Solvability & nilpotency of Kac–Paljutkin's finite quantum group & Sekine quantum groups.

Masato Tanaka, D1

Nagoya University.

§1. Introduction.

§2. QA & FQGs

§3. Solvability & nilpotency

(§4. Appendix A.)

§1. Introduction

- **Finite Quantum Groups** (FQGs for short) is something like finite groups.

- $\{\text{FQGs}\} \supsetneq \{\text{finite groups}\}$
  - $\exists$ nilpotency & solvability
  - nilp. $\Rightarrow$ solv.
  - Burnside's $p^a q^b$ theorem.

Question

$\exists^{?}$ Solv. & nilp. for FQGs

History of this challenge

- Etingof, Nikshych & Ostrik (2009)

— A def of solv. & nilp. for "Fusion Categories" ($\doteqdot$ representation Cats of FQGs)

— nilp. $\not\Rightarrow$ solv. in general.

(Rem 4.6. in their paper)

- Cohen-Westreich (2016)

— A (new) def of solv. & nilp. for FQGs.

— nilp. $\Longrightarrow$ solv.

— Burnside's $p^a q^b$ theorem.

— However

$\{\text{Solv. or nilp. FQGs in CW}\} \setminus \{\text{FGs}\} = \emptyset$.

- Glowacki, Hattori & T. (2024)

— $\exists^{\aleph_0}$ examples of solv. & nilp.

FQGs which does not arise

from FGs.

operator alg

§2. OA & FQGs

5/23

We don't define this today

Thm (Gelfand–Naimark)

Any commutative unital "C^*-alg" is of the form $\boxed{C(G)}$ for some cpct Haus sp G.

The alg of cts fcns from G to $\mathbb{C}$.

Thm' (Gelfand-Naimark)

the Cat of cpct Haus sps & cts fcns

and

the Cat of commu. unital C^*-algs & unital $*$-homs

are equivalent.

$$\overset{\text{sp}}{G} \longmapsto \overset{\text{alg}}{C(G)}$$

$$\underset{\text{sp}}{\Omega(A)=\{\tau : A \to \mathbb{C} \,;\, \text{non zero } *\text{-hom}\}} \longleftarrow\!\shortmid \underset{\text{alg}}{A}$$

the correspondence

Philosphy ("Def") via G-N thm.

- A cpct grp = A commu. unital C^*-alg with some axioms

- A cpct quantum group (<u>CQG</u>) = A unital C^*-alg with some axioms

Def (FQGs) finite quantum grps 8/23

A **FQG** is a f.d. CQG.

More precisely, a **FQG** is a f.d. C^*-alg A with a (unital) $*$-hom $\Delta : A \to A \otimes A$, a (unital) $*$-hom $\varepsilon : A \to \mathbb{C}$ & an anti-hom $S : A \to A$

satisfying these axioms

$$
\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "id \otimes \Delta"] \\
A \otimes A \arrow[r, "\Delta \otimes id"'] & A \otimes A \otimes A
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
A \otimes A \arrow[d, "\varepsilon \otimes id"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "id"] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "id \otimes \varepsilon"] \\
\mathbb{C} \otimes A \arrow[r, "\sim"] & A & A \otimes \mathbb{C} \arrow[l, "\sim"']
\end{tikzcd}
$$

&

$$
\begin{tikzcd}
A \otimes A \arrow[d, "id \otimes S"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\varepsilon(\cdot)1"] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "S \otimes id"] \\
A \otimes A \arrow[r, "\mu"'] & A & A \otimes A \arrow[l, "\mu"] \\
 & xy & x \otimes y \arrow[l, mapsto]
\end{tikzcd}
$$

.

10/23

EX1.

G : a fin. grp

$$A := \underset{\text{grp-alg}}{\mathbb{C}[G]} = \left\{ \sum_{g \in G} a_g \lambda_g \;\middle|\; a_g \in \mathbb{C} \right\}$$

$\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ ⟵ cocommutative

i.e. $\Delta = \Delta^{op}$

$:= (x \otimes y \mapsto y \otimes x) \circ \Delta$

$\varepsilon(\lambda_g) = 1$

$S(\lambda_g) = \lambda_{g^{-1}}$

$\rightsquigarrow A$: FQG

Ex2.

11/23

G : as above

$A := C(G) \leftarrow$ commu.

$$\Delta : A \longrightarrow A \otimes A \cong C(G \times G)$$

$$f \longmapsto \Delta(f) \, (: (x,y) \longmapsto f(xy))$$

$$\varepsilon : A \longrightarrow \mathbb{C} \ ; \ f \longmapsto f(1)$$

$$S : A \longrightarrow A \ ; \ f \longmapsto f(\bullet^{-1})$$

$\rightsquigarrow A$: FQG.

12/23

Facts

$\Delta = \Delta^{op}$

- Any cocommu. FQG is of the form $\mathbb{C}[G]$.
- Any commu. FQG is of the form $C(G)$.

$\rightsquigarrow \overset{?}{\exists}$ NC & NCC FQGs

Ex 3. (Kac-Paljutkin, 1966)

$$A = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus M_2$$ ← Non commu.

$$(e_i^2 = e_i = e_i^*)$$

$$\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4$$
$$+ \frac{1}{2} a_{11} \otimes a_{11} + \frac{1}{2} a_{12} \otimes a_{12} + \frac{1}{2} a_{21} \otimes a_{21} + \frac{1}{2} a_{22} \otimes a_{22}$$

$$\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3$$
$$+ \frac{1}{2} a_{11} \otimes a_{22} + \frac{1}{2} a_{22} \otimes a_{11} - \frac{\sqrt{-1}}{2} a_{12} \otimes a_{21} + \frac{\sqrt{-1}}{2} a_{21} \otimes a_{12}$$

← Non cocommu.

and more.

Ex4. (Sekine, 1996)

$k \geq 3$ int

$$A_k = \underbrace{\mathbb{C}d_{0,0} \oplus \cdots \oplus \mathbb{C}d_{k-1,k-1}}_{k^2 \text{ terms}} \oplus M_k$$

NC

$(d_{ij}^2 = d_{ij} = d_{ij}^*)$

$\eta := \exp(2\pi\sqrt{-1}/k)$ k-th root of unity

Non cocommu.

$$\Delta(d_{i,j}) = \sum_{m,n \in \mathbb{Z}_k} d_{m,n} \otimes d_{i-m,j-n} + \frac{1}{k} \sum_{m,n \in \mathbb{Z}_k} \eta^{i(m-n)} e_{mn} \otimes e_{m+j,n+j}$$

$$\Delta(e_{ij}) = \sum_{m,n \in \mathbb{Z}_k} \eta^{m(i-j)} d_{-m,-n} \otimes e_{i-n,j-n} + \sum_{m,n \in \mathbb{Z}_k} \eta^{m(j-i)} e_{i-n,j-n} \otimes d_{mn}$$

Mat units

§3. Solv. & nilp.

15/23

- Review: A grp G is called solvable (nillpotent)

  if $\exists$ a chain of subgrps

$$\{e\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_t = G.$$

s.t. ...

$$\mathbb{C} \subsetneq \mathbb{C}[H_1] \subsetneq \cdots \subsetneq \mathbb{C}[G]$$

"**Def**" (Cohen-Westreich, Def 3.5, Prop 3.8)

A FQG A is <u>solvable</u> if

$\exists$ a chain of subalgs (coideals)

$$\mathbb{C} = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_t = A.$$

s.t. $\cdots$.

In a similar way, we define <u>nilp.</u>

Thm. (Cohen–Westreich)

A cocommu. FQG ($= \mathbb{C}[G]$) is nilp (resp. solv.) iff G is nilp. (resp. solv.)

Thm. (Cohen–Westreich)

nilp. $\Longrightarrow$ solv.

Thm. (Glowacki – Hattori – T.)

KP & Sekines are nilp.
(and hence solv.)

"<u>pf</u>" $\nexists$ difficult things. All we have to do is to compute & to show that 18/23

$$
\begin{array}{c}
\text{KP} \\
A \\
| \\
\mathrm{span}\{e_1+e_4,\ e_2+e_3,\ a_{11},\ a_{22}\} \\
| \\
\mathrm{span}\{e_1+e_2+e_3+e_4,\ a_{11}+a_{22}\} \\
| \\
\mathbb{C}
\end{array}
\qquad
\begin{array}{c}
\text{Sek.} \\
A_k \\
| \\
\mathrm{span}\left\{\sum_{r=0}^{k-1} d_{p-r,q+r},\ \sum_{r=0}^{k-1} \eta^{r(q-p)} e_{p-s,q-s} \,\middle|\, p,q \in \mathbb{Z}_k\right\} \\
| \\
\mathrm{span}\left\{\sum_{r,s=0}^{k-1} d_{r,s},\ \sum_{s=0}^{k-1} e_{s,s}\right\}. \\
| \\
\mathbb{C}
\end{array}
$$

are the chains we want. $\square$

§4. Appendix A

In our paper, we computed the univ. R-mats of $K\underline{P}$.

Sol's of QYBE.

$R \in A \otimes A$ inv. s.t. $\cancel{(*)}$

$$\begin{cases} (id \otimes \Delta)(R) = R_{13} R_{12} \\ (\Delta \otimes id)(R) = R_{13} R_{23} \\ R \Delta(\cdot) R^{-1} = \Delta^{op} := (x \otimes y \mapsto y \otimes x) \circ \Delta \end{cases}$$

$$\left(\begin{array}{l} (a \otimes b)_{12} = a \otimes b \otimes 1 \\ (a \otimes b)_{13} = a \otimes 1 \otimes b \end{array} \right)$$

<u>Strategy</u> $\nexists$ difficult points.

Put $R = \sum A_{ij}\, e_i \otimes e_j + \sum B_{ijk}\, e_i \otimes a_{jk}$

$+ \sum C_{ijk}\, a_{ij} \otimes e_k + \sum D_{ijkl}\, a_{ij} \otimes a_{kl}$.

And compute.

$$\rightsquigarrow R = \sum A_{ij}\, e_i \otimes e_j$$

$$+ B_1 e_1 \otimes a_{11} + B_1 e_1 \otimes a_{22} + B_2 e_2 \otimes a_{11} - B_2 e_2 \otimes a_{22}$$

$$+ B_3\ {}_3 \otimes a_{11} - B_3 e_3 \otimes a_{22} + B_4 e_4 \otimes a_{11} + B_4 e_4 \otimes a_{22}$$

$$+ C_1 a_{11} \otimes e_1 + C_1 a_{22} \otimes e_1 + C_2 a_{11} \otimes e_2 - C_2 a_{22} \otimes e_2$$

$$+ C_3 a_{11} \otimes e_3 - C_3 a_{22} \otimes e_3 + C_4 a_{11} \otimes e_4 + C_4 a_{22} \otimes e_4$$

$$+ D_{1111} a_{11} \otimes a_{11} + D_{1122} a_{11} \otimes a_{22} + D_{1212} a_{12} \otimes a_{12}$$

$$+ D_{1221} a_{12} \otimes a_{21} + D_{1221} a_{21} \otimes a_{12} + D_{1212} a_{21} \otimes a_{21}$$

$$- D_{1122} a_{22} \otimes a_{11} + D_{1111} a_{22} \otimes a_{22}.$$

<u>Thm.</u> (Glowacki - Hattori - T., Suzuki, Wakui).

Our coeff's are the following:

$$[A_{ij}]_{ij} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- By hand
  → used 2~3 months
- Mathematica
  → we need <u>5 sec.</u>

1) $$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} \pm 1 \\ \pm 1 \\ 0 \\ 0 \end{bmatrix}$$

2) $$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} \pm 1 \\ \mp 1 \\ 0 \\ 0 \end{bmatrix}$$

3) $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda \\ -\sqrt{-1}\lambda \end{bmatrix}$ $(\lambda^2 = \sqrt{-1})$

4) $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda \\ \sqrt{-1}\lambda \end{bmatrix}$ $(\lambda^2 = \sqrt{-1})$.

Moreover R is unitary ($R^* R = 1 \otimes 1 = R R^*$)

& R in 3), 4) are minimal in the sense of Radford.

References (Please see refs therein also)

1. M. Cohen, S. Westreich, Solvability for semisimple Hopf algebras via integrals, J. Algebra Vol 472, 67-94 (2017)

2. P. Etingof, D. Nikshych, V. Ostrik, Weakly group theoretical and solvable fusion categories, Adv. Math. Vol 226(1), 176-205 (2011)

3. G. Glowacki, M. Hattori, M. Tanaka, Examples of solvable and nilpotent finite quantum groups, preprint, arXiv 2402.00706 (2024)

Thank you for your attention and your precious time in your life.

M. Tanaka