Introduction to Mirror Symmetry

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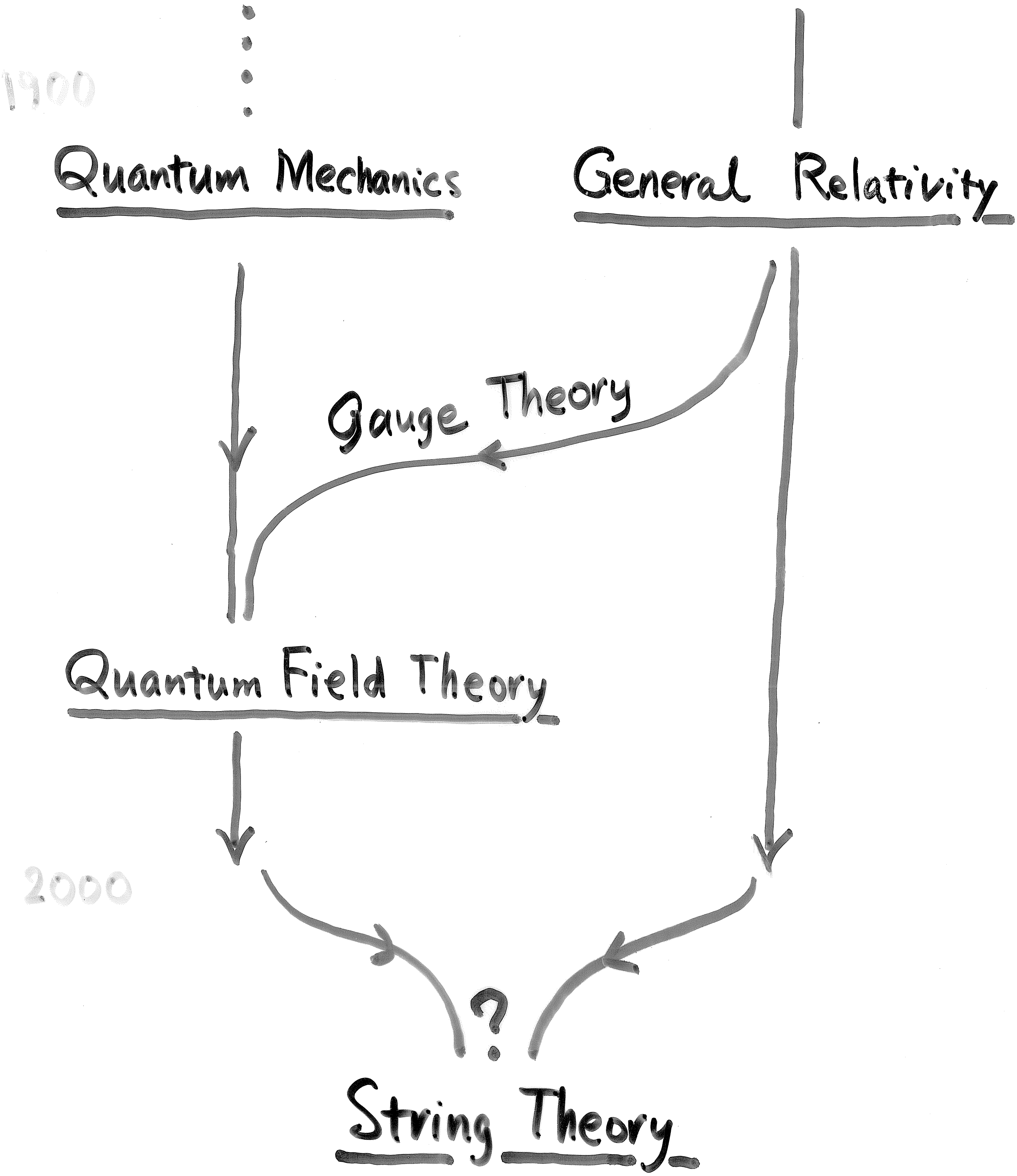

1900
Quantum Mechanics
General Relativity
Gauge Theory
Quantum Field Theory
2000
?
String Theory

ミラー対称性の代表的な例

X : Calabi-Yau 多様体

…… $c_1(T_X)=0$ なる ケーラー多様体 ＜ 複素構造 / シンプレクティック構造

$\mathcal{M}_C$ = 複素構造のモジュライ空間

(J_C, g_C) : 自然な複素構造／計量

$\mathcal{M}_K$ = (複素化された) シンプレクティック構造の弦による量子補正を受けたモジュライ空間

(J_K, g_K) : 量子補正を受けた複素構造／計量

ミラー対称性

$$X \longleftrightarrow \tilde{X}$$

$$(\mathcal{M}_C, J_C, g_C) = (\tilde{\mathcal{M}}_K, \tilde{J}_K, \tilde{g}_K)$$

$$(\mathcal{M}_K, J_K, g_K) = (\tilde{\mathcal{M}}_C, \tilde{J}_C, \tilde{g}_C)$$

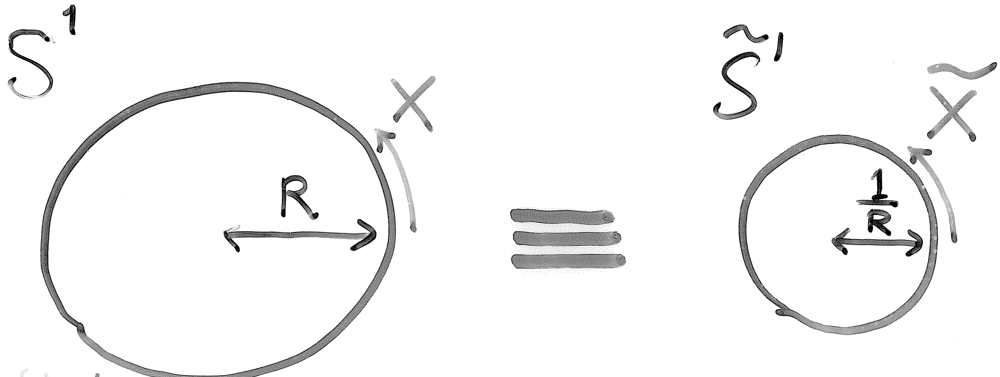
T-duality

Closed string on S^1 (target) of radius R

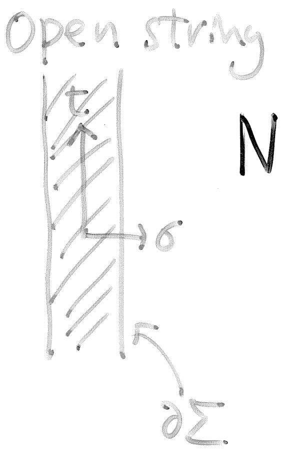
momentum $\frac{\ell}{R}$

$$E^2 = P^2 + M^2 = \left(\frac{\ell}{R}\right)^2 + (Rm)^2$$

... invariant under $R \leftrightarrow \frac{1}{R}$, $\ell \leftrightarrow m$



winding #		momentum
$\partial_\sigma X$	$=$	$\partial_t \tilde{X}$
momentum		winding #
$\partial_t X$	$=$	$\partial_\sigma \tilde{X}$



Neumann b.c.		Dirichlet b.c.
$\partial_\sigma X = 0$ on $\partial\Sigma$	$\Longleftrightarrow$	$\partial_t \tilde{X} = 0$ on $\partial\Sigma$

Open string boundary condition = D-brane

M target space
$\cup$
γ : a submanifold

A : a connection of a $U(1)$ bundle on γ.

(γ, A) specifies a boundary condition

$$X : \Sigma \to M$$

- Dirichlet b.c. in normal direction

$$X|_{\partial\Sigma} \subset \gamma .$$

- Neumann b.c. in tangent direction

$$\partial_n X \perp T\gamma .$$

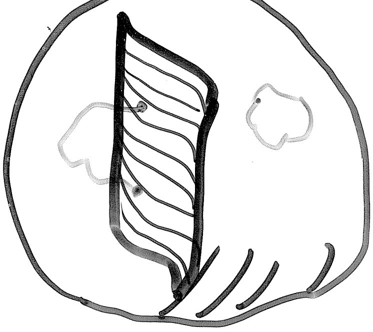
- boundary interaction

$$S_{\partial\Sigma} = \int_{\partial\Sigma} X^* A .$$

—— D-brane boundary condition

D-branes

…. • Objects on which a string can end.



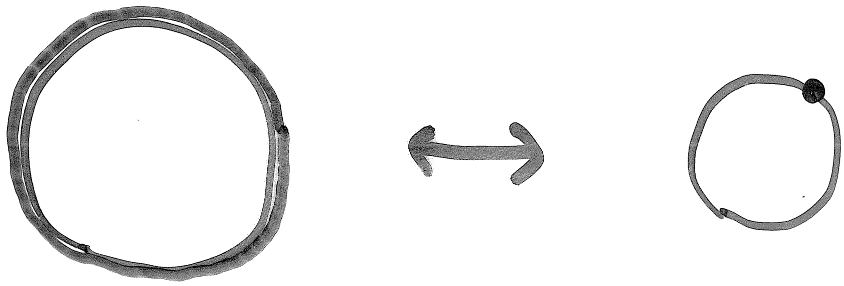
• Support U(1) gauge potentials (string end points are charged)

<u>back to S^1</u>

$$\text{Neumann b.c.} \overset{T}{\longleftrightarrow} \text{Dirichlet b.c.}$$

$\Updownarrow$ $\Updownarrow$

D1 brane wrapped on S^1 D0 brane at a pt of $\tilde{S}^1$



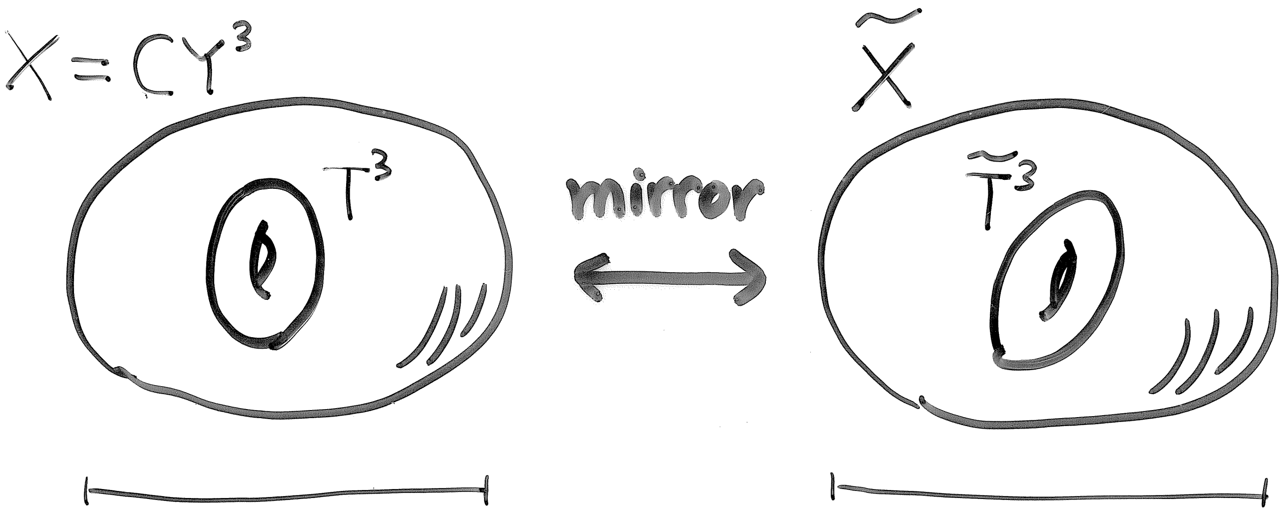
U(1) holonomy $\oint_{S^1} A \longleftrightarrow$ position in $\tilde{S}^1$

$$\tilde{S}^1 = \{\text{D0-brane}\} = \{\text{D1-brane}\} = \underbrace{H^1(S^1, U(1))}_{}$$ space of flat U(1) conn on S^1

$\therefore$ <u>T-dual space = dual space</u> !

$\rightsquigarrow$ higher dim. $\tilde{T}^n = H^1(T^n, U(1))$.

Strominger-Yau-Zaslow



$$\left\{ \begin{array}{l} \text{D3 brane wrapped on} \\ \text{Special Lagrangian } T^3 \subset X \end{array} \right\} = \tilde{X}$$

$$X = \left\{ \begin{array}{c} \text{D3 brane wrapped on} \\ \text{SLAG } \tilde{T}^3 \subset \tilde{X} \end{array} \right\}$$

Mirror Symmetry

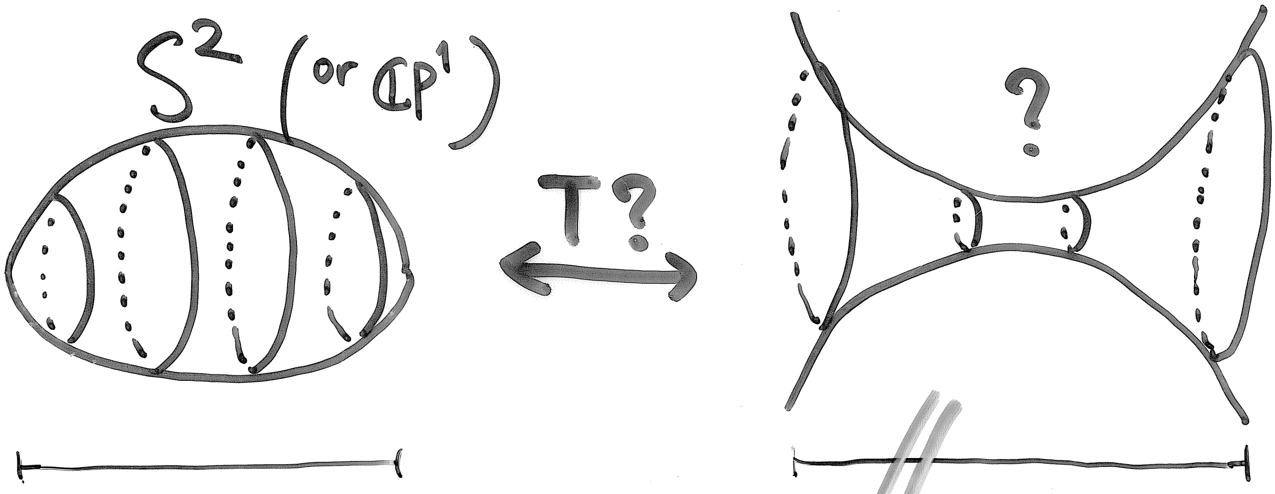
= T-duality + Quantum Correction

$\sim T^3$ degeneration

Gross
W.D. Ruan
D. Morrison
⋮

} topological construction of the Mirror

A Simpler Situation with degenerate fibres:



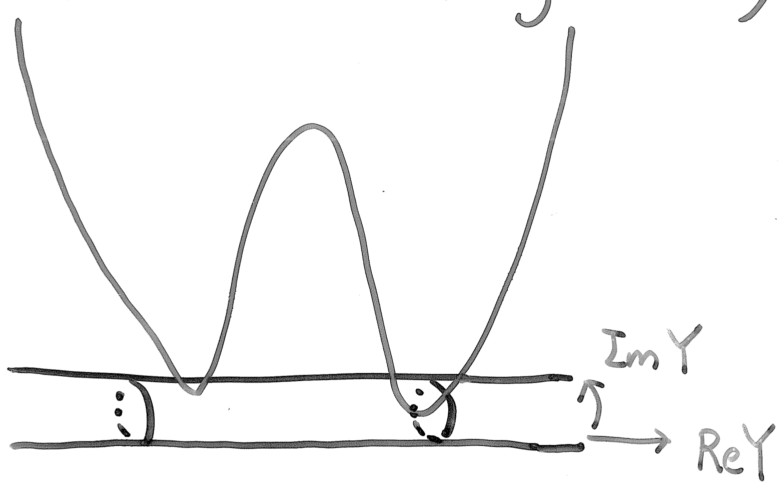
True Story

$r = \text{area}(S^2)$

$\theta =$ 'B-field'

$t := r - i\theta$

Potential Theory
(Landau-Ginzburg Model)



Superpotential $W = e^{-Y} + e^{-t+Y}$

.... derived by an exact analysis of
QFT on the worldsheet Hori-Vafa

σ-Model

$$\phi : \Sigma \to (X, g)$$

$$\psi_\pm \in \Gamma(\Sigma, \phi^* T^{1,0}X \otimes S_\pm), \ \overline{\psi}_\pm \in \Gamma(\Sigma, \phi^* T^{0,1}X \otimes S_\pm)$$

$$S_{\sigma M} = \int \Big\{ \|d\phi\|^2 + i\overline{\psi}_-(D_t + D_\sigma)\psi_- + i\overline{\psi}_+(D_t - D_\sigma)\psi_+ + R_{i\bar{j}k\bar{l}}\, \psi_+^i \psi_-^k \overline{\psi}_-^{\bar{j}} \overline{\psi}_+^{\bar{l}} \Big\}$$

LG-model

$W : X \to \mathbb{C}$ holomorphic

$$S = S_{\sigma M} - \int \Big\{ \underbrace{\|\partial W\|^2}_{\substack{\uparrow \\ \text{Potential term}}} + \underbrace{D_i \partial_j W \psi_+^i \psi_-^j + D_{\bar{i}} \partial_{\bar{j}} \overline{W} \overline{\psi}_-^{\bar{i}} \overline{\psi}_+^{\bar{j}}}_{\substack{\uparrow \\ \text{Yukawa (or mass) term}}} \Big\}$$

— Generalization:

toric manifold $\leftrightarrow$ LG, W = (affine) Toda-type

... explains earlier observations:

Quantum Cohomology Batyrev, Givental

Topological Strings Eguchi-H-Yang, EH-Xiong

Scattering Matrix Fendley-Intriligator

— The method can be applied to derive the mirror symmetry between Green-Plesser Batyrev

CY hypersurfaces/Complete-intersections in toric Fano.

— $X = \mathbb{CP}^1, \mathbb{CP}^2, \ldots$

$$W\Big|_{Y \to \frac{\partial}{\partial\mu}} = \text{Lax Operator}$$

for the integrable system associated with top. string on X

Eguchi-Yang, EHY, EHX, ...

Instead of repeating the QFT derivation, we present some consequences of this Mirror duality, especially on D-branes.

We will see that D-branes can <u>see</u> or <u>reproduce</u> the duality itself, as SYZ conjectured.

Supersymmetry & D-branes

σ-model (LG-model) on a Kählermfd
has <u>(2, 2) supersymmetry</u>

$$Q_\pm, \overline{Q}_\pm \qquad \{Q_\pm, \overline{Q}_\pm\} = H \pm P$$

Mirror Symmetry exchanges $Q_- \leftrightarrow \overline{Q}_-$

We focus on <u>D-branes</u> preserving
<u>a half of (2,2) SUSY</u>

A-branes $\begin{cases} Q_A = \overline{Q}_+ + Q_- \\ Q_A^\dagger = Q_+ + \overline{Q}_- \end{cases}$

Mirror Symmetry

B-branes $\begin{cases} Q_B = \overline{Q}_+ + \overline{Q}_- \\ Q_B^\dagger = Q_+ + Q_- \end{cases}$

X Kähler mfd $\begin{cases} \omega & \text{symplectic structure} \\ J & \text{complex structure} \end{cases}$

(LG-model : superpotential $W : X \to \mathbb{C}$ holo)

A D-brane wrapped on $\gamma \subset X$
supporting a $U(1)$ gauge potential A is

an A-brane if $\gamma \subset (X, \omega)$ **Lagrangian**
A : **flat** $(F_A = 0)$
(**Im W = constant** on γ)

a B-brane if $\gamma \subset (X, J)$ **complex submanifold**
A : **holomorphic** $(F_A^{2,0} = 0)$
(**W = constant** on γ)

Mirror Symmetry

non-linear σ-model on X^n_{toric} $\longleftrightarrow$ Landau-Ginzburg model $W : (\mathbb{C}^\times)^n \to \mathbb{C}$

e.g.

$X = \mathbb{CP}^n \longleftrightarrow W = e^{-Y_1} + \cdots + e^{-Y_n} + e^{-t+Y_1+\cdots+Y_n}$

A-branes in X $\longleftrightarrow$ B-branes in LG

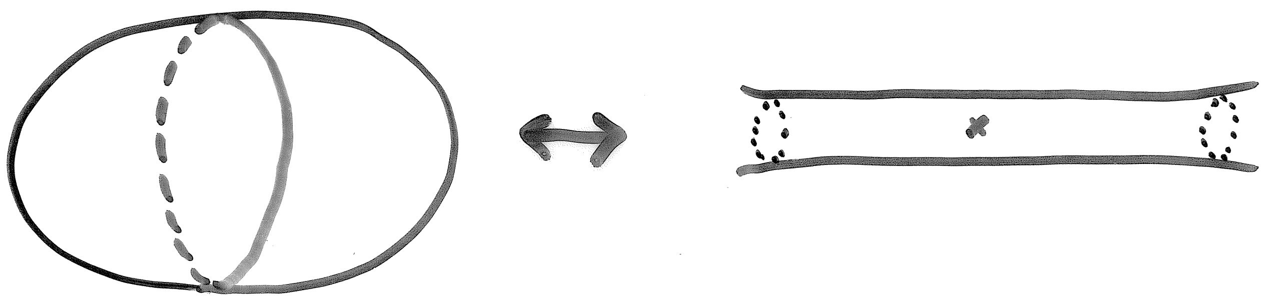
Lagrangian submfds — complex submfds, $W=$ const

B-branes in X — A-branes in LG

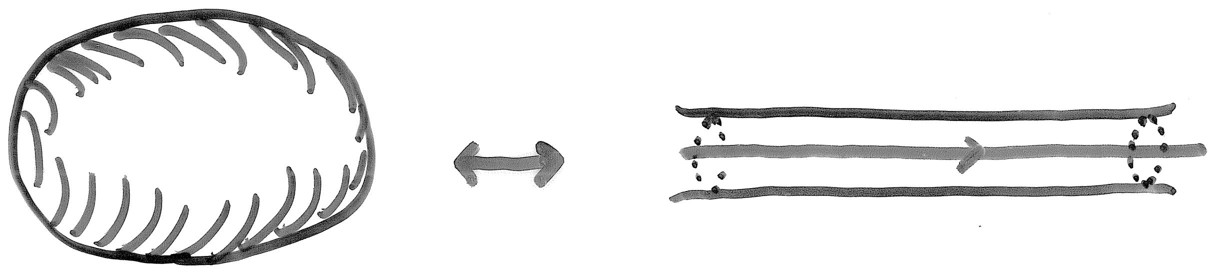
Coherent sheaves — Lagrangian submfds, $\operatorname{Im} W =$ const

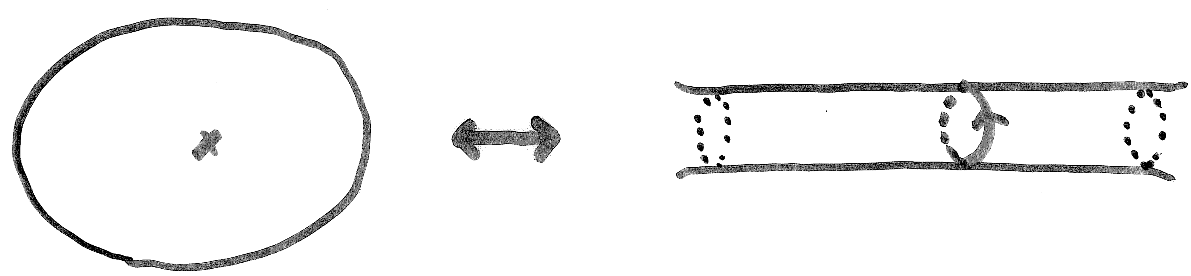
$$X = \mathbb{CP}^1 \longleftrightarrow W = e^{-Y} + e^{-t+Y}$$

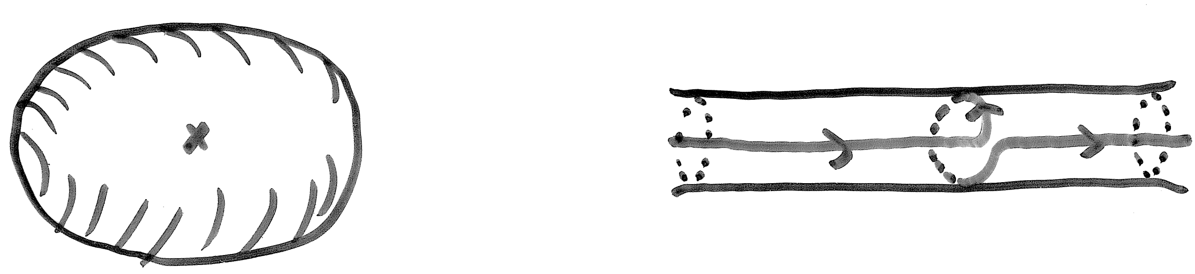
A-branes in $\mathbb{CP}^1$ B-branes in LG



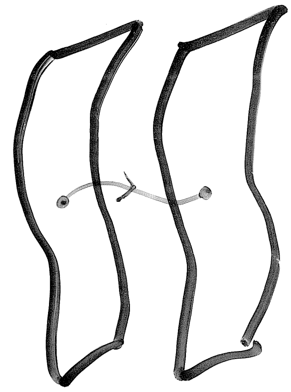
B-branes in $\mathbb{CP}^1$ A-branes in LG







First question to ask:



What are the ground states of the open string stretched between D-branes?, especially the **supersymmetric ground states**?

... states annihilated by Q & $Q^\dagger$ $(Q = Q_A \text{ or } Q_B)$

'usually'

$$\{Q, Q^\dagger\} = H$$
$$Q^2 = 0$$

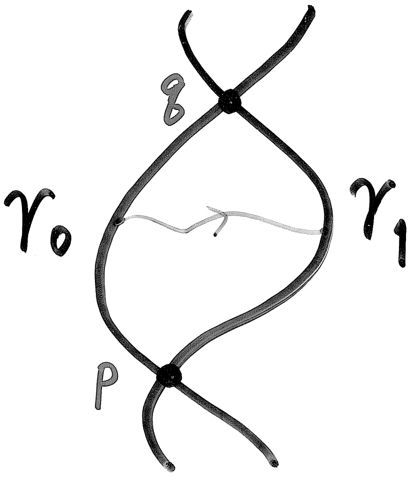
$\rightsquigarrow$ $\mathcal{H}_{SUSY} = Q$-cohomology group

However, it may happen (for Open strings) that

$$Q^2 \neq 0$$

$\rightsquigarrow$ No cohomological characterization of $\mathcal{H}_{SUSY}$

<u>A-branes in X</u> ··· Lagrangian submanifolds



Classical SUSY ground states

$\Longleftrightarrow$ constant maps to $\gamma_0 \cap \gamma_1$

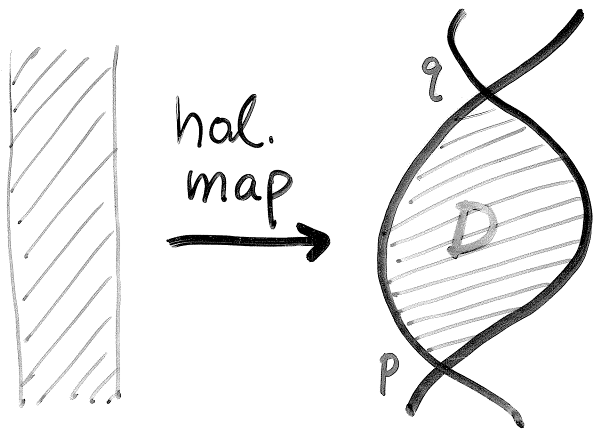
--- possibly lifted by Quantum Tunneling Effect

$$\mathcal{H}_{SUSY} = HF^\bullet(\gamma_0, \gamma_1) \quad \text{Floer `cohomology'}$$

$\sim$ a model of Q_A-'cohomology'

$$C^\bullet = \bigoplus_{p \in \gamma_0 \cap \gamma_1} \mathbb{C}p$$

$$Q_A p = \sum_q e^{-A(D)} q$$



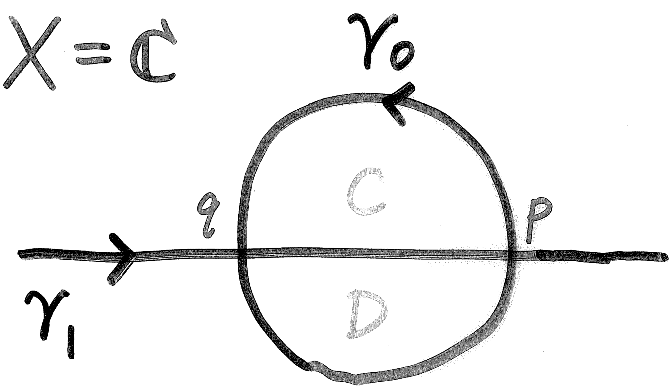
This is the tunneling configuration.

Fukaya-Oh-Ohta-Ono :

$Q_A^2 = 0$ not always true !

$(C^\bullet, Q_A)$ not always a complex.

e.g. $X = \mathbb{C}$



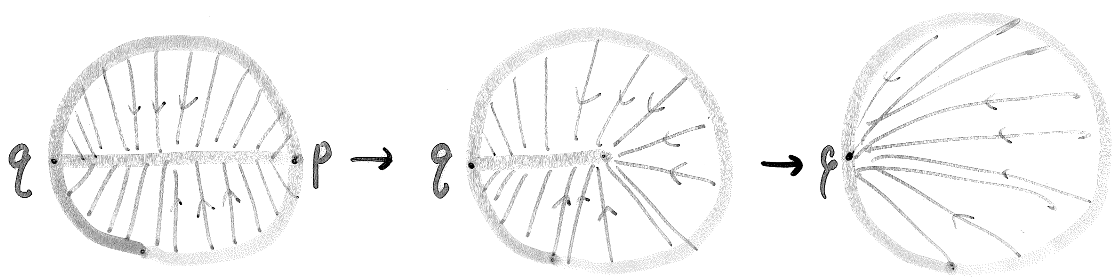
$$Q_A q = e^{-A(C)} p$$

$$Q_A p = e^{-A(D)} q$$

$$Q_A^2 q = e^{-A(C)} \cdot e^{-A(D)} q = e^{-A(C \cup D)} q \neq 0 \ !$$

$\Longleftrightarrow$ $\exists$ 1-para. family of $q \to q$ tunneling config.

s.t.



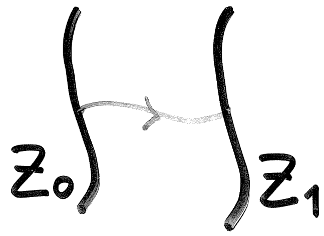
Composition
$(q \to p) \# (p \to q)$

holomorphic disc

… <u>bubble off of a holomorphic disc.</u>

B-branes in LG

$$Q_B = i\int_0^{\pi} d\sigma \left\{ (\bar\psi_- + \bar\psi_+)\partial_t\phi - (\bar\psi_- - \bar\psi_+)\partial_\sigma\phi + i(\psi_- - \psi_+)W' \right\}$$



$$\boxed{Q_B^2 = W|_{Z_0} - W|_{Z_1}} \quad \text{exact}$$

If $W|_{Z_0} \neq W|_{Z_1}$, $Q_B^2 \neq 0$: the mirror of FOOO's $Q_A^2 \neq 0$.

If $W|_{Z_0} = W|_{Z_1}$, $(\mathcal{H}, Q_B)$ indeed a complex.

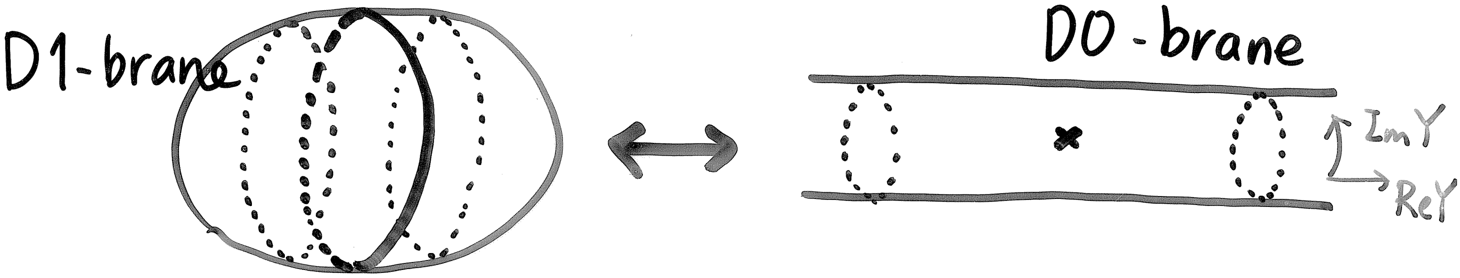
$\exists$ finite dim. model

$$\begin{cases} C^i = \bigoplus_{p-q=i} \Omega^{0,p}(Z_0 \cap Z_1, \wedge^q N_{Z_0 \cap N_{Z_1}}) \\ Q_B = \bar\partial + \partial W \cdot \end{cases}$$

If Z_0 & Z_1 are points, $Z_0 \cap Z_1 \neq \phi$ iff $Z_0 = Z_1 =: p$

then $$\mathcal{H}_{SUSY}^{\cdot} = H_{Q_B}^{\cdot} = \begin{cases} \wedge^{\cdot} T_p X & \text{if } \partial W = 0 \text{ at } p \\ 0 & \text{if } \partial W \neq 0 \text{ at } p. \end{cases}$$

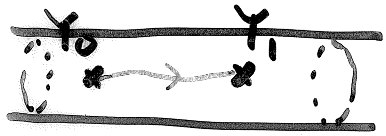
Back to the Mirror



$$\text{Area}(\;) = \text{Re}\, Y_{\times}$$

$$\oint A = \text{Im}\, Y_{\times}$$

For two D0's in LG



$$Q_B^2 = W(Y_0) - W(Y_1) = e^{-Y_0} + e^{-t+Y_0} - e^{-Y_1} - e^{-t+Y_1}$$

$$\therefore\; \underline{Q_B^2 = 0 \quad \text{iff} \quad Y_1 = Y_0 \quad \text{or} \quad Y_1 = t - Y_0}$$

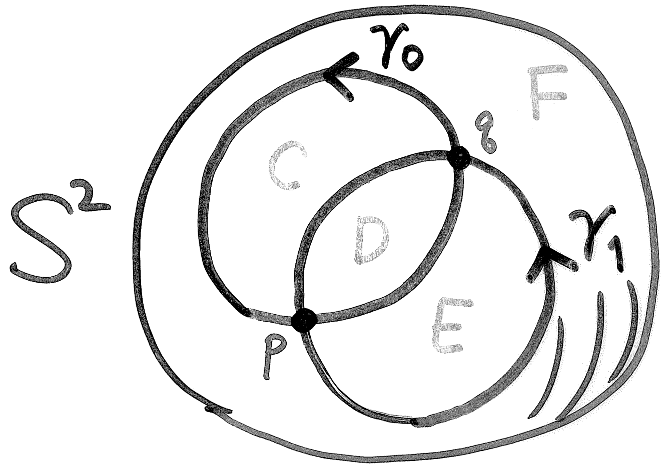
Then
$$\mathcal{H}_{SUSY} = \begin{cases} \Lambda^{\bullet}\mathbb{C} = \mathbb{C} \oplus \mathbb{C} & \text{if } Y_0 = Y_1 = \text{a crit pt} \\ 0 & \text{otherwise} \end{cases}$$

Crit pt of W :

$$0 = \partial_Y W = -e^{-Y} + e^{-t+Y}$$

$$\therefore\; e^{-Y} = \pm e^{-t/2} \quad \longleftrightarrow \quad \begin{cases} \text{Area}(\;) = \dfrac{r}{2} \\ \oint A = 0 \quad \text{or} \quad \pi \end{cases}$$

Direct Computation in σ-model F000



$C \cup D =: D_0$

$D \cup E =: D_1$

$$Q_A q = e^{-A(D)} p - e^{-A(F)} p$$

$$Q_A p = e^{-A(C)} q - e^{-A(E)} q$$

$$\therefore\ Q_A^2 q = \left[e^{-A(D_0)} - e^{-A(D_1)} - e^{-r+A(D_1)} + e^{-r+A(D_0)} \right] q$$

$$\underline{Q_A^2 = 0 \quad \text{iff} \quad A(D_1) = A(D_0) \quad \text{or} \quad A(D_1) = r - A(D_0)}$$

Suppose $A(D_1) = A(D_0)$ $\quad (\Rightarrow \gamma_1 \sim \gamma_0)$

$$Q_A q = e^{-A(D)} (1 - e^{-r+2A(D_0)}) p$$

$$Q_A p = 0$$

$$HF^{\bullet}(\gamma_0, \gamma_1) = \begin{cases} \mathbb{C}[q] \oplus \mathbb{C}[p] & A(D_0) = \frac{r}{2} = A(D_1) \\ 0 & \text{otherwise} \end{cases}$$

The prediction of Mirror Symmetry was indeed correct.

The analysis extends to general toric manifolds :

Direct computation becomes technically more involved, while computation in the mirror LG side remains easy algebraic manipulation.

— <u>Practical aspect of Mirror Symmetry</u>

cf. Confirmed by Cho-Oh

More important aspect.

Compare

$$Q_A^2 q = \left[e^{-A(D_0)} + e^{-r+A(D_0)} - e^{-A(D_1)} - e^{-r+A(D_1)} \right] q$$

$$Q_B^2 = W|_{Y_0} - W|_{Y_1} = e^{-Y_0} + e^{-t+Y_0} - e^{-Y_1} - e^{-t+Y_1}$$

Identical!, as they should be.

Note: $Q_B^2 = W|_{Z_0} - W|_{Z_1}$ has enough information to determine W itself, up to constant addition.

Thus, direct computation of Q_A^2 can reproduce the superpotential of the Mirror LG model.

There, Quantum Correction associated with degen. fibres, open string (disc) instantons, are taken into account.

SYZ program is realized & completed!