

Solvability & nilpotency of Kac-  
Paljutkin's finite quantum group  
& Sekine quantum groups.

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§1. Introduction.

§2. OA & FQGs

§3. Solvability & Illegality

( §4. Appendix A. )

# §1. Introduction

• Finite Quantum Groups (FQGs for short)

is something like finite groups.

•  $\{ \text{FQGs} \} \not\supseteq \{ \text{finite groups} \}$

↑

•  $\exists$  nilpotency & solvability

• nilp.  $\Rightarrow$  solv.

• Burnside's  $p^a q^b$  theorem.

# Question

$\exists$ ? Solv. & nilp. for FQGs

## History of this challenge

- Etingof, Nikshych & Ostrik (2009)

— A def of solv. & nilp. for

"Fusion Categories" ( $\equiv$  representation cats of FQGs)

— nilp.  $\not\Rightarrow$  solv. in general. <sup>3/23</sup>

(Rem 4.6. in their paper)

• Cohen-Weserich (2016)

— A (new) def of solv. & nilp. for

FQGs.

— nilp.  $\Rightarrow$  solv.

— Burnside's  $p^a q^b$  theorem.

— However

{ Solv. or nlp. FQGs in CW } \ { FGs } = \emptyset

• Glowacki, Hattori & T. (2024)

—  $\exists$  No examples of solv. & nlp.

FQGs which does not arise

from FGs.

operator alg

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## §2. OA & FQGs

Thm (Gelfand-Naimark)

We don't define this today

Any commutative unital " $\mathbb{C}^*$ -alg" is

of the form  $\boxed{\mathbb{C}(G)}$  for some cpct

Haus sp  $G$ .

the alg of cts fns  
from  $G$  to  $\mathbb{C}$ .

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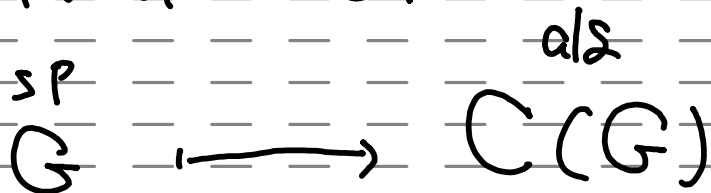
# Thm' (Gelfand-Naimark)

the Cat of cpct Haus sps & cts fns

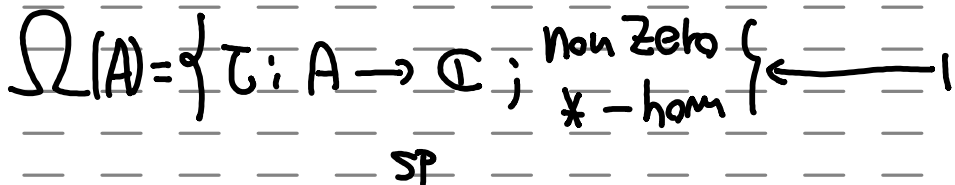
and

the Cat of commu. unital  $C^*$ -algs & unital  $*$ -homs

are equivalent.



the correspondence



$A$   
algs



Philosophy ("Def") via G-N thm.  $\frac{7}{23}$

• A cpct grp  $\equiv$  A commu. unital  $C^*$ -alg  
with some axioms

• A cpct quantum group (CQG)  
 $\equiv$  A unital  $C^*$ -alg  
with some axioms

Def (FQG) finite quantum steps 8/23

A FQG is a f.d. CQG.

More precisely, a FQG is a

f.d.  $C^*$ -alg  $A$  with a  $*$ -hom  $\Delta: A \rightarrow A \otimes A$ , a  $*$ -hom  $\varepsilon: A \rightarrow \mathbb{C}$  (unital)

& an anti-hom  $S: A \rightarrow A$

Satisfying these axioms

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$$\begin{array}{ccc}
 A \xrightarrow{\Delta} A \otimes A & & A \otimes A \xleftarrow{\Delta} A \xrightarrow{\Delta} A \otimes A \\
 \Delta \downarrow \circlearrowright & & \varepsilon \otimes \text{id} \downarrow \circlearrowright \quad \text{id} \otimes \varepsilon \downarrow \circlearrowright \\
 A \otimes A \xrightarrow{\Delta \otimes \text{id}} A \otimes A \otimes A & & \mathbb{C} \otimes A \xrightarrow{\cong} A \xleftarrow{\cong} A \otimes \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 A \otimes A \xleftarrow{\Delta} A \xrightarrow{\Delta} A \otimes A \\
 \text{id} \otimes \varepsilon \downarrow \circlearrowright & & \varepsilon \otimes \text{id} \downarrow \circlearrowright \\
 A \otimes A \xrightarrow{\mu} A \xleftarrow{\mu} A \otimes A
 \end{array}$$

EX1.

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$G$ : a fin. grp

$$A := \mathbb{C}[G]_{\text{grp-alg}} = \left\{ \sum_{g \in G} a_g \lambda_g \mid a_g \in \mathbb{C} \right\}$$

$$\Delta(\lambda_g) = \lambda_g \otimes \lambda_g \quad \leftarrow \text{cocommutative.}$$

$$\varepsilon(\lambda_g) = 1$$

i.e.  $\Delta = \Delta^{\text{op}}$

$$:= (x \otimes y \mapsto y \otimes x) \circ \Delta$$

$$S(\lambda_g) = \lambda_{g^{-1}}$$

$$\leadsto A : \text{FQG}$$

Ex 2.

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$G$ : as above

$A := C(G)$  ← commu.

$\Delta: A \rightarrow A \otimes A \cong C(G \times G)$

$f \mapsto \Delta(f) (x, y) \mapsto f(xy)$

$\varepsilon: A \rightarrow \mathbb{C} ; f \mapsto f(1)$

$\eta: A \rightarrow A ; f \mapsto f(\bullet^{-1})$

$\leadsto A: FQG$ .

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## Facts

$$\Delta = \Delta^{op}$$

• Any cocommu. FQG is of the form  $\mathbb{C}[G]$ .

• Any commu. FQG is of the form  $\mathbb{C}(G)$ .

$\leadsto \exists ? NC \& NCC \text{ FQGs}$

Ex 3. (Kac-Paljutkin, 1966)

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$$A = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{M}_2$$

$(e_i^2 = e_i = e_i^*)$

Non commu.

$$\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4$$

$$+ \frac{1}{2} a_{11} \otimes a_{11} + \frac{1}{2} a_{12} \otimes a_{12} + \frac{1}{2} a_{21} \otimes a_{21} + \frac{1}{2} a_{22} \otimes a_{22}$$

$$\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3$$

$$+ \frac{1}{2} a_{11} \otimes a_{22} + \frac{1}{2} a_{22} \otimes a_{11} - \frac{\sqrt{-1}}{2} a_{12} \otimes a_{21} + \frac{\sqrt{-1}}{2} a_{21} \otimes a_{12}$$

and more.

Non cocommu.

# Ex4. (Serre, 1996)

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$k \geq 3$  int

$$A_k = \underbrace{\mathbb{C}d_{0,0} \oplus \dots \oplus \mathbb{C}d_{k-1,k-1}}_{k^2 \text{ terms}} \oplus M_k$$

NC

$$(d_{ij}^2 = d_{ij} = d_{ij}^*)$$

$k^2$  terms

Non commu.

$\eta := \exp(2\pi i/k)$   $k$ -th root of unity

$$\Delta(d_{ij}) = \sum_{m,n \in \mathbb{Z}_k} d_{m,n} \otimes d_{i-m, j-n} + \frac{1}{k} \sum_{m,n \in \mathbb{Z}_k} \eta^{i(m-n)} e_{m,n} \otimes e_{m+j, n+j}$$

$$\Delta(e_{ij}) = \sum_{m,n \in \mathbb{Z}_k} \eta^{m(i-j)} d_{-m, -n} \otimes e_{i-n, j-n} + \sum_{m,n \in \mathbb{Z}_k} \eta^{m(j-i)} e_{i-n, j-n} \otimes d_{m,n}$$

Mat units



## §3. Solv. & nilp.

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- Review A grp  $G$  is called solvable (nilpotent)

if  $\exists$  a chain of subgrps

$$\{e\} = H_0 \subsetneq H_1 \subsetneq \dots \subsetneq H_t = G \quad \bullet$$

s.t. ...



$$\mathbb{C} \subsetneq \mathbb{C}[H_1] \subsetneq \dots \subsetneq \mathbb{C}[G]$$

"Def" (Cohen-Westreich, Def 3.5, Prop 3.8) 16/23

A FQG  $A$  is solvable if

$\exists$  a chain of subalgs (ideals)

$$\mathbb{Q} = L_0 \subsetneq L_1 \subsetneq \dots \subsetneq L_t = A.$$

s.t.  $\dots$

In a similar way, we define niip.

Thm. (Cohen-Westreich)

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A cocommu. FQG (=  $\mathbb{C}[G]$ ) is nilP  
(resp. solv.) iff  $G$  is nilP. (resp. solv.)

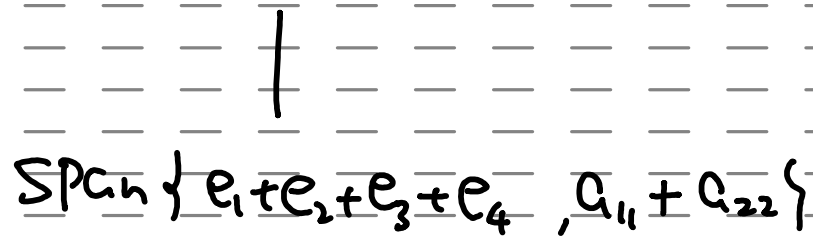
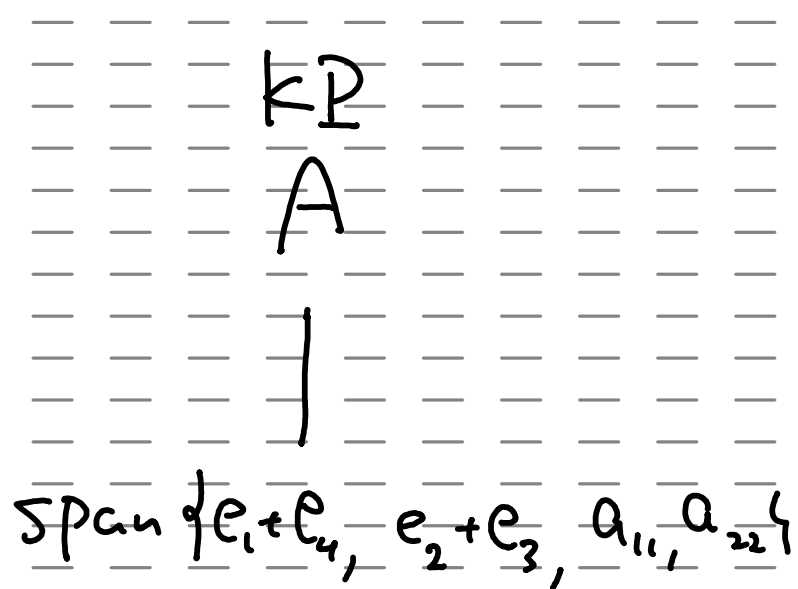
Thm. (Cohen-Westreich)

nilP.  $\Rightarrow$  Solv.

Thm. (Glowacki-Hattori-T.)

KP & Serines are nilP.  
(and hence solv.)

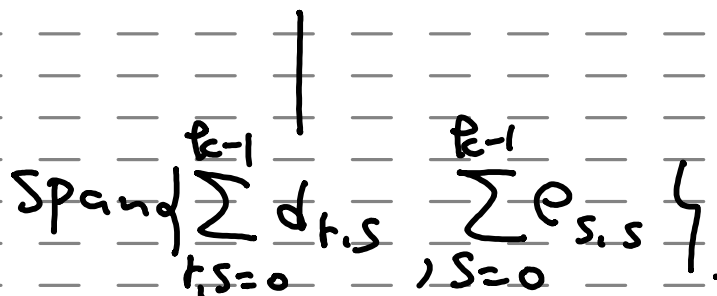
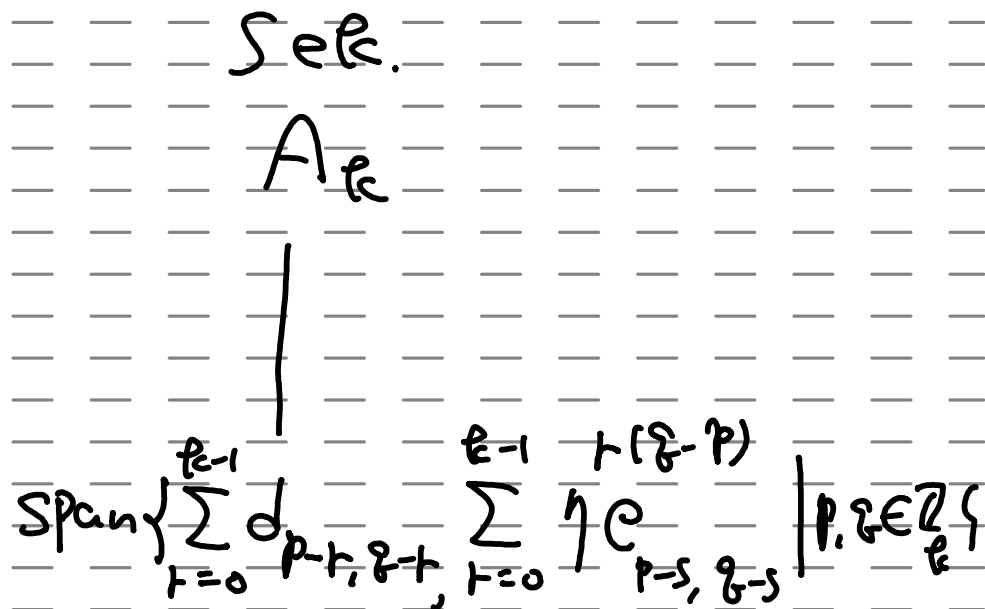
"pf" ~~is~~ difficult things. All we 18/23  
 have to do is to compute & to show that



|

$\mathbb{C}$

are the chains we want.



# § 4. Appendix A

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In our paper, we computed the univ. R-  
mat's of KP.

Sols of QrBE.

$R \in A \otimes A$  inv. s.t.

$$\left\{ \begin{array}{l} (\text{id} \otimes \Delta)(R) = R_{13} R_{12} \\ (\Delta \otimes \text{id})(R) = R_{13} R_{23} \\ R \Delta(\cdot) R^{-1} = \Delta^{\text{op}} := (x \otimes y \mapsto y \otimes x) \circ \Delta \end{array} \right. \quad \left( \begin{array}{l} \text{X1} \\ (a \otimes b)_{12} = a \otimes b \otimes 1 \\ (a \otimes b)_{13} = a \otimes 1 \otimes b \end{array} \right)$$

Strategy

~~A~~ difficult points.

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$$\text{Put } R = \sum A_{ij} e_i \otimes e_j + \sum B_{ijk} e_i \otimes a_{jk}$$

$$+ \sum C_{ijk} a_{ij} \otimes e_k + \sum D_{ijke} a_{ij} \otimes a_{ke}.$$

And compute.

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$$\leadsto R = \sum A_{ij} e_i \otimes e_j$$

$$+ B_1 e_1 \otimes a_{11} + B_1 e_1 \otimes a_{22} + B_2 e_2 \otimes a_{11} - B_2 e_2 \otimes a_{22}$$

$$+ B_3 e_3 \otimes a_{11} - B_3 e_3 \otimes a_{22} + B_4 e_4 \otimes a_{11} + B_4 e_4 \otimes a_{22}$$

$$+ C_1 a_{11} \otimes e_1 + C_1 a_{22} \otimes e_1 + C_2 a_{11} \otimes e_2 - C_2 a_{22} \otimes e_2$$

$$+ C_3 a_{11} \otimes e_3 - C_3 a_{22} \otimes e_3 + C_4 a_{11} \otimes e_4 + C_4 a_{22} \otimes e_4$$

$$+ \bigoplus_{1111} a_{11} \otimes a_{11} + \bigoplus_{1122} a_{11} \otimes a_{22} + \bigoplus_{1212} a_{12} \otimes a_{12}$$

$$+ \bigoplus_{1221} a_{12} \otimes a_{21} + \bigoplus_{1221} a_{21} \otimes a_{12} + \bigoplus_{1212} a_{21} \otimes a_{21}$$

$$- \bigoplus_{1122} a_{22} \otimes a_{11} + \bigoplus_{1111} a_{22} \otimes a_{22}$$

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Thm. (Głowacki-Hattori-T., Suzuki, Watui).

Our coeff's are the following:

$$[A_{ij}]_{ij} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

• By hand  
→ Used 2~3 months

• Mathematica

→ we need 5 sec.

$$1) \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ + \\ - \\ 1 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ + \\ - \\ + \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} \pm 1 \\ \pm 1 \\ 0 \\ 0 \end{bmatrix}$$

$$2) \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} \pm 1 \\ \mp 1 \\ 0 \\ 0 \end{bmatrix}$$





$$3) \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ + \\ - \\ + \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ - \\ + \\ - \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda \\ -\sqrt{-1}\lambda \end{bmatrix} \quad (\lambda^2 = \sqrt{-1}) \quad 23/23$$

$$4) \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ - \\ + \\ - \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ + \\ - \\ - \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda \\ \sqrt{-1}\lambda \end{bmatrix} \quad (\lambda^2 = \sqrt{-1})$$

Mahedon  $R$  is unitary ( $R^*R = 1 \otimes 1 = RR^*$ )

&  $R$  in 3), 4) are minimal in the sense of Radford.

## References. (Please see refs therein also)

1. M. Cohen, S. Westreich, Solvability for semisimple Hopf algebras via integrals, *J. Algebra* Vol 472, 67-94 (2017)

2. P. Etingof, D. Nikshych, V. Ostrik, Weakly group-theoretical and solvable fusion categories, *Adv. Math.* Vol 226(1), 176-205 (2011)

3. G. Glowacki, M. Hattori, M. Tanaka, Examples of solvable and nilpotent finite quantum

groups, preprint, arXiv 2402.001706 (2024)

Thank you for your attention  
and your precious time in  
your life.

M. Tanaka