Surface operators
and AdS/CFT correspondence

Satoshi Yamaguchi
(Seoul National University)

based on
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Introduction
We want to address the operator localized on a sub-manifold
Motivation

- Phase structure of quantum field theories
- To understand “branes” in string theory via AdS/CFT
Classification

Non-local operators localized on a submanifold can be classified by the dimension of the submanifold.

In a 4-dimensional field theory

- **0 dim** Local operator
- **1 dim** Line operator (Ex. Wilson loop)  
  Introduce test particle
- **2 dim** Surface operator  
  Introduce test string
- **3 dim** Interface operator  
  (Can connect two different CFTs)  
  Introduce test membrane (wall)
AdS/CFT correspondence

AdS

\[ \text{type IIB superstring} \]

\[ \text{AdS5 x S5} \]

CFT

\[ \text{4dim N=4} \]

\[ \text{Super YM theory SU(N)} \]

Some objects

• Field fluctuation
• F-string
• D-brane probe
• NS5-brane probe
• Gravity solution
  etc.

(local or non-local) operators
  ex. surface operator

GKPW prescription

\[ \Rightarrow \text{correlation function} \]
AdS/CFT correspondence and surface operators

What kind of surface operators exist?

What is the gravity dual

GKPW

correlation function

surface operator

object

correlation function

AdS

CFT
Summary of our work [Koh, SY '08, '09]

- 1/4 BPS surface operators in N=4 SYM
- 1/2 BPS surface operators in Klebanov-Witten theory

Propose a gravity dual of these surface operators
Check the supersymmetry
Calculate the correlation function with local operators

Support AdS/CFT correspondence
Plan

- Review of 1/2 BPS surface operators in N=4 SYM
  - definition and symmetry
  - Gravity dual

- 1/4 BPS surface operators in N=4 SYM
  - definition and symmetry
  - Gravity dual
  - correlation function

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  - correlation function
4-dim N=4 SYM
1/2 BPS surface operator
Definition of a operator

Not all the operators can be written as functions of the fields in the Lagrangian

Example: 2-dim massless compact free boson

\[ S = \frac{1}{2\pi} \int d^2 z \partial_z \phi \partial_{\bar{z}} \phi \]

\[ \phi \approx \phi + 2\pi R \]

Vertex operator of momentum \( p \)

\[ O(z) = \exp(ip\phi) \]

Winding modes?
Winding mode operator can be written in terms of boundary condition (or OPE) as

\[ \phi(z) \tilde{O}(0) \sim \frac{wR}{2i} (\log z - \log \bar{z}) \tilde{O}(0) \]

- Correlation function can be defined by the path-integral under this boundary condition

Classical solution with singularity

\[ \phi(z) = \frac{wR}{2i} (\log z - \log \bar{z}) \]

an operator localized at the singularity
4-dim N=4 super Yang-Mills theory

- fields
  \( A_\mu, \mu = 0, 1, 2, 3 \quad \psi : 16 \text{ components} \quad \phi_i, i = 4, \ldots, 9 \)

- action
  \[
  S_{YM} = \frac{2N}{\lambda} \int d^4 x \text{ tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \cdots \right]
  \]
  \[
  \lambda = g_{YM}^2 N : \text{'t Hooft coupling}
  \]
1/2 BPS surface operator

4-dim N=4 SYM

\[ \Phi = \text{diag} \left( \frac{\beta}{z}, 0, 0, \ldots, 0 \right) \]

\[ \Phi := \phi_4 + i \phi_5 \]

\[ z = x^2 + ix^3 \]

\[ \beta : \text{constant} \]

- This configuration is a classical solution

- singular locus \[ z = 0 \]

parallel to \( x^0, x^1 \) direction

[Operator localized at \( z = 0 \)]

[Gukov, Witten '06]
Supersymmetry of the classical solution

The classical solution preserves half of the supersymmetry

\[ \delta \psi = D_\mu \phi_I \Gamma^{\mu I} \epsilon = 0 \rightarrow (1 + \Gamma^{2345}) \epsilon = 0 \]

\[ \rightarrow \text{1/2 BPS} \]
Generalization

\( M \) : integer

\( N_i, \ i = 1, \ldots, M \) : partition of \( N \)

\[ \sum_{i=1}^{M} N_i = N \]

\[ \Phi = \frac{1}{z} \text{diag} \]

\[ A_z = \frac{1}{2\pi iz} \text{diag} \]
Gravity dual of 1/2 BPS surface operator

[Constable, Erdmenger, Guralnik, Kirsch '02], [Gukov, Witten '06],
[Gomis, Matsuura '07], [Drukker, Gomis, Matsuura '08],
[Lin, Lunin, Maldacena '04], [Lin, Maldacena '05]

- D3-brane probe

AdS3 x S1 shaped

SO(2,2) x SO(2) x SO(4)
supersymmetry

\[ \Phi = \text{diag} \left( \frac{\beta}{z}, 0,0,\ldots, 0 \right) \]
AdS5 coordinates \((y, r, \phi, x_1, x_2)\)

Coordinate of a great circle in S5 \(\theta\)

\[
ds^2 = \frac{1}{y^2} (dy^2 + dr^2 + r^2 \, d\phi^2 + dx_0^2 + dx_1^2) + d\theta^2
\]

D3-brane

\(\kappa \, y = r, \quad \theta = \phi\)
Comments

• Global symmetry is checked
  [Constable, Erdmenger, Guralnik, Kirsch '02], [Gukov, Witten '06],
  [Gomis, Matsuura '07]

• Correlation functions are calculated
  [Drukker, Gomis, Matsuura '08]

• Bubbling geometry as gravity dual
  [Lin, Lunin, Maldacena '04], [Lin, Maldacena '05],
  [Gomis, Matsuura '07]
Plan

- Review of 1/2 BPS surface operators in N=4 SYM
  - definition and symmetry
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- 1/4 BPS surface operators in N=4 SYM
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  - correlation function

- 1/2 BPS surface operators in Klebanov-Witten theory
  - definition and symmetry
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  - correlation function
1/4 BPS
surface operator
1/2 BPS surface operator

4 dim N=4 SYM

\[ \Phi = \text{diag} \left( \frac{\beta}{z_1}, 0, 0, \ldots, 0 \right) \]

\[ \Phi := \phi_4 + i \phi_5 \]

\[ z^1 = x^2 + ix^3 \]

\[ \beta : \text{constant} \]

- Supersymmetry

\[ \delta \psi = D_\mu \phi_I \Gamma^\mu I \epsilon = 0 \quad \Rightarrow \quad (1 + \Gamma^{2345}) \epsilon = 0 \quad 1/2 \text{ BPS} \]

Holomorphy is important to preserve the supersymmetry!

- Dilatation symmetry

\[ \Phi \quad \text{has conformal dimension 1} \]

Degree (-1) is important to preserve the dilatation symmetry
$1/4$ BPS surface operator

$\Phi \sim \frac{1}{\sqrt{z^1 z^2}}$

Multi-valued

$z^1 = x^2 + ix^3$
$z^2 = x^0 + ix^1$

Well-defined ??
Yes, in the following way.

\[ \Phi = \text{diag} \left( \frac{\beta}{\sqrt{z^1 z^2}}, -\frac{\beta}{\sqrt{z^1 z^2}}, 0, \ldots, 0 \right) , \quad A_\mu = 0 , \]

For example for fixed \( z^2 \), there is monodromy around \( z^1 = 0 \)

\[ z^1 \rightarrow z^1 e^{2 \pi i} \]

Cancel the monodromy by the gauge holonomy

\[ g = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & I_{N-2} \end{pmatrix} \]

Cancel the monodromy and become a consistent configuration
Gravity dual = a configuration of D3-brane

AdS5 x S5
complex coordinates

\[ ds^2 = \frac{1}{y^2} (|dz^1|^2 + |dz^2|^2) + y^2 \sum_{a=1}^{3} |d\omega^a|^2 \]

\[ y^{-2} := \sum_{a=1}^{3} |\omega^a|^2 \]

D3-brane wrapping the surface

\[ z_1 z_2 (\omega^1)^2 - \kappa^2 = 0, \quad \omega^2 = \omega^3 = 0, \]

\[ \kappa : \text{constant related to } \beta \text{ by } \kappa = \frac{2\pi/\beta}{\sqrt{\lambda}} \]
\( z^1 z^2 (\omega^1)^2 - \kappa^2 = 0, \quad \omega^2 = \omega^3 = 0, \)

\[
\kappa y = \sqrt{r_1 r_2}, \quad \theta = \frac{1}{2} (\phi_1 + \phi_2)
\]

\[
\omega = y^{-1} e^{i \theta}, \quad z_j = r_j e^{i \phi_j}.
\]
Supersymmetry of the gravity dual

- Kappa symmetry projection
- 12 dimensional formulation

[Mikhailov '00], [Kim, Lee '06]
Correlation function with a local operator—gauge theory side

\[ O(z) = C^{I_1 \cdots I_\Delta} \text{tr} [\phi_{I_1} \cdots \phi_{I_\Delta}] \]

Traceless, symmetric tensor

(For 1/2 BPS case [Drukker, Gomis, Matsuura])
The result in the gauge theory side (classical)

\[
\frac{\langle O_{\Sigma} \cdot O(\zeta) \rangle}{\langle O_{\Sigma} \rangle} = \frac{1}{\langle O_{\Sigma} \rangle} \int_{\text{boundary condition}} [DAD\psi D\phi] O(\zeta) e^{-S}
\]

\[
\Rightarrow \quad O|_{\Sigma}(\zeta)
\]

classical approximation simply insert the classical solution

The result in the gauge theory side (classical)

\[
\frac{\langle O_\beta(\Sigma) O_{\Delta,k}(\zeta) \rangle}{\langle O_\beta(\Sigma) \rangle} = \frac{(8\pi^2)^{\Delta/2}}{\lambda^{\Delta/2} \sqrt{\Delta}} C_{\Delta,k} \frac{\beta^\Delta}{(\zeta^1 \zeta^2)^{(\Delta-k)/2}} \frac{1}{(\zeta^1 \zeta^2)^{(\Delta+k)/2}} (1 + (-1)^\Delta)
\]
Correlation function with a local operator—gravity side

Some field fluctuation of metric and RR4-form

Chiral primary operators

GKPW: calculate the classical action of the solution with source inserted at boundary.

D3-brane is treated as probe

Action of the gravity side

\[ S_{\text{gravity}} = S_{\text{IIB sugra}} + S_{D3} \]

\[ S_{D3} = S_{\text{DBI}} - S_{\text{WZ}}, \quad S_{\text{DBI}} = T_{D3} \int d^4 \xi \sqrt{\det G_{mn}}, \quad S_{\text{WZ}} = T_{D3} \int_{\Sigma_4} C_4. \]
\[
\frac{\left\langle O_\beta (\Sigma) O_{\Delta, k} (\zeta) \right\rangle}{\left\langle O_\beta (\Sigma) \right\rangle} = \frac{\delta S_{\text{gravity}}}{\delta s_0 (\zeta)} \bigg|_{s_0 = 0} = \frac{\delta S_{D3}}{\delta s_0 (\zeta)} \bigg|_{s_0 = 0}
\]

\[
= -2 \Delta T_{D3} c(\Delta) C_{\Delta, k} \int d^4 z \frac{\omega^{-\frac{\Delta-k}{2}} (z) \bar{\omega}^{-\frac{\Delta+k}{2}} (\bar{z})}{L^{\Delta+2}} \frac{\left| \zeta^m \partial_m \omega (z) \right|^2}{\left| \omega (z) \right|^2}
\]

\[
L \equiv \sum_{m=1,2} \left| z^m - \zeta^m \right|^2 + \left| \omega \right|^{-2} \quad \omega (z) = \frac{K}{\sqrt{z_1 z_2}}
\]

It is not easy to evaluate exactly this integral
Approximation $\kappa \to \infty$

The integrand has a SHARP PEAK in this limit!
The result in the gravity side \( \kappa \to \infty \)

\[
\frac{\langle O_\beta (\Sigma) O_{\Delta,k} (\zeta) \rangle}{\langle O_\beta (\Sigma) \rangle} = \frac{2^{\Delta/2}}{\sqrt{\Delta}} C_{\Delta,k} \frac{\kappa^\Delta}{(\zeta_1 \zeta_2)^{(\Delta-k)/2}(\zeta_1^\Delta \zeta_2^\Delta)^{(\Delta+k)/2}}(1+(-1)^\Delta)
\]

The result in the gauge theory side

\[
\left(\frac{\langle O_\beta (\Sigma) O_{\Delta,k} (\zeta) \rangle}{\langle O_\beta (\Sigma) \rangle} = \frac{(8\pi^2)^{\Delta/2}}{\lambda^{\Delta/2}\sqrt{\Delta}} C_{\Delta,k} \frac{\beta^\Delta}{(\zeta_1 \zeta_2)^{(\Delta-k)/2}(\zeta_1^\Delta \zeta_2^\Delta)^{(\Delta+k)/2}}(1+(-1)^\Delta)\right)
\]

Agree with the classical calculation in the gauge theory side with the identification

\[
\kappa = \frac{2\pi \beta}{\sqrt{\lambda}}
\]
Correction

\[ \frac{\langle O_{\beta}(\Sigma) O_{\Delta,k}(\zeta) \rangle}{\langle O_{\beta}(\Sigma) \rangle} = (leading) \left[ 1 + \frac{\lambda}{4\pi^2 \beta^2} \frac{\Delta^2 - k^2}{16(\Delta - 1)} \left( \frac{|\zeta_1|^2 + |\zeta_2|^2}{|\zeta_1 \zeta_2|} \right) + \ldots \right] \]

This expression is positive power in $\lambda$!

The situation is similar to plane wave limit of BMN

Large $\beta$ mimics the perturbative expansion in $\lambda$

To compare this term with the perturbative Yang-Mills calculation is an interesting problem.
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- 1/2 BPS surface operators in Klebanov-Witten theory
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1/2 BPS surface operators in the Klebanov-Witten Theory
An example of N=1 AdS/CFT

\[ \text{AdS} \quad \bigg/ \bigg/ \quad \text{CFT} \]

- type IIB superstring
- AdS5 x T1,1
- Klebanov-Witten Theory
- N=1 superconformal

[Klebanov, Witten '98]
Klebanov-Witten theory

N=1 gauge theory

- gauge group: $\text{SU}(N) \times \text{SU}(N)$
- matters: $A_1, A_2$ \quad $(N, \bar{N})$
  \quad $B_1, B_2$ \quad $(\bar{N}, N)$
- superpotential: $W = \text{Tr} \left[ A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1 \right]$

This theory has non-trivial fixed point
1/2 BPS surface operator

\[ A, B \quad : \text{dimension } \frac{3}{4} \]

To preserve the scale invariance

\[ A, B \sim \frac{1}{z^{3/4}} \]

Example:

\[ A_1 = B_1 = \frac{\beta}{z^{3/4}} \text{diag} (1, i, 0, \cdots, 0), \quad A_2 = B_2 = 0. \]

Holonomy around \( z=0 \) \( \rightarrow \) Well-defined configuration

\[ (g, \tilde{g}) \in SU(N) \times SU(N) \]

\[ g = \begin{pmatrix} \sigma_1 & 0 \\ 0 & e^{\frac{\pi i}{N-2}} I_{N-2} \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} i \sigma_2 & 0 \\ 0 & I_{N-2} \end{pmatrix}, \]
Supersymmetry

\[ A_1 = B_1 = \frac{\beta}{z^{3/4}} \text{diag}(1, i, 0, \cdots, 0), \quad A_2 = B_2 = 0. \]

\[ F = 0, \quad D = 0 \]

(variation of fermions) = 0

Non-trivial condition

\[ \sigma^\mu \partial_\mu A_1 \epsilon = 0 \]

\[ (\sigma^1 + i \sigma^2) \epsilon = 0 \]

1/2 BPS
Gravity dual

$AdS_5 \times T^{1,1}$

d$s^2 = \frac{1}{y^2} (dy^2 + dr^2 + r^2 d\phi^2 + dx_0^2 + dx_1^2) + ds^2_{T^{1,1}}$

d$s^2_{T^{1,1}} = \frac{1}{9} (d\psi + \cos \theta_1 d\nu_1 + \cos \theta_2 d\nu_2)^2 + \frac{1}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2 \theta d\nu_i^2)$

D3-brane wrapping on the surface

$\kappa y = r, \quad \psi = -3 \phi, \quad \theta_i = \pi, \quad \nu_i = 0$

Induced metric = $AdS_3 \times S1$
Correlation function with chiral primary operator

Chiral primary operators

\[ O^I_n = p^I_n C^I (i_1, \cdots, i_n) (j_1, \cdots, j_n) \text{Tr} [ A_{i_1} B_{j_1} A_{i_2} B_{j_2} \cdots A_{i_n} B_{j_n} ] \]

Conformal dimension

\[ \Delta = \frac{3}{2} n \]

Normalized as

\[ \langle \bar{O}^I_n (x) O^I_n (0) \rangle = \frac{1}{|x|^2} \]

Operators

\[ O_n = p_n \text{Tr} [(A_1 B_1)^n] \]

have non-trivial correlation function with the surface operator
Gravity side GKPW calculation

\[
\frac{\langle O(\Sigma)O_n(\zeta) \rangle}{\langle O(\Sigma) \rangle} = \frac{\sqrt{3}}{4} \frac{2 \Delta + 3}{\sqrt{\Delta(\Delta + 1)(\Delta + 2)}} \frac{\kappa^\Delta}{\zeta^\Delta} (1 + (-1)^n)
\]

Classical approximation (?) in the gauge theory

\[
\frac{\langle O(\Sigma)O_n(\zeta) \rangle}{\langle O(\Sigma) \rangle} = p_n \frac{\beta^{2n}}{\zeta^\Delta} (1 + (-1)^n)
\]

- The position dependence is the same
- The relations (up to overall scaling of A,B)

\[
\kappa = \beta^{4/3}, \quad p_n = \frac{\sqrt{3}}{4} \frac{2 \Delta + 3}{\sqrt{\Delta(\Delta + 1)(\Delta + 2)}}
\]
Summary
1/4 BPS surface operators in N=4 SYM

- Definition.

- The gravity dual as a certain configuration of a probe D3-brane.

- Correlation functions with chiral primary operators agree!

1/2 BPS surface operators in Klebanov-Witten theory

- We defined a 1/2 BPS surface operators in KW.

- Gravity dual, correlation function.
Thank you