# Nahm-like construction for M2-M5

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25 September 2023

at International Seminar-Type Online Workshop on Topological Solitons

mainly based on the papers:

"Integrability of BPS equations in ABJM theory "

K. Sakai and ST, JHEP11(2013)002,

"M5-branes in ABJM theory and Nahm equation"

T.Nosaka and ST, Phys.Rev. D86 (2012) 125027

**(11dim.) M-theory is "defined" by**

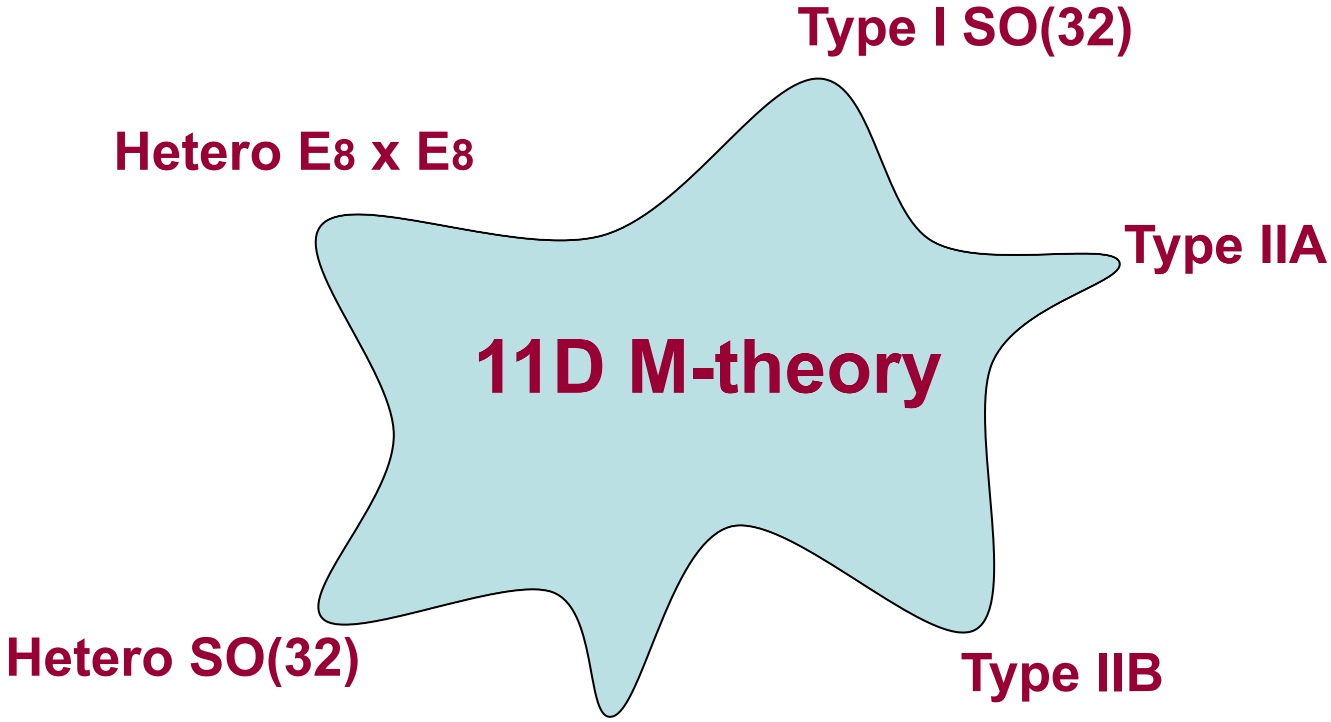
**Non-perturbative (10 dim.) IIA string theory**

**More precisely,**

**M-theory on $S^1$ = type IIA string**

**radius of $S^1$ ~ string coupling constant**

# It was claimed:
# **“All string theories are unified”**




**Any string theory = a compactification of M-theory**

**Various string dualities are manifest in M-theory**

**M-theory is described by**

**11d supergravity in a low energy limit**

**Fields in 11d sugra**
**are**
**3-forms $C_{\mu\nu\rho}$ and metric only.**

(except gravitino)

↓

**Only *M2*-branes and *M5*-branes**
**are (charged) branes**

**Here, *M2*-brane is (2+1)d object**
**and *M5*-brane is (5+1)d object.**

**Let us concentrate on M2-brane, which has been significantly understood recently.**

**The low energy effective theory on M2-branes is given by ABJM(Aharony-Bergman-Jafferis-Maldacena) theory, which is just a (2+1)d Chern-Simons-matter theory after the important works by Basu-Harvey and Bagger-Lambert (and Gustavsson).**

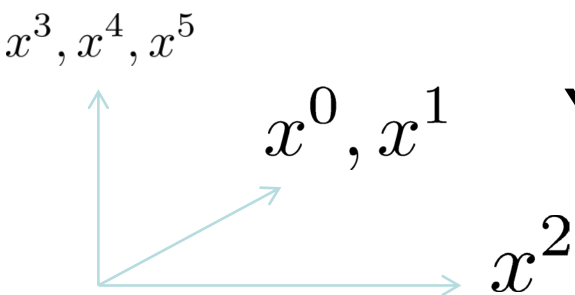
**M5-branes may be more interesting, but, mysterious.**

**Nevertheless, we expect that M2-branes know M5-branes.**

**Why?**

# Now, remember that in type IIA string theory, D2-branes know D4-branes.

D2-branes (no boundary)




**Yang-Mills** theory describes the D2-branes

# ADHM(N) construction

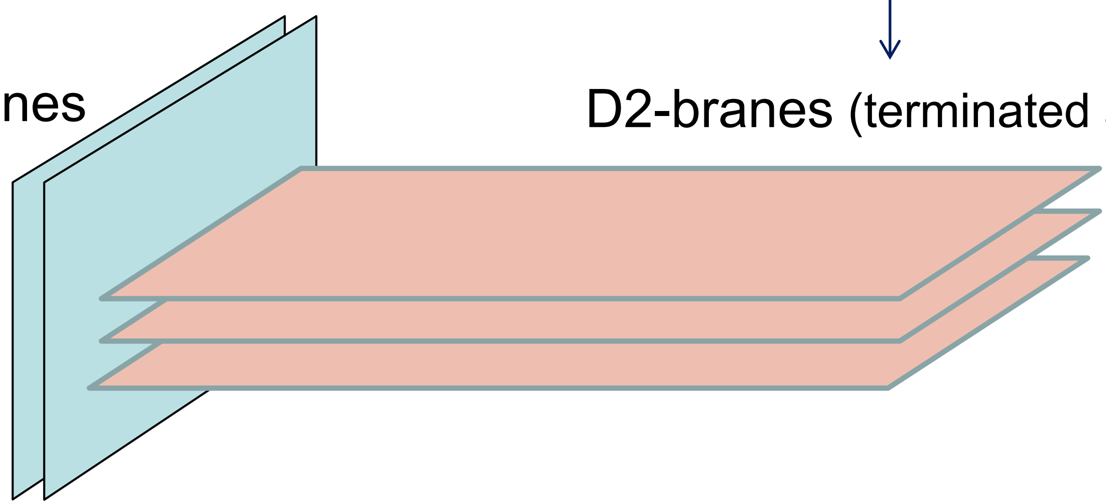
**Monopole equation** ⟺ **Nahm equation**

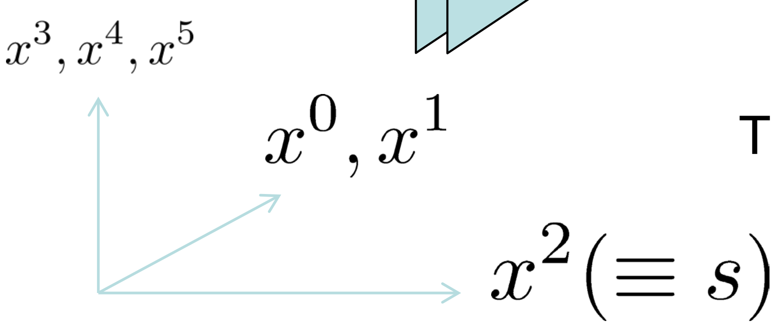
$$\frac{1}{2}\epsilon_{IJK}F^{IJ} = D_K\Phi$$

$$\dot{T}^I = i\epsilon_{IJK}T^J T^K$$

D4-branes

D2-branes (terminated at D4-branes)






This is a soliton solution in YM theory

**IIA string :** **D2-brane** **D4-brane**

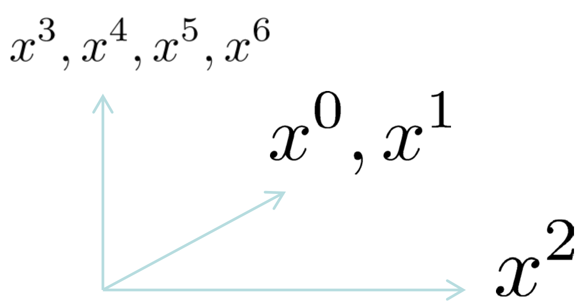
↓ ↓ ↓

**M-theory (on $S^1$) :** **$M$2-brane** **$M$5-brane**

# We expect that

# the M2-branes know the M5-branes.

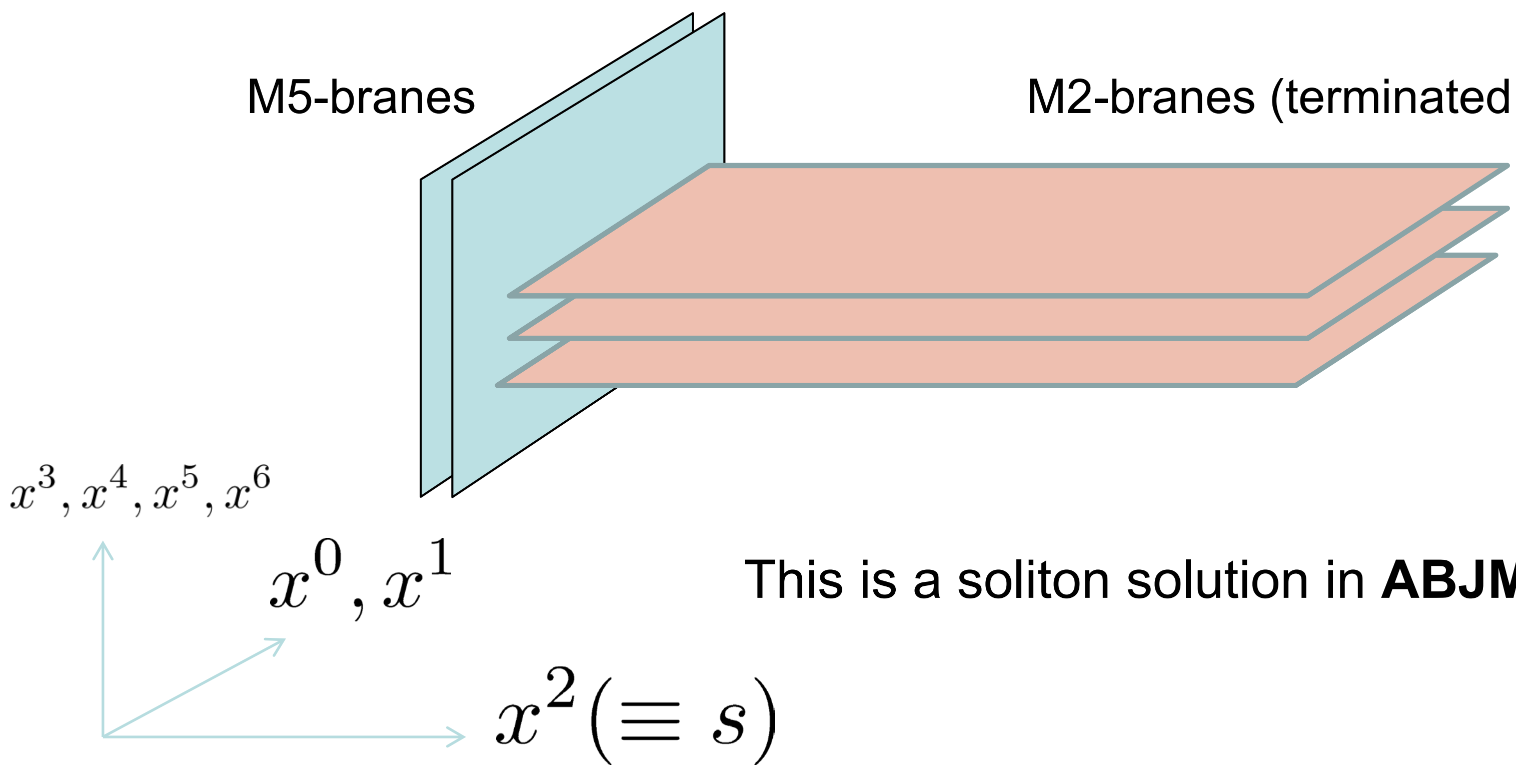
M2-branes (no boundary)




ABJM theory describes these M2-branes

# We expect that

# the M2-branes know the M5-branes.




This is a soliton solution in **ABJM** theory

# In ABJM theory, BPS equations for this M5-brane-M2-brane bound state is

$$0 = \frac{dY^1}{dx^2} + \frac{2\pi}{k}(Y^2 Y_2^\dagger Y^1 - Y^1 Y_2^\dagger Y^2),$$

$$0 = \frac{dY^2}{dx^2} + \frac{2\pi}{k}(Y^1 Y_1^\dagger Y^2 - Y^2 Y_1^\dagger Y^1),$$

In ABJM theory,
BPS equations for
this M5-brane-M2-brane bound state is

$$
\begin{aligned}
0 &= \frac{dY^1}{dx^2} + \frac{2\pi}{k}(Y^2 Y_2^\dagger Y^1 - Y^1 Y_2^\dagger Y^2), \\
0 &= \frac{dY^2}{dx^2} + \frac{2\pi}{k}(Y^1 Y_1^\dagger Y^2 - Y^2 Y_1^\dagger Y^1),
\end{aligned}
$$

Cubic interaction, not quadratic,
not commutator like Nahm eq, ADHM, instanton,,,
new type of soliton equation!

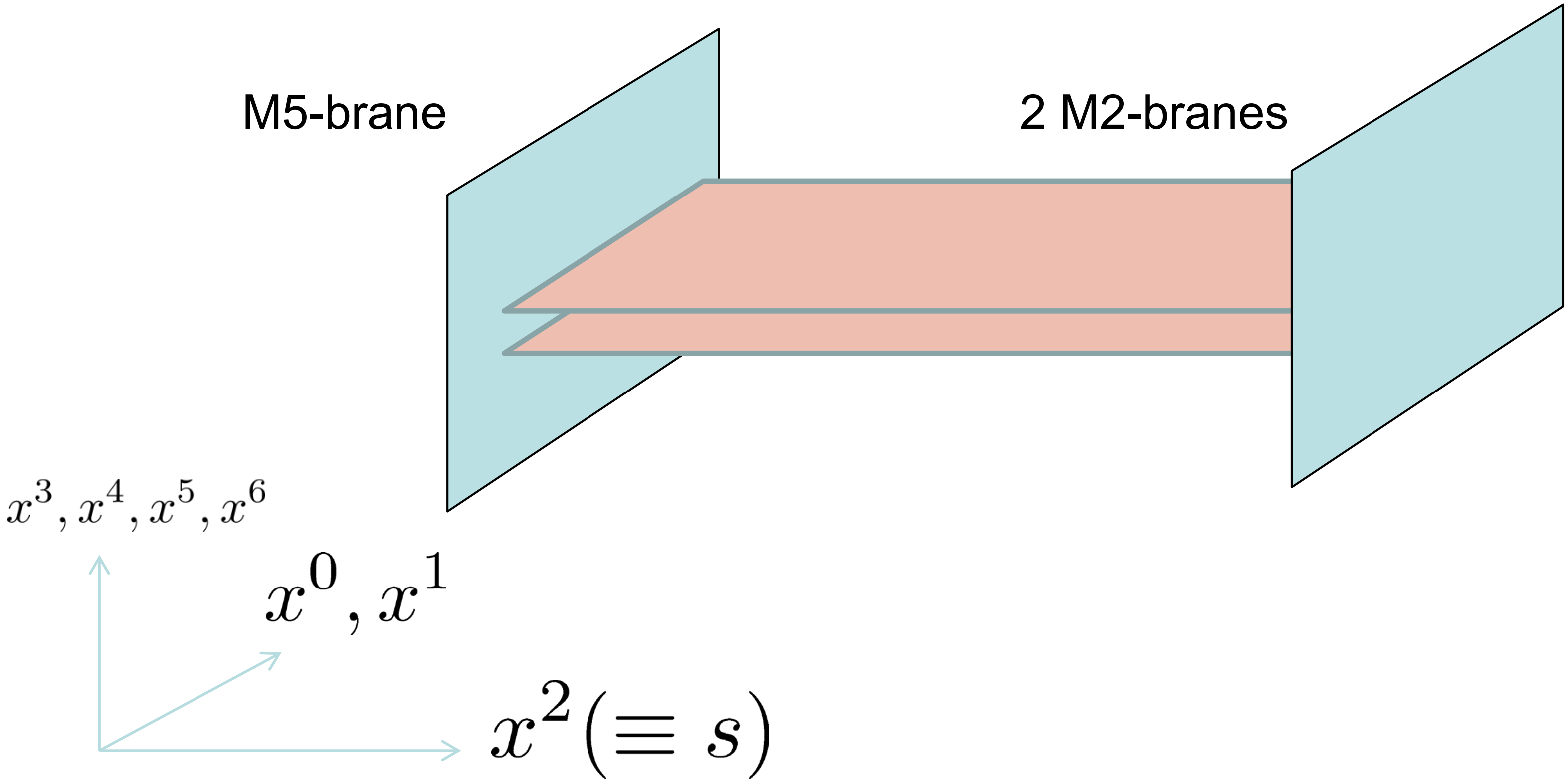
**Thus,**
**the solutions of these BPS eq.**
**should represent the M5-branes.**
**Those will be important!**

**(for example, in order to try to find**
**a Nahm-like transformation to**
**the BPS solutions in the M5-brane action).**

**However,**

**only a few solutions had been known.**

# We found all solutions for two M2-branes:



# The KEY facts:

## The BPS equations

$$
\begin{aligned}
0 &= \frac{dY^1}{dx^2} + \frac{2\pi}{k}(Y^2 Y_2^{\dagger} Y^1 - Y^1 Y_2^{\dagger} Y^2), \\
0 &= \frac{dY^2}{dx^2} + \frac{2\pi}{k}(Y^1 Y_1^{\dagger} Y^2 - Y^2 Y_1^{\dagger} Y^1),
\end{aligned}
$$

**equivalent!**

## Lax equation $\dot{A} = [A, B]$ for Lax pair

$$
A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix},
$$

$$
B(s;\lambda) = \begin{pmatrix} \lambda^{-1} Y^1 Y^{2\dagger} + \lambda Y^2 Y^{1\dagger} & O \\ O & \lambda Y^{1\dagger} Y^2 + \lambda^{-1} Y^{2\dagger} Y^1 \end{pmatrix}
$$

**Using this integrable structure rather tricky,**

**we find all solutions for two *M2*-branes.**

**This could be a step toward understanding *M5*-branes**

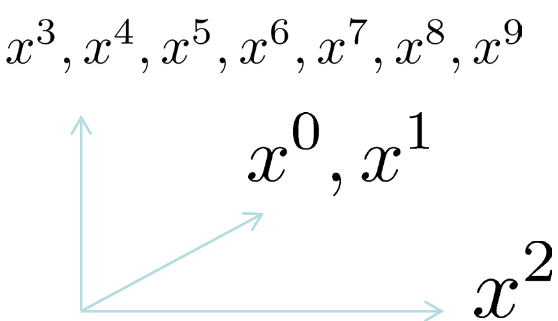
**I will talk about this.**

# Plan

- **Introduction**
- **D2-D4 and Nahm eqution**
- **ABJM theory and the BPS equations**
- **The Lax pair**
- **The solutions for two M2-branes**

# D2-D4 and Nahm equation

# N D2-branes in IIA string


SU(N) YM theory describes these D2-branes

D2-brane effective action (3 dim. field theory)
is super Yang-Mills theory with maximal SUSY

- 7 scalars = location of D2-brane in 10d spacetime
- 16 SUSY and SO(7) global symmetry
- Not conformal

We will consider N D2-branes, which means SU(N) gauge

# Fields in SU(N) SYM action:

7 real scalars ($I$=1,2,3,4,5,6,7)
adjoint rep. of *U(N)*

$$T^I$$

**location of D2-branes**

4 (2+1)d Dirac spinors
adjoint rep. of *U(N)*

$$\psi_A \; , \; \psi^{A\dagger}$$

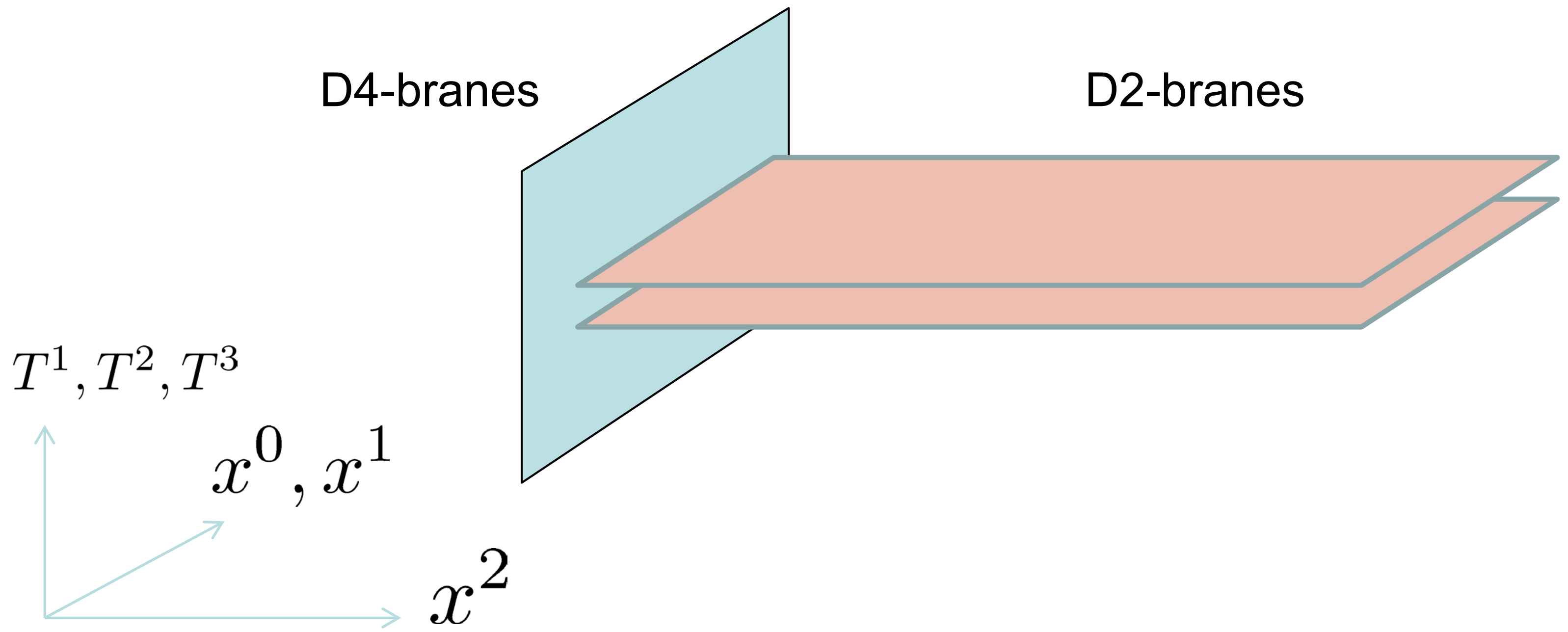
(2+1)d *U(N)* gauge fields

$$A_\mu$$

# D2-D4-brane bound state in super YM

We assume

$$\begin{cases} T^1 = T^1(x^2) \\ T^2 = T^2(x^2) \\ T^3 = T^3(x^2) \\ T^4 = T^5 = T^6 = T^7 = 0 \\ A_\mu = 0 \end{cases}$$

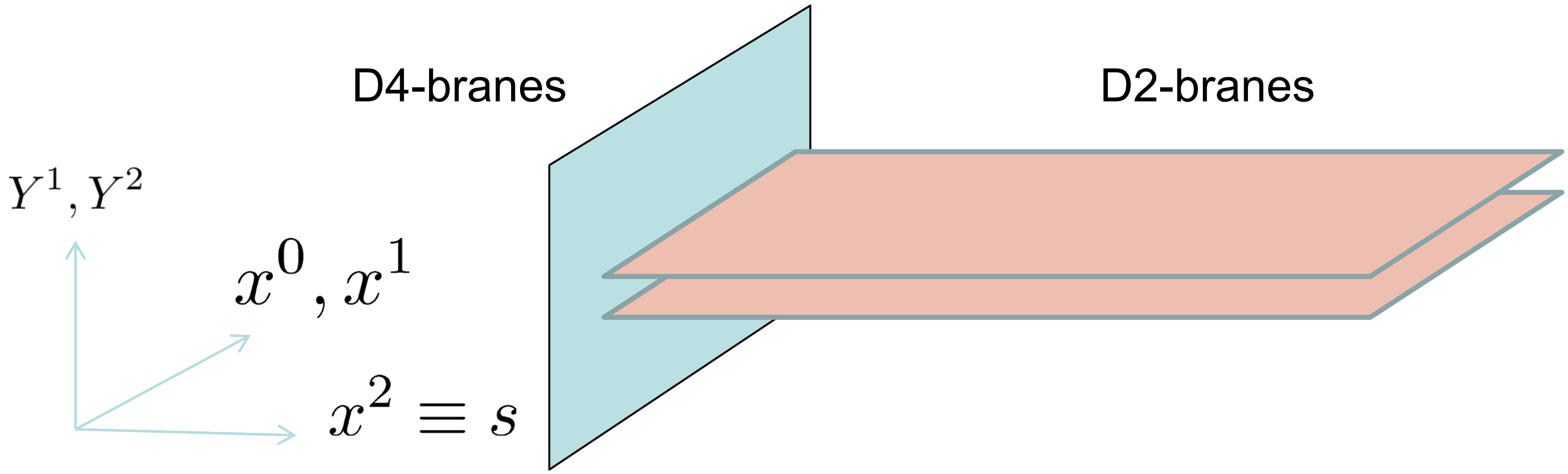


Then,

½ BPS equations for D2-D5 bound stare is Nahm eq.

$$\dot{T}^I = i\epsilon_{IJK} T^J T^K \qquad \text{where } \dot{T}^I = \frac{dT^I}{ds}$$

Diaconescu




If $T^I(c) = \infty$ , there will be D4-brane at $x^2 = c$

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Actually, any solution will becomes
the following basic solution near the D4-branes:

$$T^I \sim \frac{1}{x^2} R^I$$

where
R is N dim. representation of SU(2) (N x N matrices)

This correspond to the fuzzy 2 sphere

# The ABJM theory and the BPS equations

# M2-brane effective action (3 dim. field theory) should have

- 8 scalars = location of M2-brane in 11d spacetime
- 16 SUSY and SO(8) global symmetry
- Conformal symmetry (→ not Yang-Mills theory)

# Fields in ABJM action:

4 complex scalars ($A$=1,2,3,4)
bi-fundamental rep. of *U(N) x U(N)*

$$Y^A \,,\; Y_A^\dagger$$

**location of M2-branes**

4 (2+1)d Dirac spinors
bi-fundamental rep. of *U(N) x U(N)*

$$\psi_A \,,\; \psi^{A\dagger}$$

(2+1)d *U(N) x U(N)* gauge fields

$$A_\mu \,,\; \hat{A}_\mu$$

# ABJM action
is
U(N) x U(N) Chern-Simons-matter theory
which has

- 12 SUSY (N=6)
- SU(4) x U(1) global symmetry
- Conformal symmetry

This action describes *N* M2-branes on $\mathbf{C}^4/\mathbf{Z}_k$

$$Y^A \to e^{2\pi i/k} Y^A$$

# ABJM action:

$$S = \int d^3x \left[ \frac{k}{4\pi} \varepsilon^{\mu\nu\lambda} \mathrm{Tr} \left( A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right.$$
$$\left. - \mathrm{Tr} D_\mu Y_A^\dagger D^\mu Y^A - i \mathrm{Tr}\, \psi^{A\dagger} \gamma^\mu D_\mu \psi_A - V_{\mathrm{bos}} - V_{\mathrm{ferm}} \right]$$

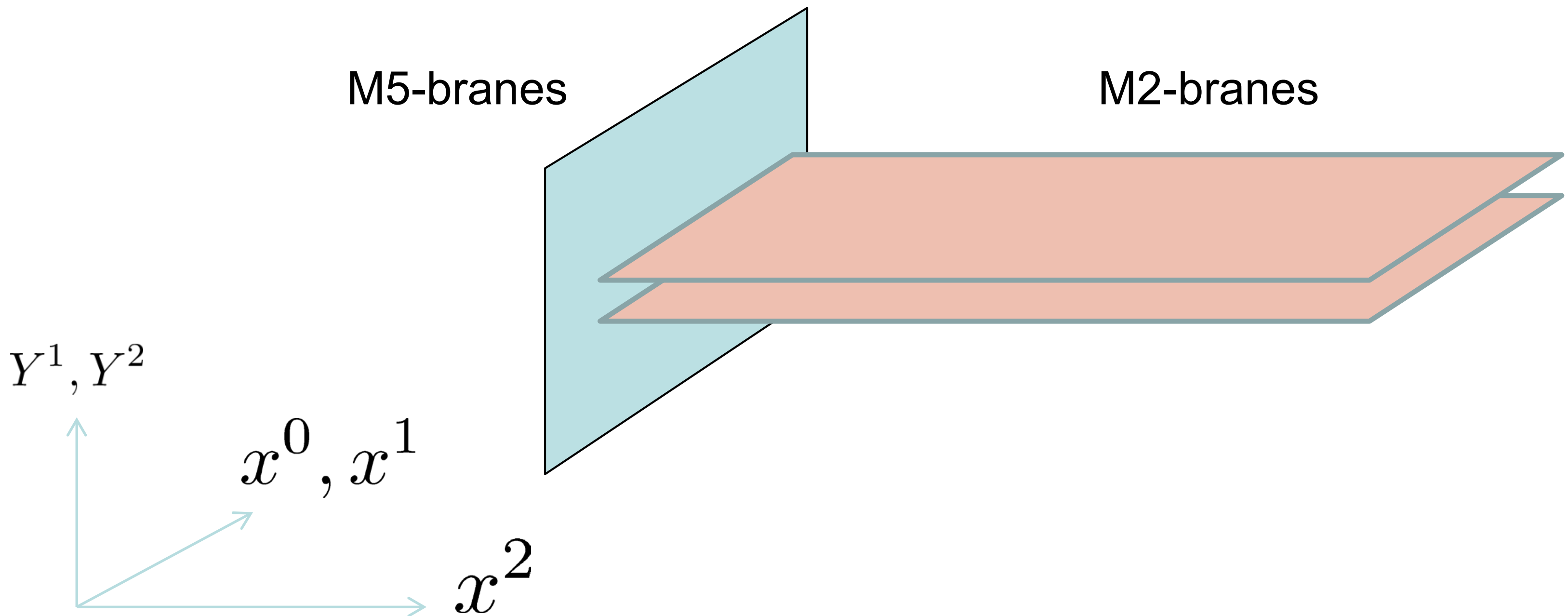
$$V_{bos} = -\frac{4\pi^2}{3k^2} \mathrm{Tr} \left( Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \right.$$
$$\left. + 4 Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6 Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger \right)$$

$$V_{ferm} = -\frac{2i\pi}{k} \mathrm{Tr} \left( Y_A^\dagger Y^A \psi^{B\dagger} \psi_B - \psi^{B\dagger} Y^A Y_A^\dagger \psi_B - 2 Y_A^\dagger Y^B \psi^{A\dagger} \psi_B + 2 \psi^{B\dagger} Y^A Y_B^\dagger \psi_A \right.$$
$$\left. + \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D - \epsilon_{ABCD} Y^A \psi^{B\dagger} Y^C \psi^{D\dagger} \right),$$

# M2-M5-brane bound state in ABJM

We assume

$$
\begin{cases}
Y^1 = Y^1(x^2) \\
Y^2 = Y^2(x^2) \\
Y^3 = Y^4 = 0 \\
A_\mu = \hat{A}_\mu = 0
\end{cases}
$$

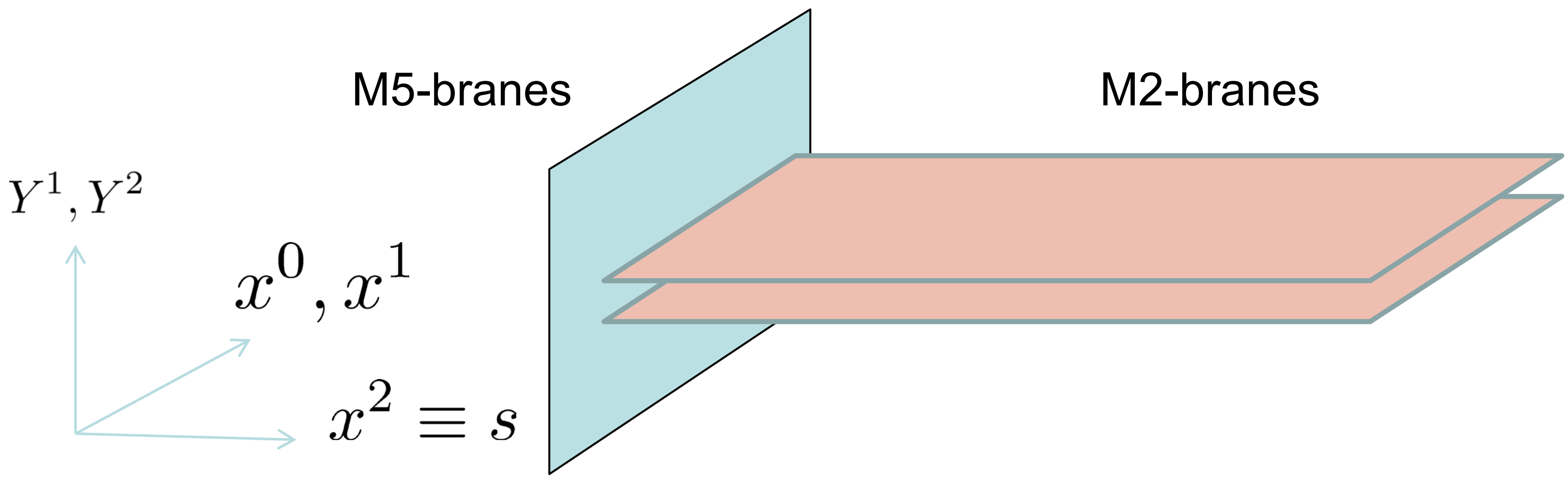


Then,

½ BPS equations for M2-M5 bound stare is given by

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

where $\dot{Y} \equiv \frac{dY}{ds}$ $\quad (a = 1, 2)$ ST

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If $Y^a(c) = \infty$, there will be M5-brane at $x^2 = c$

Actually, any solution will becomes
the following basic solution near the M5-branes:

$$Y^a = \sqrt{\frac{k}{4\pi x^2}} S^a$$

where S are constant N x N matrices satisfying

$$\begin{aligned} S^1 &= S^2 S^{2\dagger} S^1 - S^1 S^{2\dagger} S^2 \\ S^2 &= S^1 S^{1\dagger} S^2 - S^2 S^{1\dagger} S^1 \end{aligned}$$

This can be solved by digonalizing $S^1$ by U(N) x U(N) gauge symmetry

$$(S^1)_{ij} = \delta_{i,\,j-1}\sqrt{i}, \quad (S^2)_{ij} = \delta_{ij}\sqrt{N-i} \quad (i,j = 1,\cdots,N)$$

$$\begin{pmatrix} 0 & \sqrt{N-1} & & & \\ & 0 & \ddots & & \\ & & \ddots & \sqrt{2} & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \sqrt{N-2} & \\ & & & & \sqrt{N-1} \end{pmatrix}$$

**This should represent an Fuzzy 3-sphere**

Because $\begin{cases} Y^a = \sqrt{\dfrac{k}{4\pi x^2}} S^a \\ \dfrac{1}{N} Tr\left((S^a)^\dagger S_a\right) = N - 1 \end{cases}$

radius of 3-sphere is $r \sim \sqrt{kN/(4\pi x^2)}$

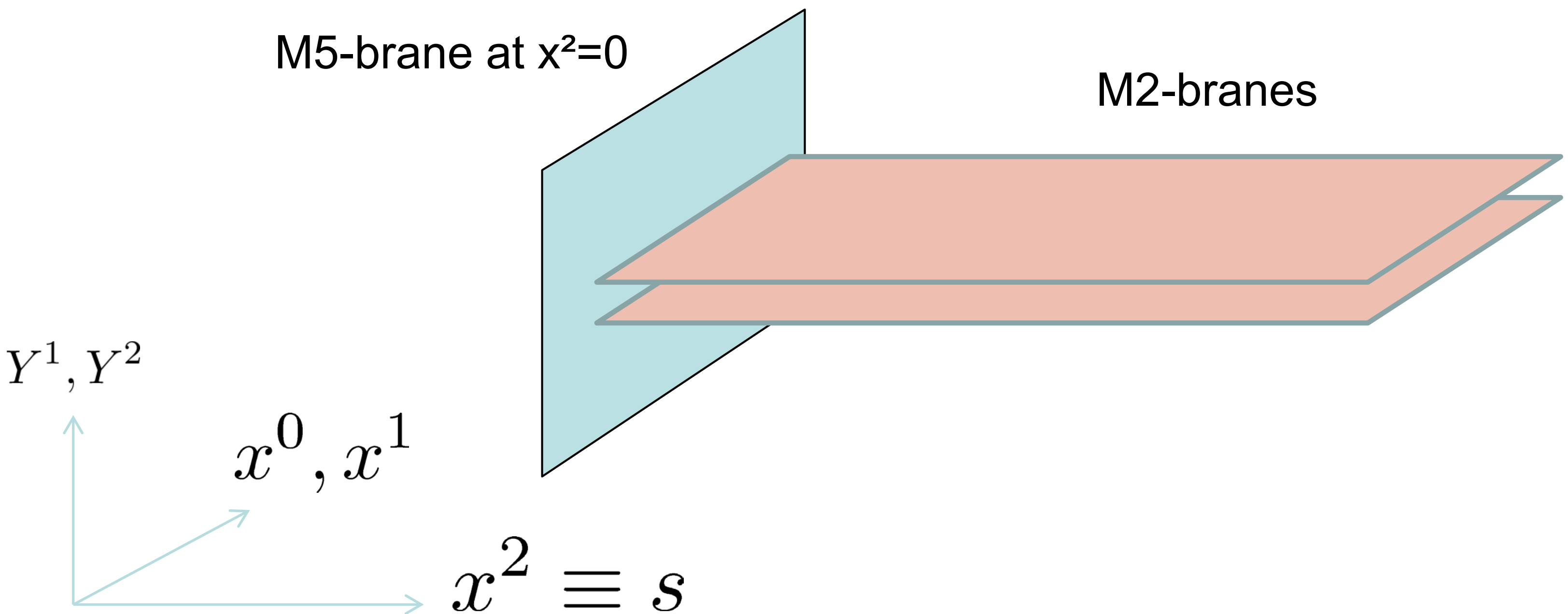
The action is evaluated as

$$S \sim -2 \int d^3x \mathrm{Tr} D_\mu Y_a^\dagger D^\mu Y^a \sim -2 \int d^3x \frac{k}{16\pi(x^2)^3} \mathrm{Tr}(S^a (S^a)^\dagger)$$

$$\sim - \int dx^0 dx^1 \boxed{drr^3} \frac{2\pi}{k}$$

Correct tension of M5-brane!

The basic solution $Y^a = \sqrt{\frac{k}{4\pi x^2}} S^a$




Adding to this, there are some few solutions had been known

## Translational symmetry

If $T$ is a solution of the Nahm equation $\dot{T}^I = i\epsilon_{IJK}T^J T^K$

then, $T^I + \text{const}.1$ is also a solution.

But, even if $Y$ is a solution of the BPS equation

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

$Y^a + \text{const}.1$ is NOT necessary a solution.

Actually, translational symmetry is broken by the orbifolding.
This is an origin of additional difficulty to solve the equation.

# 3-bracket

Bagger and Lambert showed that
ABJM action also has Lie 3-algebra structure defined by

$$[\tilde{Y}_A, \tilde{Y}_B, \tilde{Y}_C] \equiv \tilde{Y}_A \tilde{Y}_B \tilde{Y}_C - \tilde{Y}_C \tilde{Y}_B \tilde{Y}_A \qquad \tilde{Y}_A = \begin{pmatrix} 0 & Y^A \\ Y_A^\dagger & 0 \end{pmatrix}$$

Structure constant: $f^{abcd}$
which satisfy (i) and (ii)

(i) fundamental identities

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d.$$

(ii) NOT total anti-symmetric

$$f^{abcd} \neq f^{[abcd]}$$

The BPS eqution is written as $\frac{d}{ds}\tilde{Y}_A = [\tilde{Y}_B, \tilde{Y}_B, \tilde{Y}_A]$

# The Lax pair

# Lax pair for Nahm equation

$$\dot{T}^I = i\epsilon_{IJK} T^J T^K$$

⇕ equivalent!

Hitchin

The Lax equation $\dot{A} = [A, B]$

$$\begin{cases} A = T^3 + \frac{\lambda}{2}(T^1 - iT^2) - \frac{1}{2\lambda}(T^1 + iT^2) \\ B = \frac{\lambda}{2}(T^1 - iT^2) + \frac{1}{2\lambda}(T^1 + iT^2) \end{cases}$$

$\lambda$ is a arbitrary constant parameter

Because of $\dot{A} = [A, B]$

$\mathrm{Tr} A^m$ are "conserved charge"

These are summarized to

the spectral curve $P(\mu, \lambda) = 0$

defined by

$$P = \det(\eta \mathbf{1}_N - A)$$

We also introduce a star-conjugate:

$$\mathcal{M}^\star(\lambda) := \mathcal{M}(-\bar{\lambda}^{-1})^\dagger$$

Then, we find

$$A^\star = A, \qquad B^\star = -B$$

Now, we will consider the so-called linear problem:

$$A(s;\lambda)\psi(s;\lambda) = \eta(\lambda)\psi(s;\lambda),$$
$$B(s;\lambda)\psi(s;\lambda) = -\dot{\psi}(s;\lambda).$$

We define an N x N matrix and an N x N matrix:

$$\Psi = (\psi_1, \psi_2, \ldots . \psi_N)$$

$$D = \text{diag}(\eta_1, \eta_2, \ldots . \eta_N)$$

Then, the linear problems are written as:

$$A\Psi = \Psi D \qquad B\Psi = -\dot{\Psi}$$

We have assumed there are N linearly independent solutions for $A\psi = \eta\psi$

Then, we can reconstruct A(s;λ) from Ψ(s;λ) and D. Indeed, we find

$$A(s;\lambda) = \Psi(s;\lambda)C(\lambda)\Psi^{\star}(s;\lambda)$$

$$C := D\mathcal{N}^{-1}$$

$$\mathcal{N} := \Psi^{\star}\Psi$$

# The Lax pair for ABJM

Let us consider the BPS equations:

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

$$(a = 1, 2)$$

The symmetry of the BPS equations is

$$Y^a \to Y'^a = e^{i\varphi} \Lambda^a{}_b U Y^b V^\dagger$$

$$U, V \in \mathrm{SU}(N), \quad (\Lambda^a{}_b) \in \mathrm{SU}(2), \quad e^{i\varphi} \in \mathrm{U}(1)$$

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

⇕ equivalent!

Sakai-ST

The Lax equation $\dot{A} = [A, B]$

$$A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix},$$

$$B(s;\lambda) = \begin{pmatrix} \lambda^{-1} Y^1 Y^{2\dagger} + \lambda Y^2 Y^{1\dagger} & O \\ O & \lambda Y^{1\dagger} Y^2 + \lambda^{-1} Y^{2\dagger} Y^1 \end{pmatrix}$$

$\lambda$ is a arbitrary constant parameter

Because of $\dot{A} = [A, B]$

$\mathrm{Tr}A^m$ are "conserved charge"

These are summarized to
the spectral curve $P(\mu, \lambda) = 0$
defined by

$$
\begin{aligned}
P &:= \det(\eta \mathbf{1}_{2N} - A) \\
&= \det\left[\eta^2 \mathbf{1}_N - \left(Y^1 + \lambda Y^2\right)\left(Y^{1\dagger} - \lambda^{-1} Y^{2\dagger}\right)\right] \\
&= \det\left[\eta^2 \mathbf{1}_N - \left(Y^{1\dagger} - \lambda^{-1} Y^{2\dagger}\right)\left(Y^1 + \lambda Y^2\right)\right]
\end{aligned}
$$

$$\mu := \eta^2$$

We introduce a "chirality" matrix

$$\Gamma := \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{pmatrix}$$

Then, we find

$$\{A, \Gamma\} = 0, \qquad [B, \Gamma] = 0$$

$$\begin{cases} A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix} \\ B = \lambda \dfrac{\partial}{\partial \lambda} A^2 \end{cases}$$

We also introduce a star-conjugate:

$$\mathcal{M}^{\star}(\lambda) := \mathcal{M}(-\bar{\lambda}^{-1})^{\dagger}$$

Then, we find

$$A^{\star} = A, \qquad B^{\star} = -B$$

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Now, we will consider the so-called linear problem:

$$A(s;\lambda)\psi(s;\lambda) = \eta(\lambda)\psi(s;\lambda),$$
$$B(s;\lambda)\psi(s;\lambda) = -\dot{\psi}(s;\lambda).$$

If $\psi$ is an eigen vector with eigen value $\eta$

then $\Gamma\psi$ is an eigen vector with eigen value $-\eta$

Then, we will take

$$\psi_{N+m} = \Gamma\psi_m, \qquad \eta_{N+m} = -\eta_m, \qquad m = 1, \ldots, N$$

We define an N x 2N matrix and an 2N x 2N matrix:

$$\Psi := (\psi_1, \ldots, \psi_{2N}) = (\psi_1, \ldots, \psi_N, \Gamma\psi_1, \ldots, \Gamma\psi_N),$$

$$D := \mathrm{diag}(\eta_1, \ldots, \eta_{2N}) = \mathrm{diag}(\eta_1, \ldots, \eta_N, -\eta_1, \ldots, -\eta_N)$$

Then, the linear problems are written as:

$$A\Psi = \Psi D \qquad B\Psi = -\dot{\Psi}$$

We have assumed there are 2N linearly independent solutions for $A\psi = \eta\psi$

Then, we can reconstruct A(s;λ) from Ψ(s;λ) and D. Indeed, we find

$$A(s;\lambda) = \Psi(s;\lambda)C(\lambda)\Psi^{\star}(s;\lambda)$$

$$C := D\mathcal{N}^{-1}$$
$$\mathcal{N} := \Psi^{\star}\Psi$$

# The relation to two Nahm equations

Each of these 3 matrices

$$T_1^I := (\sigma^I)_{ab} Y^a Y^{b\dagger}$$

$$T_2^I := (\sigma^I)_{ab} Y^{b\dagger} Y^a$$

satisfy the Nahm equation $\dot{T}^I = i\epsilon_{IJK} T^J T^K$

Nosaka-ST

(We do not know why two Nahm eq. appears.)

Nahm equation $\dot{T}^I = i\epsilon_{IJK} T^J T^K$

also has a Lax representation

$$\dot{A}_\alpha = [A_\alpha, B_\alpha]$$

$$A_\alpha := T^3_\alpha + \frac{\lambda}{2}\left(T^1_\alpha - iT^2_\alpha\right) - \frac{1}{2\lambda}\left(T^1_\alpha + iT^2_\alpha\right)$$
$$B_\alpha := \frac{\lambda}{2}\left(T^1_\alpha - iT^2_\alpha\right) + \frac{1}{2\lambda}\left(T^1_\alpha + iT^2_\alpha\right).$$

relation to Lax pair for the BPS equations in ABJM?

Indeed, a simple relation:

$$A^2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

similar to "Dirac equation"!

## These relations means

$$P = \det(\mu \mathbf{1}_N - A_1) = \det(\mu \mathbf{1}_N - A_2)$$

## The linear problems for the Nahm equations:

$$A_\alpha \Psi_\alpha = \Psi_\alpha M$$

$$B_\alpha \Psi_\alpha = -\dot{\Psi}_\alpha$$

$$M = \mathrm{diag}(\mu_1, \ldots, \mu_N)$$

# Now consider the original BPS eq. in ABJM and the linear problem $A\Psi = \Psi D$

If we express $$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_1 & \Psi_1 \\ \Psi_2 & -\Psi_2 \end{pmatrix}$$

$\Psi_1, \Psi_2$ are the eigenvectors of A for the corresponding Nahm equations

$$D = \begin{pmatrix} H & O \\ O & -H \end{pmatrix}, \qquad H = \mathrm{diag}(\eta_1, \ldots, \eta_N)$$

$$H^2 = M$$

# Next, we assume eigenvectors for Nahm data are given:

$$\Psi_1, \Psi_2, M$$

⇓

$$D = \begin{pmatrix} H & O \\ O & -H \end{pmatrix}, \qquad H = \mathrm{diag}(\eta_1, \ldots, \eta_N)$$

$$H^2 = M$$

# A candidate for Ψ should be

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_1 & \Psi_1 \\ \Psi_2 & -\Psi_2 \end{pmatrix}$$

## For this Ψ, we can reconstruct A as

$$A = \begin{pmatrix} O & \Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star \\ \Psi_2 H \mathcal{N}_1^{-1} \Psi_1^\star & O \end{pmatrix}$$

**Comparing with** $A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix}$

Cond.1 $$\frac{\partial^2}{\partial \lambda^2} \left[ \Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star \right] = 0$$

Cond.2 $$H \mathcal{N}_1 = \mathcal{N}_2 H \quad \Longleftarrow \quad A^\star = A$$

## With these conditions, this A gives the solutions of the BPS equations!

# The solutions for two M2-branes

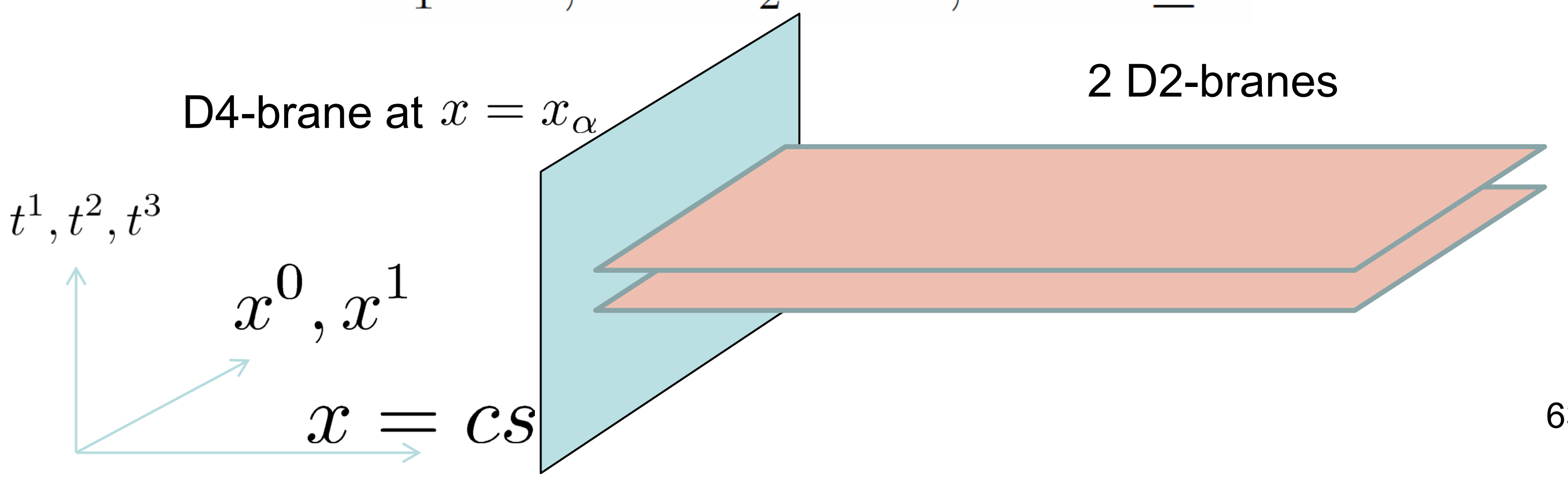
# General solutions of the Nahm equations for N=2 case (with an M5-brane)

$$T^1_\alpha = \frac{c}{\sinh(x - x_\alpha)}\frac{\sigma^1}{2} + t^1 \mathbf{1}_2, \qquad T^2_\alpha = \frac{c}{\sinh(x - x_\alpha)}\frac{\sigma^2}{2} + t^2 \mathbf{1}_2$$

$$T^3_\alpha = \frac{c}{\tanh(x - x_\alpha)}\frac{\sigma^3}{2} + t^3 \mathbf{1}_2.$$

$$x = cs, \qquad c \geq 0$$

$$x_1 = 0, \qquad x_2 = -l, \qquad l \geq 0$$



# For this, the Lax pair is written as

$$A_\alpha = \left(\tanh\frac{x - x_\alpha}{2}\right)^{-\rho^1} M \left(\tanh\frac{x - x_\alpha}{2}\right)^{\rho^1}$$

$$\begin{cases} M = \left(\frac{c}{2}\sigma^3 + t_\lambda \mathbf{1}_2\right) \\ \rho^1 = \frac{\lambda + \lambda^{-1}}{4}\sigma^1 + \frac{\lambda - \lambda^{-1}}{4i}\sigma^2 = \frac{1}{2}\begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix} \\ t_\lambda = t^3 + \frac{\lambda}{2}(t^1 - it^2) - \frac{1}{2\lambda}(t^1 + it^2) \end{cases}$$

$$\Rightarrow \quad \Psi_\alpha = \left(\tanh\frac{x - x_\alpha}{2}\right)^{-\rho^1} D_\alpha$$

$$D_\alpha^\star D_\alpha = \mathbf{1}_2$$

# We can compute the candidate of A as

$$\Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star = \left(\tanh\frac{x-x_1}{2}\right)^{-\rho^1} D_1 M^{1/2} D_2^\star \left(\tanh\frac{x-x_2}{2}\right)^{\rho^1}$$

$$\left(\tanh\frac{x}{2}\right)^{\pm\rho^1} = \frac{1}{\sqrt{2\sinh x}} \begin{pmatrix} e^{x/2} & \mp\lambda^{-1}e^{-x/2} \\ \mp\lambda e^{-x/2} & e^{x/2} \end{pmatrix}$$

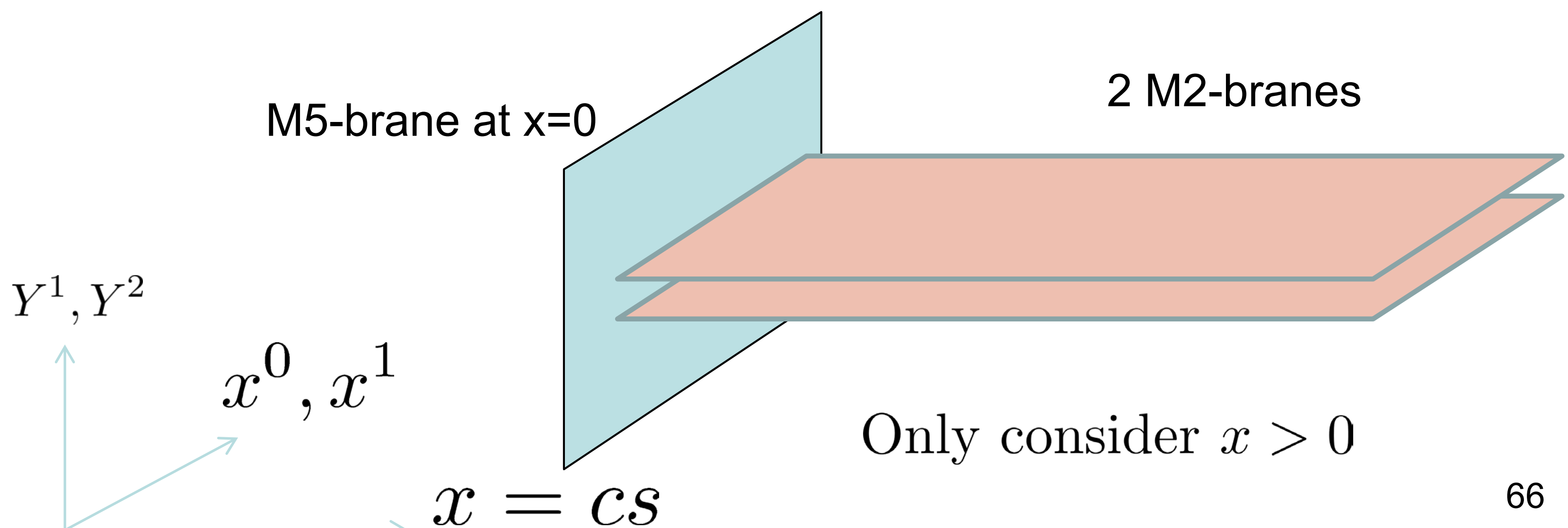
# Then, we require the conditions 1,2

$$\frac{\partial^2}{\partial\lambda^2}\left[\Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star\right] = 0$$

$$A^\star = A$$

# Finally, we find general solution for N=2

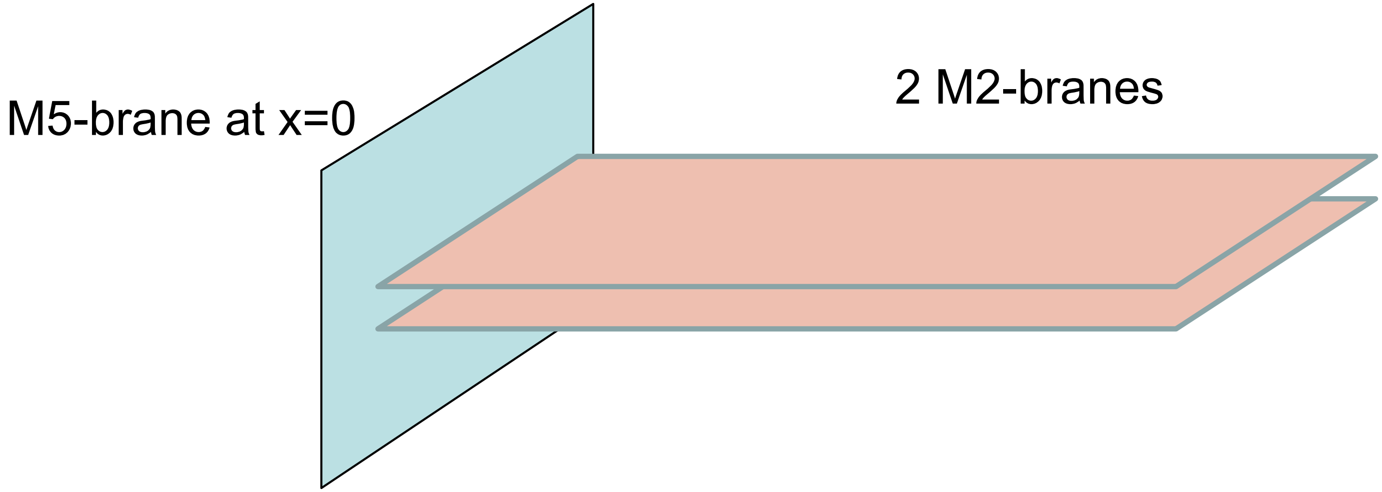
$$
\begin{cases}
Y^1 = \sqrt{\dfrac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh(x+l)\cos\frac{\theta}{2}e^{i\phi} & \sinh l \sin\frac{\theta}{2} \\ 0 & \sinh x \cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \\
Y^2 = \sqrt{\dfrac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh x \sin\frac{\theta}{2} & 0 \\ \sinh l \cos\frac{\theta}{2}e^{i\phi} & \sinh(x+l)\sin\frac{\theta}{2} \end{pmatrix}
\end{cases}
$$



$$\begin{cases} Y^1 = \sqrt{\dfrac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh(x+l)\cos\frac{\theta}{2}e^{i\phi} & \sinh l \sin\frac{\theta}{2} \\ 0 & \sinh x \cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \\ Y^2 = \sqrt{\dfrac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh x \sin\frac{\theta}{2} & 0 \\ \sinh l \cos\frac{\theta}{2}e^{i\phi} & \sinh(x+l)\sin\frac{\theta}{2} \end{pmatrix} \end{cases}$$

$$x \to 0 \qquad Y^a \to \text{basic solution}$$

$$x \to \infty \quad \begin{cases} Y^1 \to \sqrt{\dfrac{c}{2\sinh l}} \cos\dfrac{\theta}{2} e^{i\phi} \ \mathrm{diag}(e^{l/2}, 1) \\ Y^2 \to \sqrt{\dfrac{c}{2\sinh l}} \sin\dfrac{\theta}{2} \ \mathrm{diag}(1, e^{l/2}) \end{cases}$$



# We can rewrite the solution as

$$Y^1 = \frac{1}{2}\left(f_1 \sin\frac{\theta}{2}\,\sigma^1 + f_2 \sin\frac{\theta}{2}\,i\sigma^2 + f_3 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^3 - f_0 e^{i\phi}\cos\frac{\theta}{2}\,\mathbf{1}_2\right)$$

$$Y^2 = \frac{1}{2}\left(f_1 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^1 - f_2 e^{i\phi}\cos\frac{\theta}{2}\,i\sigma^2 - f_3 \sin\frac{\theta}{2}\,\sigma^3 - f_0 \sin\frac{\theta}{2}\,\mathbf{1}_2\right)$$

$$f_1 = f_2 = \sqrt{\frac{c\sinh l}{2\sinh x\sinh(x+l)}}, \qquad f_3 = \frac{\cosh(x+l/2)}{\cosh(l/2)}f_1, \qquad f_0 = -\frac{\sinh(x+l/2)}{\sinh(l/2)}f_1$$

## We can show that

$$Y^1 = \frac{1}{2}\left( f_1 \sin\frac{\theta}{2}\,\sigma^1 + f_2 \sin\frac{\theta}{2}\, i\sigma^2 + f_3 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^3 - f_0 e^{i\phi}\cos\frac{\theta}{2}\,\mathbf{1}_2 \right)$$
$$Y^2 = \frac{1}{2}\left( f_1 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^1 - f_2 e^{i\phi}\cos\frac{\theta}{2}\, i\sigma^2 - f_3 \sin\frac{\theta}{2}\,\sigma^3 - f_0 \sin\frac{\theta}{2}\,\mathbf{1}_2 \right)$$

**with**

$$\dot{f}_i = f_j f_k f_l \qquad \text{where } \epsilon^{ijkl} \neq 0$$

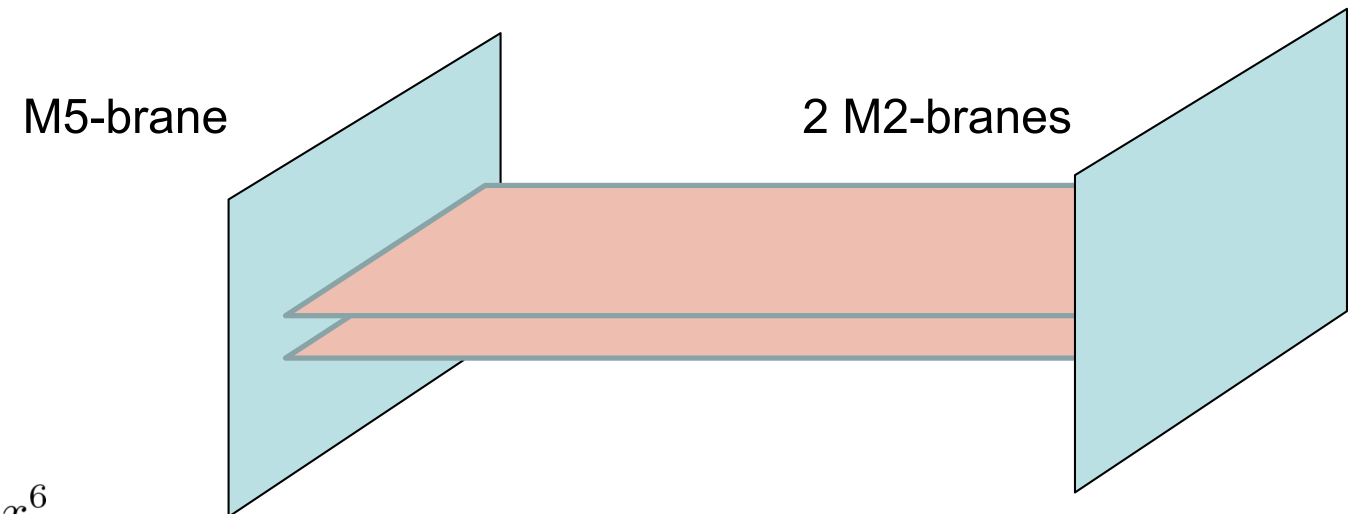
**are the solutions of the BPS equations.**

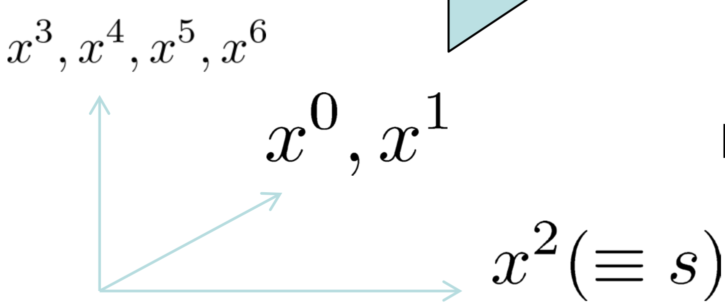
$$\Longrightarrow \quad f_I^2 - f_0^2 \text{ are constants}$$

Nosaka-ST

**It reduces to a first-order differential equation, and can be solved by elliptic integral.**

# These include
# all solutions for two M2-branes:




**represented by Elliptic functions**

# explicit form 1

$$f_i = \frac{\vartheta_{i+1}(u)}{\vartheta_{i+1}(u_*)} \sqrt{\frac{\pi}{2\omega_1} \frac{\vartheta_1(u_*)\vartheta_2(u_*)\vartheta_3(u_*)\vartheta_4(u_*)}{\vartheta_1(u_* + u)\vartheta_1(u_* - u)}}$$

**where**

$\vartheta_i(u) := \vartheta_i(u, \tau)$ are Jacobi theta functions

$$u = \frac{s - s_0}{2\omega_1} \qquad -u_* < u < u_*$$

$$s_0 \in \mathbb{R}, \qquad 0 < u_* < \frac{1}{2}, \qquad \omega_1 \in \mathbb{R}_{>0}, \qquad \tau \in i\mathbb{R}_{>0}$$

**4 parameters**

They diverges at $u = \pm u_*$

# explicit form 2

$$f_0 = \left( \frac{\wp_1(s_*)\wp_2(s_*)\wp_3(s_*)}{\wp(s - s_0) - \wp(s_*)} \right)^{1/2}, \qquad f_I = \frac{\wp_I(s - s_0)}{\wp_I(s_*)} f_0 \quad (I = 1, 2, 3)$$

where

$$\left\{ \begin{aligned} & s_* = 2\omega_1 u_*, \qquad 0 < s_* < \omega_1 \\ & f_I^2 - f_0^2 = \frac{\pi \vartheta_{I+1}^2}{2\omega_1} \frac{\vartheta_{J+1}(u_*)\vartheta_{K+1}(u_*)}{\vartheta_1(u_*)\vartheta_{I+1}(u_*)} \\ & \qquad = \frac{\wp_J(s_*)\wp_K(s_*)}{\wp_I(s_*)} \\ & \qquad =: a_I^2 \qquad (a_I > 0), \end{aligned} \right.$$

# explicit form 3

$$f_0 = \frac{a_3 \operatorname{sn} x}{\sqrt{\operatorname{sn}^2 x_* - \operatorname{sn}^2 x}}, \qquad f_1 = \frac{a_1 \operatorname{sn} x_* \operatorname{cn} x}{\sqrt{\operatorname{sn}^2 x_* - \operatorname{sn}^2 x}}$$

$$f_2 = \frac{a_2 \operatorname{sn} x_* \operatorname{dn} x}{\sqrt{\operatorname{sn}^2 x_* - \operatorname{sn}^2 x}}, \qquad f_3 = \frac{a_3 \operatorname{sn} x_*}{\sqrt{\operatorname{sn}^2 x_* - \operatorname{sn}^2 x}}$$

**where** $x = c(s - s_0), \qquad c = a_2\sqrt{a_1^2 - a_3^2}, \qquad \operatorname{sn} x_* = \sqrt{1 - \frac{a_3^2}{a_1^2}}$

# Corresponding Nahm data:

$$T^I_\alpha = \wp_I\left(s - s_\alpha\right)\frac{\sigma^I}{2} + \frac{n_I}{4}\left(a_I^2 - a_J^2 - a_K^2\right)\mathbf{1}_2$$

**where** $s_1 = s_0 - s_*, \qquad s_2 = s_0 + s_*$

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**or**

$$\begin{cases} \wp_1(\tilde{s}) = c\dfrac{\operatorname{cn} x}{\operatorname{sn} x}, \qquad \wp_2(\tilde{s}) = c\dfrac{\operatorname{dn} x}{\operatorname{sn} x}, \qquad \wp_3(\tilde{s}) = c\dfrac{1}{\operatorname{sn} x} \\ x = c\tilde{s} \end{cases}$$

**Two monopoles shifted in s direction.**

# Conclusion

- **New BPS equation for M5-branes in ABJM**
- **Lax representation**
- **Dirac equation like structure**
- **New solutions**
- **All solutions for 2 M2-branes**

# Future works

- **Little research has been done so far, so there remains much to be done!**
- **3-algebra structure**
- **Moduli space metric, etc**
- **reduction to Toda chain**

Fin.