Localized Fermions on $\mathbb{CP}^2$ Net-Zero Charged Topological Solitons

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September 13, 2023

International Seminar-Type Online Workshop on Topological Solitons

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I'm interested in ...

How localized fermions appear?
Can fermions feel (anti-)solitons inside a mixture?

Motivation : Dirac Fermions and Topology

Zero modes and Topology

Fermion zero modes induced by the topology appear in various contexts of theoretical physics.

- Anomalous fermion number violation [A.J.Niemi, et al., Phys.Rept.135(1986)]
- Bulk-edge correspondence [C.L.Kane, et al., Phys.Rev.B82(2010)]
- ⋮

The numbers of such modes are characterized by the topological charge of solitons through the Atiyah–Singer index theorem.

The Atiyah-Singer Index theorem [M.F.Atiyah, et al., Ann.Math(1968)]

$$n_+ - n_- = Q$$

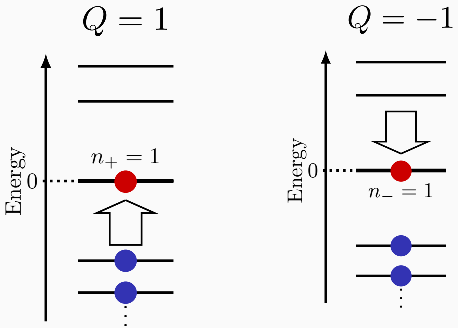
n_+ : the number of fermion zero modes

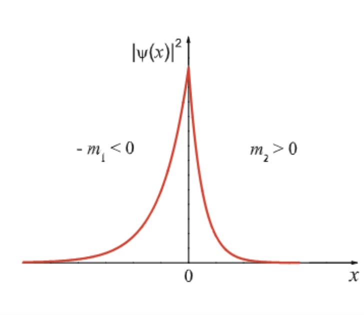
n_- : the number of anti-fermion zero modes

Q : the topological charge

Zero modes with $|Q| \geq 1$

- In a soliton background, $|Q| \geq 1$, the number of zero modes equals to the topological charge, *i.e.*, $n_\pm = |Q|$.
- In such situations, the wave function ψ is localized around the background soliton [R.Jackiw, et al., Nucl.Phys.B(1981)].





[Shun-Qing Shen, Springer(2012)]

Zero modes with $Q = 0$

What happens in the case of $Q = 0$?

The index theorem tells us, the numbers of zero modes are related as below.

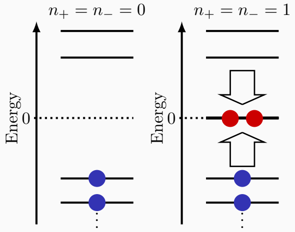
$$Q = n_+ - n_- = 0 \Leftrightarrow n_+ = n_-.$$

$\Rightarrow$ Not necessarily $n_\pm = 0$

Possibilities :

case 1. $n_+ = n_- = 0$ (trivial case).

case 2. $n_\pm \neq 0$ **(nontrivial case)**



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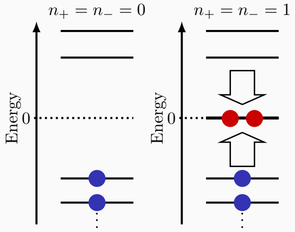

Our aim is to explore the solutions of the case 2. through a numerical calculation and how such fermion modes emerge from the $Q = 0$ solutions.

Din–Zakrzewski solutions and their Moduli space

Din–Zakrzewski solutions [A.M.Din, et al., Nucl.Phys.B(1980)]

Advantages of the solution in $2d$ $\mathbb{CP}^2$ NLSM

- Circular symmetric analytical solutions
- Two moduli parameters a, b

$$E = \int_{\mathbb{R}^2} \mathrm{d}^2x \underbrace{2\left[|D_+Z|^2 + |D_-Z|^2\right]}_{\text{density of } E \,:\, \mathcal{E}}, \quad Q = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}^2x \underbrace{2\left[|D_+Z|^2 - |D_-Z|^2\right]}_{\text{density of } Q \,:\, \mathcal{Q}}$$

$*1$ $D_\pm = \partial_\pm - Z^\dagger \cdot \partial_\pm Z$ is a covariant derivative and $x_\pm = x_1 \pm ix_2$.

$*2$ A nonlinear constraint $Z^\dagger \cdot Z = 1$ is imposed for the field $Z \in \mathbb{C}^3 \backslash \{0\}$.

The linearly independent solutions of the EOM are

1. Instanton Z_I : $Q \equiv Q_I > 0$,
2. Anti-instanton Z_A : $Q \equiv Q_A < 0$,
3. Mixture Z_M : $Q \equiv Q_M = |Q_A| + (-Q_I)$.

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How to construct the solutions

The definition of Bäcklund transformation

$$P_+ g \coloneqq \partial_+ g - \frac{g^\dagger \cdot \partial_+ g}{|g|^2} g, \ \forall g \in \mathbb{C}^3 \backslash \{0\}$$

One can construct the series of solutions from a holomorphic ansatz $f = f(x_+) \in \mathbb{C}^3 \backslash \{0\}$ as followings.

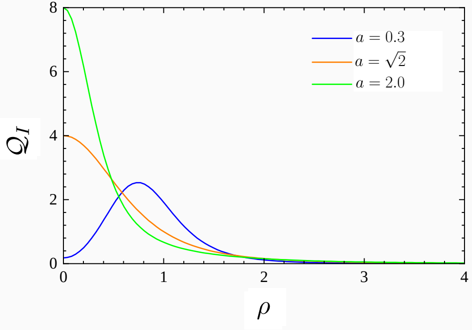
$$Z_I = \frac{f}{|f|} \overset{P_+}{\Longrightarrow} Z_M = \frac{P_+ f}{|P_+ f|} \overset{P_+}{\Longrightarrow} Z_A = \frac{P_+^2 f}{|P_+^2 f|}$$

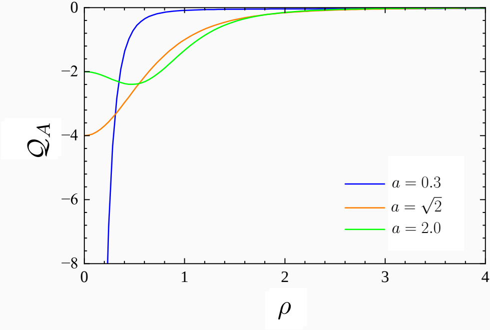
♠ Z_M can be obtained **analytically**!

- Z_I and Z_A have the energy E proportional to Q, *i.e.*, $E_I = 2\pi Q_I$, $E_A = 2\pi |Q_A|$
- Z_M has $E_M = E_I + E_A$. NOT proportional to $Q_M = |Q_A| - Q_I$!

♠ Z_M is the **mixture** of Z_I and Z_A!

- We employ a simple ansatz $f = \left(1, a x_+, b x_+^2\right)^t,\ a, b \in \mathbb{R}^+_\geq$.
- $\mathcal{M} = \mathbb{R}^+_\geq \times \mathbb{R}^+_\geq$
- In this case, the charges are $\{Q_I, Q_M, Q_A\} = \{2, 0, -2\}$.

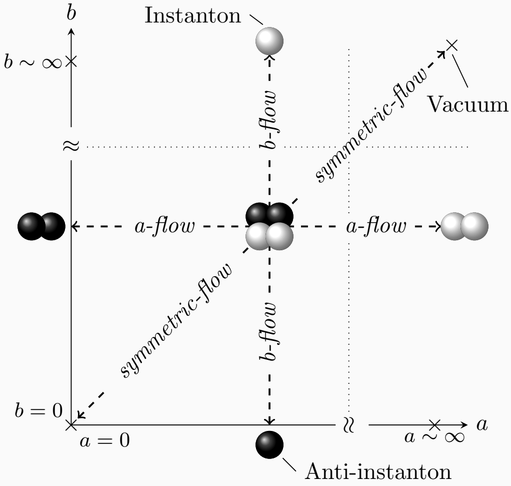




ρ : radial coordinate in $\mathbb{R}^2$, relating to $x_\pm = \rho \exp\left[\pm i\phi\right]$

The Moduli Space $\mathcal{M}$ - embedding limit

- Embedding limit... The limitting behaviour of Q in parameter space. In this situation, Z has the degree of freedom of $\mathbb{C}\mathrm{P}^1$.



Q for Mixture Solution

$$Q_M = |Q_A| - Q_I$$

$|Q_A|$: an "instanton",
$-Q_I$: an "anti-instanton" in the mixture.

- $a \to 0,\ b \to 0$: vacuum for the "instanton" $|Q_A|$
- $a \to \infty,\ b \to \infty$: vacuum for the "anti-instanton" $-Q_I$

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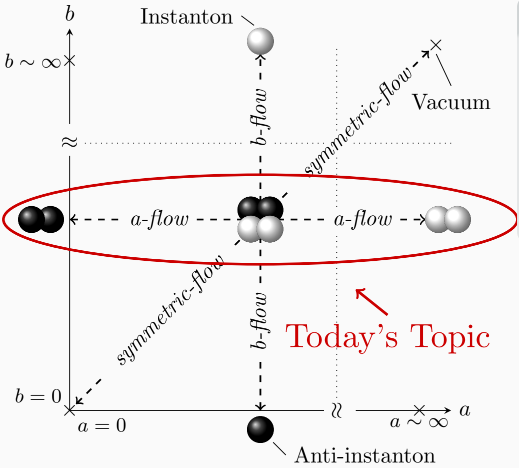

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Numerical method - Spectral-Flow

$2d$ Dirac Hamiltonian and the Eigenvalue Problem

- We forcus on the energy spectra on parameter space of background solitons.

$$\mathcal{H} = \begin{pmatrix} mX & \exp\left[-i\phi\right]\left(-\partial_\rho + \frac{i\partial_\phi}{\rho}\right) \\ \exp\left[i\phi\right]\left(\partial_\rho + \frac{i\partial_\phi}{\rho}\right) & -mX \end{pmatrix}$$

$$X \equiv I_3 - 2Z \otimes Z^\dagger,\ Z = \{Z_I, Z_M, Z_A\}$$

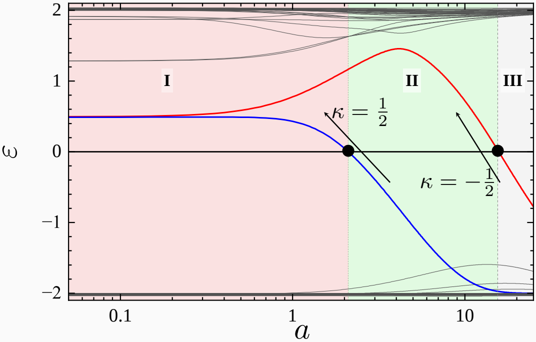
Dirac equation

$$\mathcal{H}\psi_\kappa = \varepsilon_\kappa \psi_\kappa \Leftrightarrow \det\left(H - \varepsilon I\right) = 0$$

H : matrix rep. of $\mathcal{H}$, $\kappa \in \left\{k + \frac{1}{2} | k \in \mathbb{Z}\right\}$: quantum number

The operator $\mathcal{H}$ depends on moduli parameters of Z :
$\mathcal{H} = \mathcal{H}(a, b)$. By diagonalizing H for $\{a, b\} \in \mathbb{R}^+_\geq \times \mathbb{R}^+_\geq$, we obtain a transition of the energy levels (Spectral flow).
Since ψ is a complex spinor, we forcus on $\chi \equiv \frac{1}{2\pi}\int \mathrm{d}\phi\, |\psi|^2$.

Spectral flow : Z_I background $(Q_I = 2,\ E_I = 4\pi)$ case



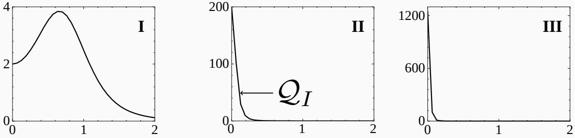
Indices

$$\begin{cases} n_+ &= 2 \\ n_- &= 0 \end{cases}$$

$$n_+ - n_- = 2$$

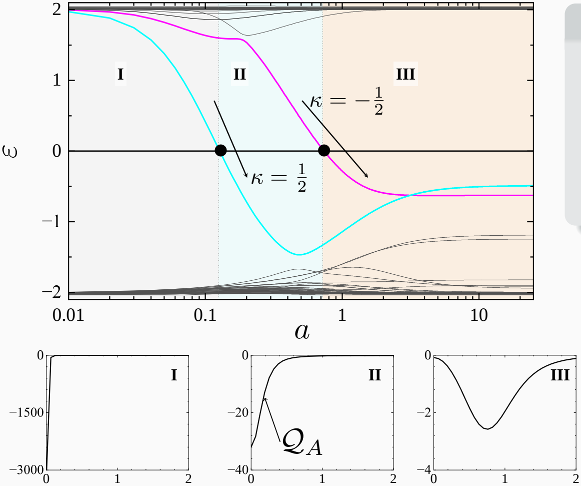
- Index theorem

  $n_+ - n_- = Q_I$
- $a \to \infty$: Vacuum for the instanton Q_I



- Red and Blue levels "respond" to the spatial structure of $\mathcal{Q}_I$.
- As $\mathcal{Q}_I$ becomes "**thick**", the two levels cross $\varepsilon = 0$ from negative continuum.

Spectral flow : Z_A background $(Q_A = -2,\ E_A = 4\pi)$ case



Indices

$$\begin{cases} n_+ &= 0 \\ n_- &= 2 \end{cases}$$

$$n_+ - n_- = -2$$

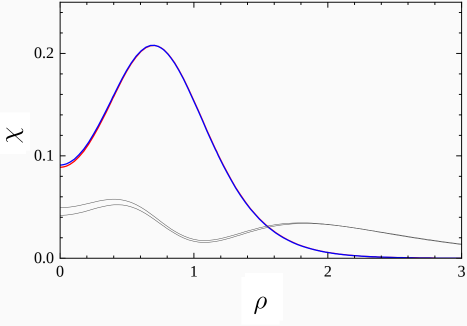
- Index theorem
  $n_+ - n_- = Q_A$
- $a \to 0$: Vacuum for the anti-instanton Q_A

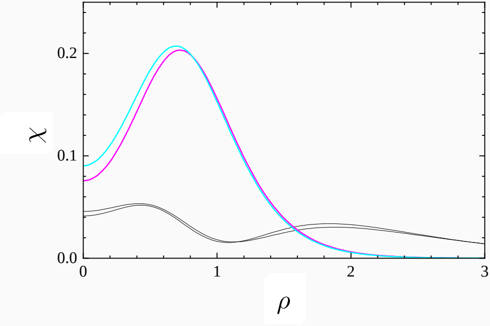
- Magenta and Cyan "respond" to the spatial structure of $\mathcal{Q}_A$.
- As $\mathcal{Q}_A$ becomes "**thick**", the two levels cross $\varepsilon = 0$ from positive continuum.

The wave functions : Z_I and Z_A background $(Q = \pm 2,\ E = 4\pi)$

The wavefunction of zero modes is localized around

the **thick** topological charge density of (anti-) instantons.

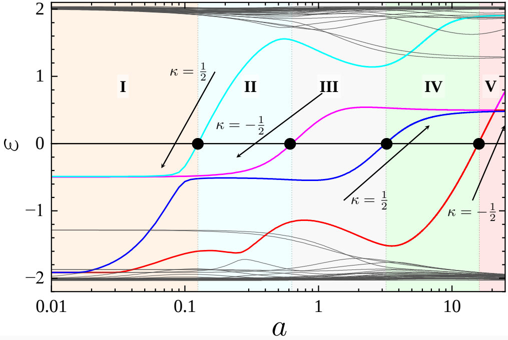


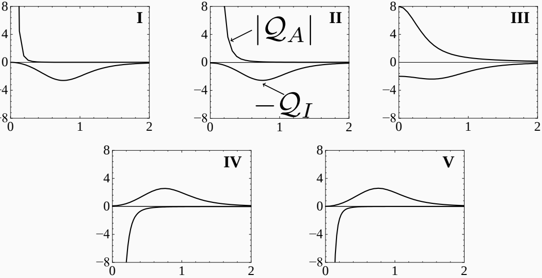


$n_{loc} = n_+ = Q_I$ $n_{loc} = n_- = |Q_A|$

The number of localized modes n_{loc} equals to $n_\pm$ and Q from the index theorem.

Spectral flow : Z_M background $(Q_M = 0,\ E_M = 8\pi)$





Indices

$$\begin{cases} n_+ = 2 \\ n_- = 2 \end{cases}$$

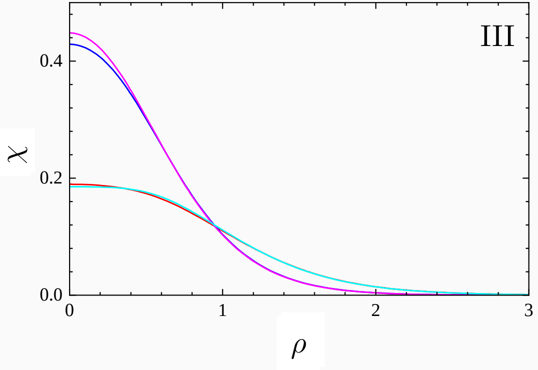
$$n_+ - n_- = 0$$

- Index theorem $n_+ - n_- = Q_M$
- $a \to 0$: Vacuum for the "instanton" $|Q_A|$
- $a \to \infty$: Vacuum for the "anti-instanton" $-Q_I$

Nontrivial zero modes!

The wave functions : Z_M background $(Q = 0,\ E = 8\pi)$

- In the intermediate region, **both** instantons and anti-instantons are "**thick**".
- Instantons and anti-instantons inside the mixture localize fermions, **respectively**.



Localized Modes

$$n_{loc} = n_+ + n_- = 4 \neq Q_M$$
$$n_+ = |Q_A| = 2$$
$$n_- = Q_I = 2$$

The number of localized modes is NOT equal to the Net topological charge.

Localized modes (Fermion+Anti-Fermion) are induced by the components of the mixture.

Summary and Outlook

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Summary

- Thrugh the spectral-flow analysis in a moduli space, we can see how zero modes appear from the vacuum and how fermions detect the topology of the background field.
- We have found four zero modes and localized modes in spite of $Q = 0$.
- We have given an interpretation to these modes, *i.e.*, **two of them are localized fermion modes**, and **the other two are localized anti-fermion modes**, induced by **components** of the mixture.

Outlook

- back reaction
- non-zero charged mixture

Thank you for your attention!