l^{\alpha(e_l)}\,|\xi_{-l}^{\alpha(e_{-l})}\rangle_{-l}\otimes\cdots\otimes|\xi_l^{\alpha(e_l)}\rangle_l \\ &\times \int \overrightarrow{d}\,\bar{\xi}_{\bar{l}}^{\alpha(e_{\bar{l}})}\cdots\int\overrightarrow{d}\,\bar{\xi}_{-\bar{l}}^{\alpha(e_{-\bar{l}})}\;{}_l\langle\bar{\xi}_{\bar{l}}^{\alpha(e_{\bar{l}})}|\otimes\cdots\otimes{}_{-l}\langle\bar{\xi}_{-\bar{l}}^{\alpha(e_{-\bar{l}})}|,\end{aligned} \tag{E.8}$$

where $\int\overleftarrow{d}\,\xi$ (resp. $\int\overrightarrow{d}\,\bar{\xi}$) denotes the integral which satisfies $\xi\int\overleftarrow{d}\,\xi = 1$ (resp. $\int\overrightarrow{d}\,\bar{\xi}\,\bar{\xi} = 1$). Now we take the partial time-reversal (6.11) acting on the region $I_1 = \sum_{1\le j\le l} e_j$ and $\bar{I}_1 = \sum_{1\le j\le l} e_{\bar{j}}$,

$$\begin{aligned}\rho_I^{\mathcal{T}_1} =&\frac{1}{\sqrt{2}} \sum_{\alpha\in Z^1(N,\mathbb{Z}_2)} \widetilde{Z}(N,\alpha)\sigma(N,\alpha;\mathrm{ord}(-l,\cdots,l,\bar{l},\cdots,-\bar{l}))(-1)^{\sum_{E_N}\alpha} \\ &\times \int \overleftarrow{d}\,\xi_{-l}^{\alpha(e_{-l})}\cdots\int\overleftarrow{d}\,\xi_l^{\alpha(e_l)}\,|\xi_{-l}^{\alpha(e_{-l})}\rangle_{-l}\otimes\cdots\otimes|\xi_{-1}^{\alpha(e_{-1})}\rangle_{-1}\otimes|i\bar{\xi}_{\bar{1}}^{\alpha(e_{\bar{1}})}\rangle_1\otimes\cdots\otimes|i\bar{\xi}_{\bar{l}}^{\alpha(e_{\bar{l}})}\rangle_l \\ &\times \int \overrightarrow{d}\,\bar{\xi}_{\bar{l}}^{\alpha(e_{\bar{l}})}\cdots\int\overrightarrow{d}\,\bar{\xi}_{-\bar{l}}^{\alpha(e_{-\bar{l}})}\;{}_l\langle i\xi_l^{\alpha(e_l)}|\otimes\cdots\otimes{}_1\langle i\xi_1^{\alpha(e_1)}|\otimes{}_{-1}\langle\bar{\xi}_{-\bar{1}}^{\alpha(e_{-\bar{1}})}|\otimes\cdots\otimes{}_{-l}\langle\bar{\xi}_{-\bar{l}}^{\alpha(e_{-\bar{l}})}|.\end{aligned} \tag{E.9}$$

which is rewritten in the fermion number basis as

$$
\begin{aligned}
\rho_I^{\mathcal{T}_1} =& \frac{1}{\sqrt{2}} \sum_{\alpha \in Z^1(N,\mathbb{Z}_2)} \widetilde{Z}(N,\alpha)\sigma(N,\alpha;\mathrm{ord}(-l,\cdots,-1,\bar{1},\cdots,\bar{l},l,\cdots,1,-\bar{1},\cdots,-\bar{l})) \\
&\times (-1)^{\sum_{E_N}\alpha} \prod_{e\in I_1\cup \bar{I}_1} (-i)^{\alpha(e)} \times (c_{-l}^\dagger)^{\alpha(e_{-l})}\cdots(c_{-1}^\dagger)^{\alpha(e_{-1})}(c_1^\dagger)^{\alpha(e_{\bar{1}})}\cdots(c_l^\dagger)^{\alpha(e_{\bar{l}})}\left|0\right\rangle \\
&\times \left\langle 0\right| c_l^{\alpha(e_l)}\cdots c_1^{\alpha(e_1)} c_{-1}^{\alpha(e_{-\bar{1}})}\cdots c_{-l}^{\alpha(e_{-\bar{l}})}.
\end{aligned}
\tag{E.10}
$$

Now we can explicitly write down the order parameter (6.9) as

$$
\begin{aligned}
\mathrm{tr}_I(\rho_I\rho_I^{\mathcal{T}_1}) =& \frac{1}{2}\sum_{\alpha\in Z^1(N,\mathbb{Z}_2)}\sum_{\alpha'\in Z^1(N',\mathbb{Z}_2)} \widetilde{Z}(N,\alpha)\widetilde{Z}(N',\alpha')\sigma(N,\alpha;\mathrm{ord}(-l,\cdots,l,\bar{l},\cdots,-\bar{l})) \\
&\times \sigma(N',\alpha';\mathrm{ord}(-l,\cdots,-1,\bar{1},\cdots,\bar{l},l,\cdots,1,-\bar{1},\cdots,-\bar{l})) \\
&\times (-1)^{\sum_{E_N}\alpha}(-1)^{\sum_{E_{N'}}\alpha'}\prod_{e\in I_1\cup\bar{I}_1}(-i)^{\alpha(e)} \\
&\times \prod_{1\le j\le l}\delta_{\alpha(e_{-j})\alpha'(e_{-\bar{j}})}\delta_{\alpha(e_j)\alpha'(e_j)}\delta_{\alpha(e_{-\bar{j}})\alpha'(e_{-j})}\delta_{\alpha(e_{\bar{j}})\alpha'(e_{\bar{j}})},
\end{aligned}
\tag{E.11}
$$

where the expression involves two copies of N evaluating $\rho_I^{\mathcal{T}_1}$ and ρ_I written as N and N' respectively. By taking the trace after partial time-reversal, the expression (E.11) looks like the form of path integral on a space X, which is obtained by gluing N, N' along their boundaries as illustrated in Fig. E.2. Namely, we identify $I_1+\bar{I}_1 = \sum_{1\le j\le l}(e_j+e_{\bar{j}})$ on ∂N and $\partial N'$, by the orientation reversing map, and $I_2+\bar{I}_2=\sum_{1\le j\le l}(e_{-j}+e_{-\bar{j}})$ by the orientation preserving map. Here, the induced map $N\sqcup N'\to X$ is restricted to each N as $N\to\widetilde{N}$, where $\widetilde{N}$ is given by identifying two boundary 0-simplices of N contained in $\partial(I_1+\bar{I}_1)$. Then, (E.11) is rewritten in the form of path integral on X as

$$
\begin{aligned}
\mathrm{tr}_I(\rho_I\rho_I^{\mathcal{T}_1}) =& \frac{1}{4}\sum_{\alpha\in Z^1(X,\mathbb{Z}_2)}\widetilde{Z}(X,\alpha)\sigma(\widetilde{N},\alpha|_{\widetilde{N}};\mathrm{ord}(-l,\cdots,l,\bar{l},\cdots,-\bar{l})) \\
&\times \sigma(\widetilde{N}',\alpha|_{\widetilde{N}'};\mathrm{ord}(-l,\cdots,-1,\bar{1},\cdots,\bar{l},l,\cdots,1,-\bar{1},\cdots,-\bar{l})) \\
&\times (-1)^{\sum_{E_{\widetilde{N}}}\alpha}(-1)^{\sum_{E_{\widetilde{N}'}}\alpha}\times\prod_{e\in I_1\cup\bar{I}_1}(-i)^{\alpha(e)}.
\end{aligned}
\tag{E.12}
$$

The expression (E.12) looks like the partition function of a Pin$^-$ theory constructed from the Grassmann integral (4.4), since both assigns $\pm i$ factor to 1-simplices where we reverse the orientation. We can actually show that these are identical up to the normalization factor $1/4$. To see this, we compare (E.12) with the Pin$^-$ theory on X in the form of (6.1). By integrating out the Grassmann variables living on boundaries of $\widetilde{N}$ and $\widetilde{N}'$, the partition

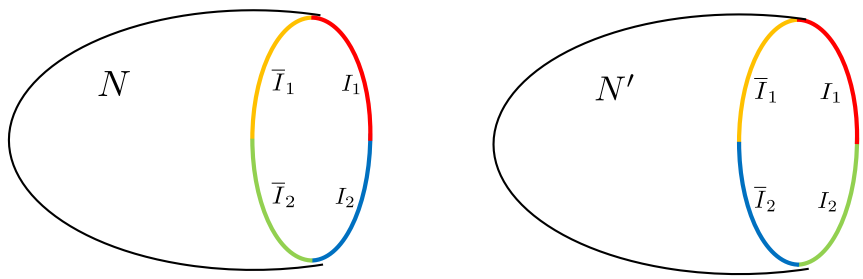

Figure E.2: Taking $\mathrm{tr}_I(\rho_I \rho_I^{\mathcal{T}_1})$ amounts to gluing N and N' along boundaries, such that the boundary 1-simplices with the same color in the figure are identified. We are gluing $I_1 + \overline{I}_1$ (red and yellow curves) by the orientation reversing map, and $I_2 + \overline{I}_2$ (blue and green curves) by the orientation preserving map. The resulting surface X has a crosscap introduced along $I_1 + \overline{I}_1$.

function for the Pin$^-$ theory (6.1) is obtained as

$$
\begin{aligned}
Z(X,\eta) = \sum_{\alpha \in Z^1(X,\mathbb{Z}_2)} & \widetilde{Z}(X,\alpha)\sigma(\widetilde{N}, \alpha|_{\widetilde{N}}; \mathrm{ord}(-l,\cdots,l,\bar{l},\cdots,-\bar{l})) \\
& \times \sigma(\widetilde{N}', \alpha|_{\widetilde{N}'}; \mathrm{ord}(-l,\cdots,-1,\bar{1},\cdots,\bar{l},l,\cdots,1,-\bar{1},\cdots,-\bar{l})) \\
& \times (-1)^{\sum_{E_X}\alpha} \prod_{e\in I_2}^{\text{even}} (-1)^{\alpha(e)} \prod_{e\in \overline{I}_2}^{\text{odd}} (-1)^{\alpha(e)} \times \prod_{e \in I_1 \cup \overline{I}_1} (-i)^{\alpha(e)},
\end{aligned} \tag{E.13}
$$

where E_X is the dual of η trivializing $w_2 + w_1^2$ on X, whose choice of representative is described in Sec. 4. By comparing (E.12) with (E.13), one can see these expression are completely the same, by checking that

$$
E_{\widetilde{N}} + E_{\widetilde{N}'} + \sum_{e_j \in I_2}^{\text{even}} e_j + \sum_{e_j \in \overline{I}_2}^{\text{odd}} e_j = E_X. \tag{E.14}
$$

Thus, we have shown that (6.9) is identical to the partition function of the Pin$^-$ theory defined in (6.1),

$$
\mathrm{tr}_I(\rho_I \rho_I^{\mathcal{T}_1}) = \frac{1}{4} Z(X,\eta) = \frac{\sqrt{2}^{\chi(X)}}{4} \mathrm{ABK}[X,\eta]. \tag{E.15}
$$

Then, the phase of the quantity $\mathrm{tr}_I(\rho_I \rho_I^{\mathcal{T}_1})$ is identified as the ABK invariant on the resulting manifold X. In particular, in our case we have $X = \mathbb{RP}^2$, thus

$$
\mathrm{tr}_I(\rho_I \rho_I^{\mathcal{T}_1}) = \frac{1}{2\sqrt{2}} e^{\pm 2\pi i/8}. \tag{E.16}
$$

E.2 The evaluation of the norm

Here, let us check that the state $|\psi\rangle$ in (6.5) is correctly normalized to have the unit norm,

$$|\psi\rangle = \sum_{\alpha \in Z^1(M,\mathbb{Z}_2)} \widetilde{Z}(M,\alpha)\sigma(M,\alpha;\mathrm{ord}(-n,\cdots,n))(-1)^{\int_M \eta\cup\alpha}(c^\dagger_{-n})^{\alpha(e_{-n})}\cdots(c^\dagger_n)^{\alpha(e_n)}|0\rangle\,, \tag{E.17}$$

$$\langle\psi| = \sum_{\alpha \in Z^1(\overline{M},\mathbb{Z}_2)} \widetilde{Z}(\overline{M},\alpha)\sigma(\overline{M},\alpha;\mathrm{ord}(\overline{n},\cdots,-\overline{n}))(-1)^{\int_{\overline{M}} \eta\cup\alpha}\langle 0| c_{\overline{n}}^{\alpha(e_{\overline{n}})}\cdots c_{-\overline{n}}^{\alpha(e_{-\overline{n}})}, \tag{E.18}$$

based on the bosonic weight for a non-closed manifold defined in (6.8),

$$\widetilde{Z}(M,\alpha) = 2^{|F|-|E|+|E_b|/2-1/2}. \tag{E.19}$$

Using the equation for gluing the Grassmann integral (E.5), we can see that

$$\langle\psi|\psi\rangle = \frac{1}{2}\sum_{\alpha\in Z^1(N,\mathbb{Z}_2)} \widetilde{Z}(N,\alpha)\sigma(N,\alpha)\prod_{1\le j\le n}^{\text{odd}}(-1)^{\alpha(e_j)}\prod_{1\le j\le n}^{\text{even}}(-1)^{\alpha(e_{-j})}(-1)^{\sum_{E_M}\alpha}(-1)^{\sum_{E_{\overline{M}}}\alpha}, \tag{E.20}$$

where $N = S^2$ is a sphere obtained by gluing two disks M and $\overline{M}$ along the boundaries. Then, it is not hard to see that

$$E_M + E_{\overline{M}} + \sum_{1\le j\le n}^{\text{odd}} e_j + \sum_{1\le j\le n}^{\text{even}} e_{-j} = E_N, \tag{E.21}$$

where $\partial E_N = S_N$, and S_N represents the dual of w_2 on N. After all, we have

$$\begin{aligned}\langle\psi|\psi\rangle &= \frac{1}{2}\sum_{\alpha\in Z^1(N,\mathbb{Z}_2)} \widetilde{Z}(N,\alpha)\sigma(N,\alpha)(-1)^{\sum_{E_N}\alpha}\\ &= \frac{\sqrt{2}^{\chi(S^2)}}{2}\mathrm{ABK}[S^2,\eta] = 1,\end{aligned} \tag{E.22}$$

where we used the definition of the ABK invariant in (6.1). This shows that $|\psi\rangle$ is correctly normalized.

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