Phase Structure of Chern-Simons Matter Theories on $S^1 \times S^2$

Tomohisa Takimi (TIFR)

cf) Jain-Minwalla-Sharma-T.T-Wadia-Yokoyama [arXiv:1301.6169]
T.T [arXiv:1304.3725]

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1-1 Non-SUSY AdS/CFT

3d Chern-Simons matter theories with (bi-)fundamental representation

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~~Parity~~ Vasiliev higher spin theory in or AdS_4 another higher spin theory

Non-SUSY AdS/CFT correspondence

Non-gauged version → [Sezgin-Sundell 2002, Klebanov-Polyakov 2002]
CS gaugedn `→ [Gaiotto-Yin, Aharony-Bergman-Jafferis-Maldacena 08], [Minwalla-Narayanan-Sharma-Umesh-Yin 11], [Giombi-Minwalla-Prakash-Trivedi-Wadia-Yin 11] [Aharony-Gur-Ari-Yacoby 11]

1-1 Non-SUSY AdS/CFT

3d Chern-Simons matter theories with (bi-)fundamental representation

~~Parity~~ Vasiliev higher spin theory in or AdS_4 another higher spin theory

Non-SUSY AdS/CFT correspondence

- Higher spin sym. may work to resolve the UV divergence even in the presence of the derivative couplings
- **A clue for the quantum gravity ?**

1-1. Usage of QFT-gravity correspondence

Our interest

[Cf) Based on Witten 1998]

Deconfinement phase transition in YM theories → **Gravitational phase transition with Black hole nucleation**

Phase structure of CS matter theories → **Phase structure of the ~~Parity~~ Vasiliev higher spin theory or some another higher spin theory**

More familiar,
Easier to analyze

1-1. Usage of QFT-gravity correspondence

Our interest

[Cf) Based on Witten 1998]

Deconfinement phase transition in YM theories → Gravitational phase transition with Black hole nucleation

Phase structure of CS matter theories → Phase structure of the ~~Parity~~ Vasiliev higher spin theory or some another higher spin theory

We have worked on the phase structure of the CS matter theory → Some clue for the phase structure

Let us study

the phase structure of CS matter theories

(Sorry I will not talk about Vasiliev theory)

[Jain-Minwalla-Sharma-T.T-Wadia-Yokoyama [arXiv:1301.6169]]

T.T [arXiv:1304.3725]

Phase structure of the CS matter theories.

- To see the property of the phase structure of the CS matter theory, it is instructive to compare with the Gross-Witten-Wadia phase transition in 2 dimensional YM theory on the lattice.

Let us see by following order.

- (1) Phase structure of 2d YM on the lattice
- (2) Phase structure of CS theory

2.Lesson of Gross–Witten–Wadia Phase transition in the 2d YM on the lattice

2-1 Path integration of 2d YM lattice at large N

Path integration (Free energy) is represented by the unitary matrix model

$$\boxed{Z \quad = \int d\alpha_m \left(2\sin\frac{\alpha_l - \alpha_m}{2}\right)\exp(-S_{eff}(\alpha))}$$

2-1 Path integration of 2d YM lattice at large N

Path integration (Free energy) is represented by the unitary matrix model

$$\boxed{Z = \int d\alpha_m \left(2\sin\frac{\alpha_l-\alpha_m}{2}\right)\exp(-S_{eff}(\alpha))}$$

In large N, this is obtained by the saddle point equation.

Minimizing $\hat{S}$

$$\hat{S} = S_{eff} - \sum_{m\neq l}\log\left(2\sin\frac{\alpha_l-\alpha_m}{2}\right)$$

Free energy ↔ Configuration

Configuration → determined set of eigenvalues

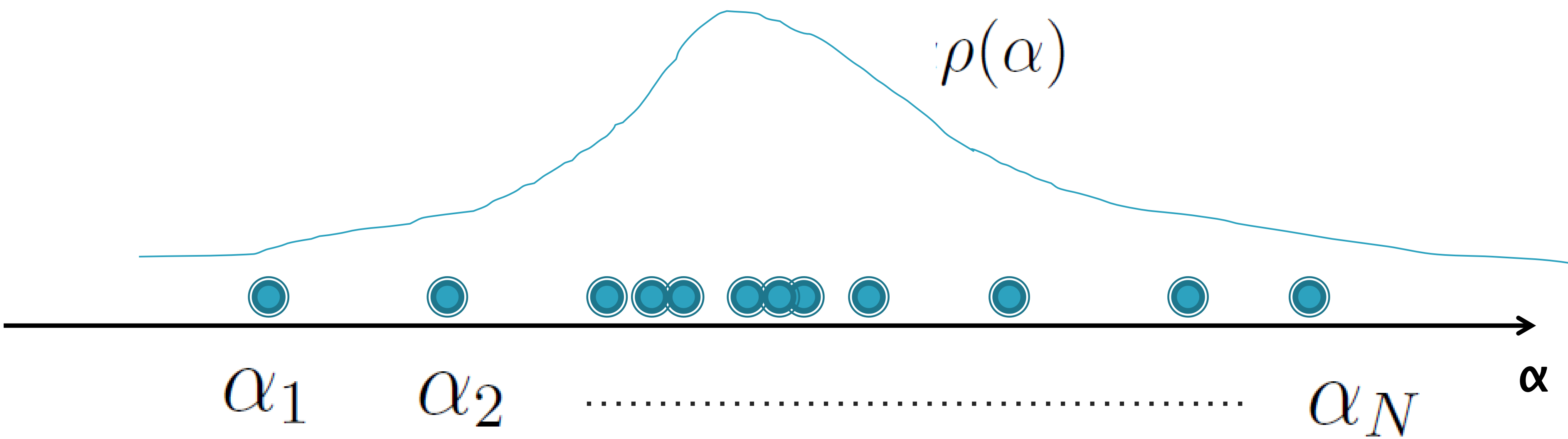
$$-\pi \leq \alpha_1 < \alpha_2 < \ldots < \alpha_N \leq \pi$$

In the large N

$$\frac{1}{N}\sum_m \to \int_0^1 dx = \int_{\alpha_{min}}^{\alpha_{max}} \frac{dx}{d\alpha} d\alpha = \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \rho(\alpha)$$

Eigenvalue density

configuration is governed by ρ(α)



In the large N

$$\frac{1}{N}\sum_{m} \to \int_0^1 dx = \int_{\alpha_{min}}^{\alpha_{max}} \frac{dx}{d\alpha} d\alpha = \int_{\alpha_{min}}^{\alpha_{max}} d\alpha\, \rho(\alpha)$$

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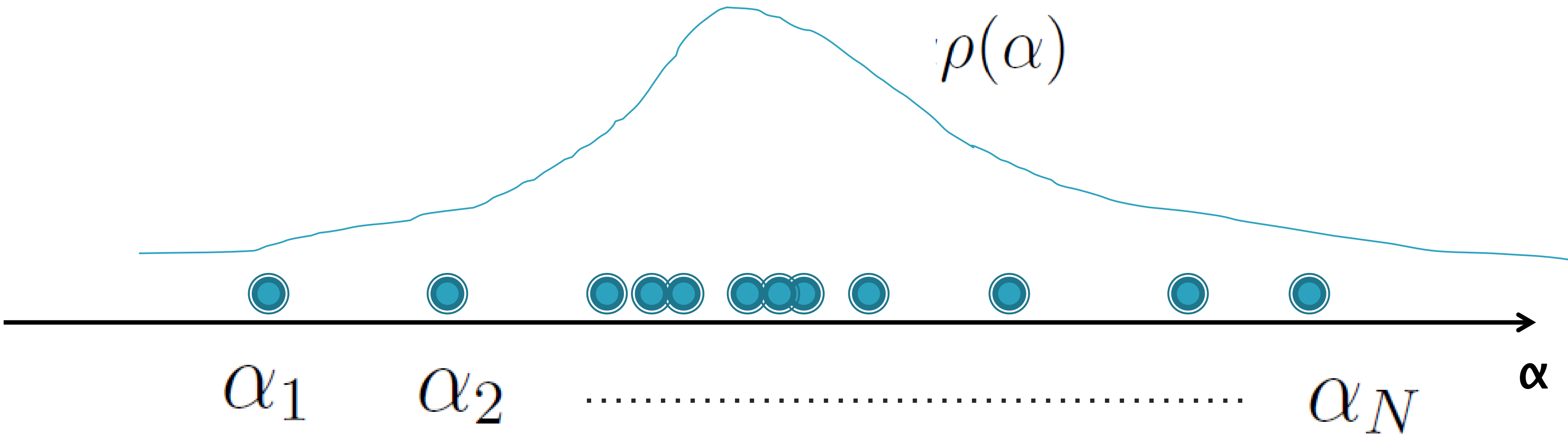


ρ(α) ↔ Configuration ↔Free energy

Let us focus on ρ(α)

In the large N

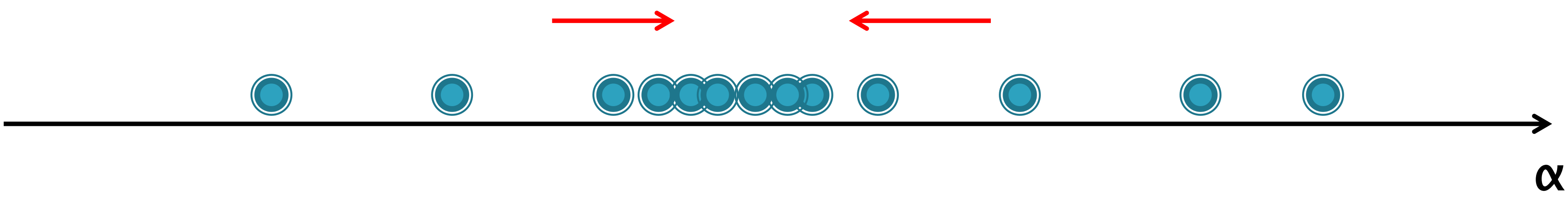
$$\frac{1}{N}\sum_m \to \int_0^1 dx = \int_{\alpha_{min}}^{\alpha_{max}} \frac{dx}{d\alpha} d\alpha = \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \rho(\alpha)$$

Eigenvalue density

Saddle point eq. In terms of ρ

$$\hat{S} = -N^2 \zeta\, v(U(x)) - N^2 \mathcal{P} \int d\alpha \int d\beta \rho(\alpha)\rho(\beta) \log\left(2 \sin\frac{\alpha-\beta}{2}\right)$$

:Indicate the location of the eigenvalues



$$\hat{S} = \boxed{-N^2\zeta\, v(U(x))} + N^2\mathcal{P}\int d\alpha \int d\beta\, \rho(\alpha)\rho(\beta)\log\left(2\sin\frac{\alpha-\beta}{2}\right)$$

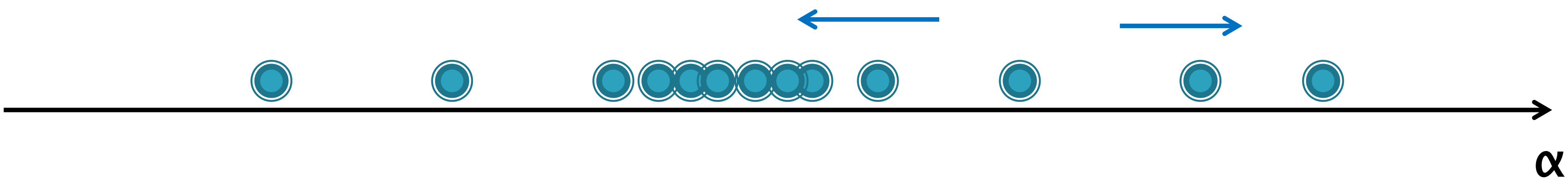
S_{eff}

Minimized when $\rho(\alpha)=\delta(\alpha)$ $\quad \alpha_1 = \ldots = \alpha_N = 0$

Attracting Force between eigenvalues,

stronger in the higher temperature

:Indicate the location of the eigenvalues



$$\hat{S} = -N^2\zeta\, v(U(x)) - \boxed{N^2\mathcal{P}\int d\alpha \int d\beta\, \rho(\alpha)\rho(\beta)\log\left(2\sin\frac{\alpha-\beta}{2}\right)}$$

Vandermond determinant

Minimized when $\rho(\alpha) = \frac{1}{2\pi}$ $\alpha_{i+1} = \alpha_i + \frac{2\pi}{N}$

Repulsive Force between eigenvalues,

stronger in the lower temperature

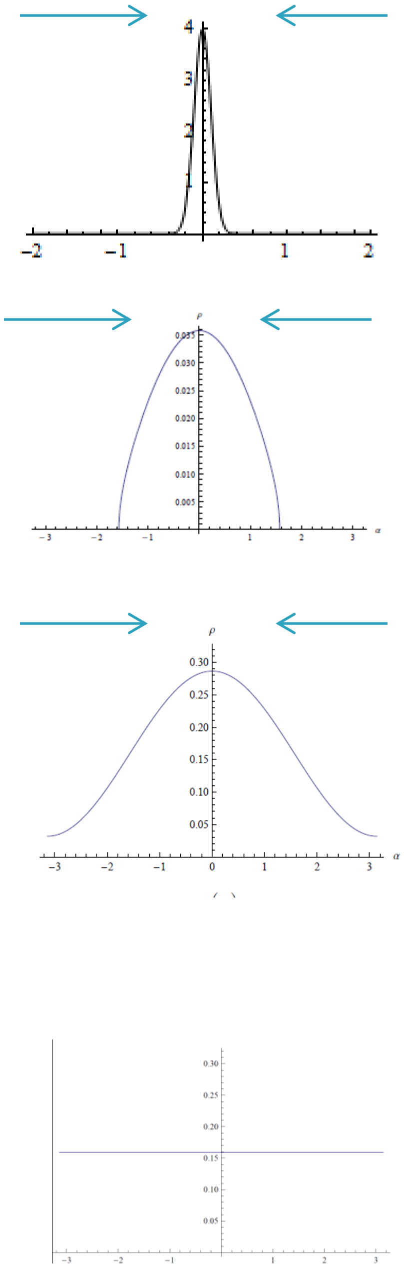
$$\hat{S} = \boxed{-N^2\zeta\, v(U(x))} + \boxed{N^2\mathcal{P}\int d\alpha \int d\beta \rho(\alpha)\rho(\beta)\log\left(2\sin\frac{\alpha-\beta}{2}\right)}$$

Attractive Force v.s Repulsive Force

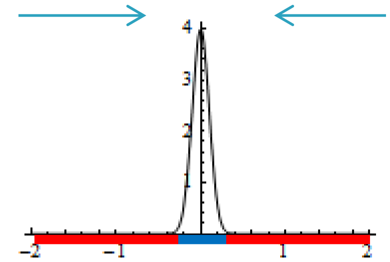
temperature ζ

Eigenvalue density function $\rho(\alpha)$ w.r.t to temperature

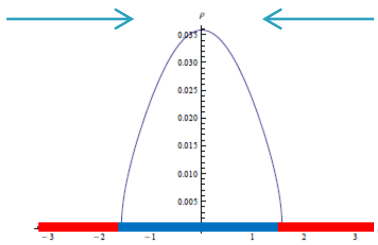
Start to clump

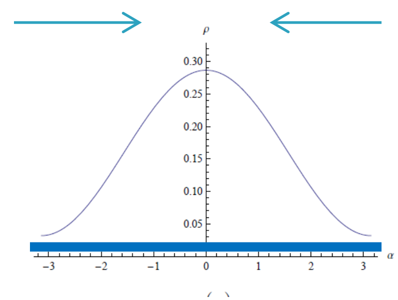


temperature ζ

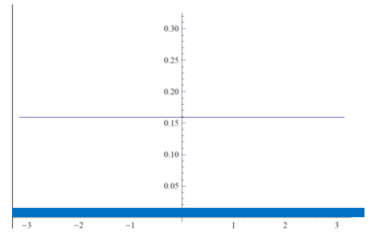


Support of $\rho(\alpha)$ in a part of the domain, there are zero points





Support of $\rho(\alpha)$ all over the domain, no zero point

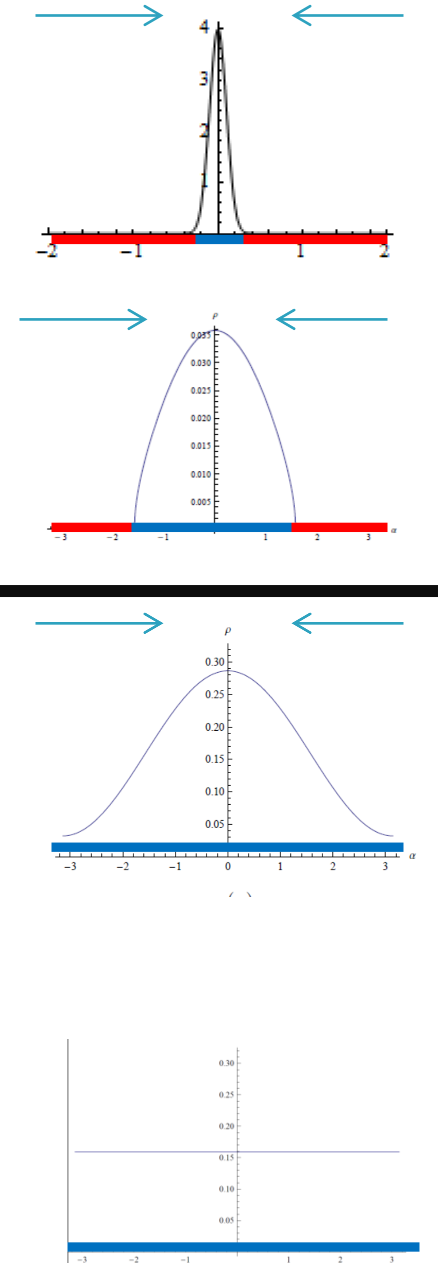


temperature ζ

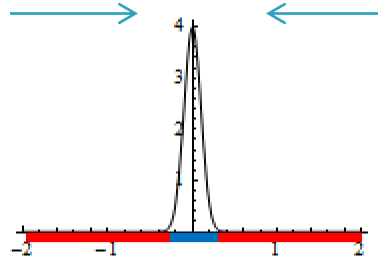
Support of $\rho(\alpha)$ in a part of the domain, there are zero points

Phase transition

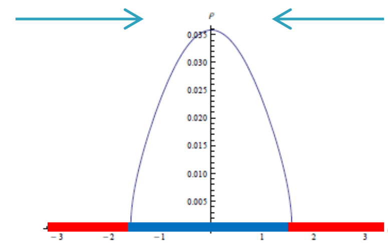
Support of $\rho(\alpha)$ all over the domain, no zero point



temperature ζ

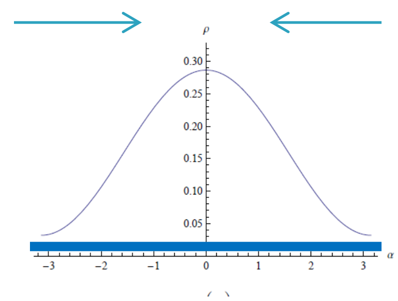


Lower gap phase



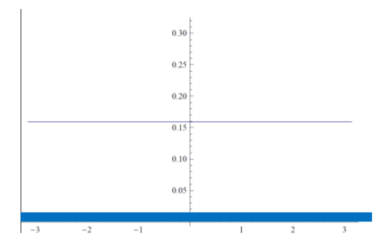
Support of ρ(α) in a part of the domain, there are zero points

Phase transition



No gap phase

Support of ρ(α) all over the domain, no zero point



Let us consider the phase structure of CS matter theory

3.How to calculate the partition function of CS matter theory.

- [Jain-Minwalla-Sharma-T.T-Wadia-Yokoyama [arXiv:1301.6169]]

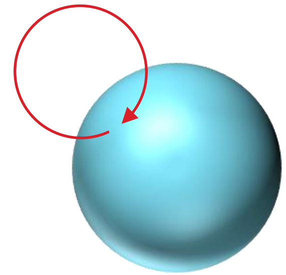
In [Jain-Minwalla-Sharma-T.T-Wadia-Yokoyama [arXiv:1301.6169]]

We give a prescription to investigate

In T.T [arXiv:1304.3725]

We have investigate the phase structure of the CS matter theory with the prescription.

3-1. Path integration of the CS matter theory on $S^1 \times S^2$



- Starting from the path integration formula,

$$Z_{\mathrm{CS}} = \int DA\,\underline{D\mu}\; e^{i\frac{k}{4\pi}\mathrm{Tr}\int\left(AdA+\frac{2}{3}A^3\right)-\underline{S_{matter}}}$$

Performing the matter integration $D\mu$

$$Z_{\mathrm{CS}} = \int DA e^{i\frac{k}{4\pi}\mathrm{Tr}\int\left(AdA+\frac{2}{3}A^3\right)-\boxed{S_{eff}}}$$

Effective potential depending on gauge fields

3-1-1 Expansion of the effective action

- Form of S_{eff}

$$S_{eff} = \int d^2x \, \left(T^2 v(U) + \mathrm{Tr}\,(\partial_i U + [A_i, U])^2 \ldots\right) \qquad (i = 1, 2)$$

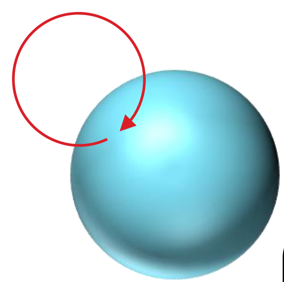
- Effective action is composed of $\boxed{A_1, A_2,}$ and the holonomy $\boxed{\oint dx_3 A_3}$

x_3 : Thermal direction,

$\boxed{U = \exp\left(\oint dx_3 A_3\right) \text{ : Holonomy}}$

$\partial_3 A_3 = 0,$ Gauge fixing

T Temperature



x_1, x_2 : coordinate on S^2

$A_1, A_2,$:Gauge fields along S^2

3-1-1 Expansion of the effective action

- Form of S_{eff}

$$S_{eff} = \int d^2x \, \left(T^2 v(U) + \mathrm{Tr} \left(\partial_i U + [A_i, U] \right)^2 \ldots \right) \qquad (i = 1, 2)$$

Mass dimension

$U \longrightarrow 0$

$\left. \begin{array}{l} T \\ A_1, A_2, \partial_i \end{array} \right\} \longrightarrow 1$

3-1-1 Expansion of the effective action

- Order of T in S_{eff}

$$S_{eff} = \int d^2x \ (\underbrace{T^2 v(U)}_{O(T^2)} + \underbrace{\mathrm{Tr}\,(\partial_i U + [A_i, U])^2}_{O(T^0)} \underbrace{\ldots}_{O(T^n)\ (n > 0)}) \qquad (i = 1, 2)$$

Depending only on holonomy

3-1-1 Expansion of the effective action

- In Large N

$$S_{eff} = \int d^2x \, \left(T^2 v(U) + \mathrm{Tr}\,(\partial_i U + [A_i, U])^2 \ldots\right) \qquad (i = 1, 2)$$

$\Rightarrow$ Order (N^1)

(1) No propagating degree of freedom of gauge fields
(2) Matter is in the fundamental representation.

3-1-1 Expansion of the effective action

- In Large N

$$S_{eff} = \int d^2x \, \left(T^2 v(U) + \mathrm{Tr}\,(\partial_i U + [A_i, U])^2 \ldots\right) \qquad (i = 1, 2)$$

$\Longrightarrow$ Order (N^1)

- Vandermond determinant contributes as order (N^2)

(We will see the Vandermond determinant later)

Phase transition

$\Longrightarrow$ by the competition of

S_{eff} v.s Vandermonde

Relatively very small

3-1-1 Expansion of the effective action

$$S_{eff} = \int d^2x \, \left(T^2 v(U) + \mathrm{Tr}\left(\partial_i U + [A_i, U]\right)^2 \ldots\right) \qquad (i = 1, 2)$$

⇒ Order (N^1)

- Vandermond determinant contributes as order (N^2)

(We will see the Vandermond determinant later)

⇒ **Phase transition can occur only when the temperature T is very high** $T^2 \sim N^1$

3-1-1 Expansion of the effective action

$$S_{eff} = \int d^2x \; \left(\boxed{T^2 v(U)} + \boxed{\cancel{\mathrm{Tr}\,(\partial_i U + [A_i, U])^2}} \ldots\right) \qquad (i = 1, 2)$$

Leading O(N²)!

Next Leading

Phase transition can occur only when the temperature T is very high $T^2 \sim N^1$

3-1-1 Expansion of the effective action

$$\boxed{S_{eff} = \int d^2x \; T^2 v(U)}$$

The effective action only depends on the holonomy along the thermal direction.

3-1-1 Expansion of the effective action

$$Z_{\rm CS} = \int DA e^{i\frac{k}{4\pi}{\rm Tr}\int\left(AdA+\frac{2}{3}A^3\right)\boxed{-S_{eff}(U)}}$$

$$= \int DA e^{i\frac{k}{4\pi}{\rm Tr}\int\left(AdA+\frac{2}{3}A^3\right)\boxed{-T^2\int d^2x\sqrt{g}\ v(U)}}$$

We can easily apply the method in Blau-Thompson Nucl.Phys. B408 (1993) 345-390.
to calculate the partition function for every CS matter theory uniformly.

3-1-2 Blau Thompson method

$$Z_{\mathrm{CS}} = \int DA e^{i\frac{k}{4\pi}\mathrm{Tr}\int\left(AdA+\frac{2}{3}A^3\right)-T^2\int d^2x\sqrt{g}\ v(U)}$$

- Gauge fixing

1. $\partial_3 A_3 = 0,$
2. Diagonalizing A_3

3-1-2 Blau Thompson method

$$Z_{\mathrm{CS}} = \int DA e^{i\frac{k}{4\pi}\mathrm{Tr}\int\left(AdA+\frac{2}{3}A^3\right)-T^2\int d^2x\sqrt{g}\ v(U)}$$

- Gauge fixing

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2. Diagonalizing A_3

- Field contents:

1. Off diagonal components of $A_{1\alpha}$, $A_{2\alpha}$
2. Off diagonal components of ghost pair $c, \bar{c}$

A. Diagonal components of A_{1d}, A_{2d}

3-1-2 Blau Thompson method

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- Gauge fixing

1. $\bar{\partial}_3 A_3 = 0,$
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A. Diagonal components of A_{1d}, A_{2d}

Integrate these first

$$Z_{\mathrm{CS}} = \int dA dc d\bar{c} \exp\left(i \int (\boxed{A_{2\alpha} D_3 A_{1\alpha} + \bar{c}_\alpha D_3 c_\alpha} + A_{2d} \partial_3 A_{1d} + \sum_m \alpha_m F_{12m}) - S_{eff} \right)$$

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$$Z_{\rm CS} = \int dA dc d\bar{c} \exp\left(i \int (A_{2\alpha} D_3 A_{1\alpha} + \bar{c}_\alpha D_3 c_\alpha + A_{2d}\partial_3 A_{1d} + \sum_m \alpha_m F_{12m}) - S_{eff} \right)$$

Euler number of S2 = 2

$$\int dA_{1,2,d} d\alpha \prod_n \underbrace{\left(\frac{2\pi i n}{\beta} + \frac{i(\alpha_m - \alpha_n)}{\beta} \right)^{\frac{1}{2}\chi_{S_2}}}_{Det(D_3)} \exp(-A_{1d}\partial_3 A_{2d} + \sum_m \alpha_m F_{12m} - S_{eff})$$

Quantized momentum along thermal circle x_3

Power of the determinant
= (# of 0-form(ghost)) – ½ (# of 1-form (gauge field))
= ½((# of 2-form) + (# of 0-form) – (# of 1-form))
= ½ (Euler number of S_2)

$$\int dA_{1,2,d}d\alpha \prod_{n=-\infty}^{\infty}\prod_{m,l=1}^{N}\left(\frac{2\pi i n}{\beta}+\frac{i(\alpha_m-\alpha_l)}{\beta}\right)^{\frac{1}{2}\chi s_2}\exp\left(i\int(A_{1d}\partial_3 A_{2d}+\sum_m \alpha_m F_{12m})-S_{eff}\right)$$

$$\boxed{\sin x = x\prod_{n\neq 0}\left(1-\frac{x^2}{n^2\pi^2}\right)}$$

$$\int dA_{1,2,d}d\alpha \left(\prod_{l\neq m} 2\sin\frac{\alpha_l-\alpha_m}{2}\right)^{\frac{1}{2}\chi s_2}\exp\left(i\int(A_{1d}\partial_3 A_{2d}+\sum_m \alpha_m F_{12m})-S_{eff}\right)$$

$$\int dA_{1,2,d}d\alpha \prod_{n=-\infty}^{\infty}\prod_{m,l=1}^{N}\left(\frac{2\pi in}{\beta}+\frac{i(\alpha_m-\alpha_l)}{\beta}\right)^{\frac{1}{2}\chi s_2}\exp\left(i\int(A_{1d}\partial_3A_{2d}+\sum_m \alpha_m F_{12m})-S_{eff}\right)$$

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A. Diagonal components of A_{1d}, A_{2d}

$$\int dA_{1,2,d}d\alpha \prod_{n=-\infty}^{\infty}\prod_{m,l=1}^{N}\left(\frac{2\pi in}{\beta}+\frac{i(\alpha_m-\alpha_l)}{\beta}\right)^{\frac{1}{2}\chi s_2}\exp\left(i\int(A_{1d}\partial_3 A_{2d}+\sum_m \alpha_m F_{12m})-S_{eff}\right)$$

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A. Diagonal components of A_{1d}, A_{2d}

(i). Massive KK momentum modes

(ii).Massless KK momentum modes

$$\int dA_{1,2,d} d\alpha \prod_{n=-\infty}^{\infty} \prod_{m,l=1}^{N} \left(\frac{2\pi i n}{\beta} + \frac{i(\alpha_m - \alpha_l)}{\beta} \right)^{\frac{1}{2}\chi s_2} \exp\left(i \int (A_{1d}\partial_3 A_{2d} + \sum_m \alpha_m F_{12m}) - S_{eff} \right)$$

$$\boxed{\sin x = x \prod_{n \neq 0} \left(1 - \frac{x^2}{n^2 \pi^2} \right)}$$

$$\int dA_{1,2,d} d\alpha \left(\prod_{l \neq m} 2 \sin \frac{\alpha_l - \alpha_m}{2} \right)^{\frac{1}{2}\chi s_2} \exp\left(i \int (\boxed{A_{1d}\partial_3 A_{2d}} + \sum_m \alpha_m F_{12m}) - S_{eff} \right)$$

KK massive modes along thermal circle

$\Rightarrow$ just the constant (ignored)

$$\int dA_{1,2,d}d\alpha \prod_{n=-\infty}^{\infty}\prod_{m,l=1}^{N}\left(\frac{2\pi i n}{\beta}+\frac{i(\alpha_m-\alpha_l)}{\beta}\right)^{\frac{1}{2}\chi s_2}\exp\left(i\int(A_{1d}\partial_3 A_{2d}+\sum_m \alpha_m F_{12m})-S_{eff}\right)$$

$$\boxed{\sin x = x\prod_{n\neq 0}\left(1-\frac{x^2}{n^2\pi^2}\right)}$$

$$\int dA_{1,2,d}d\alpha \left(\prod_{l\neq m}2\sin\frac{\alpha_l-\alpha_m}{2}\right)^{\frac{1}{2}\chi s_2}\exp\left(i\int \qquad \sum_m \alpha_m F_{12m})-S_{eff}\right)$$

$$\int dA_{1,2,d} d\alpha \prod_{n=-\infty}^{\infty} \prod_{m,l=1}^{N} \left(\frac{2\pi i n}{\beta} + \frac{i(\alpha_m - \alpha_l)}{\beta} \right)^{\frac{1}{2}\chi s_2} \exp\left(i \int (A_{1d}\partial_3 A_{2d} + \sum_m \alpha_m F_{12m}) - S_{eff} \right)$$

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$$\int dA_{1,2,d} d\alpha \left(\prod_{l \neq m} 2 \sin \frac{\alpha_l - \alpha_m}{2} \right)^{\frac{1}{2}\chi s_2} \exp\left(i \int \qquad \sum_m \boxed{\alpha_m F_{12m})} - S_{eff} \right)$$

Integration of KK massless modes along thermal circle

By fixing the residual gauge by

$$\partial_i A^i = 0 \Rightarrow A_i = \epsilon_{ij} \partial^j \chi \quad (i = 1, 2)$$

We can see

$$\int d\chi \exp\left(i \int \chi \partial_i \partial^i \alpha \right) \Rightarrow \alpha : \text{constant on} \quad S^2 \times S^1$$

$$\int dA_{1,2,d} d\alpha \prod_{n=-\infty}^{\infty} \prod_{m,l=1}^{N} \left(\frac{2\pi i n}{\beta} + \frac{i(\alpha_m - \alpha_l)}{\beta} \right)^{\frac{1}{2}\chi s_2} \exp\left(i \int (A_{1d}\partial_3 A_{2d} + \sum_m \alpha_m F_{12m}) - S_{eff} \right)$$

$$\boxed{\sin x = x \prod_{n \neq 0} \left(1 - \frac{x^2}{n^2 \pi^2} \right)}$$

$$\int dA_{1,2,d} d\alpha \left(\prod_{l \neq m} 2 \sin \frac{\alpha_l - \alpha_m}{2} \right)^{\frac{1}{2}\chi s_2} \exp\left(i \int \qquad \sum_m \boxed{\alpha_m F_{12m})} - S_{eff} \right)$$

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$$\partial_i A^i = 0 \Rightarrow A_i = \epsilon_{ij} \partial^j \chi \quad (i = 1, 2)$$

We can see

$$\int d\chi \exp\left(i \int \chi \partial_i \partial^i \alpha \right) \Rightarrow \alpha : \text{constant on} \quad S^2 \times S^1$$

$$\boxed{i \sum_m \int d^2 x \alpha_m F_{12m} = i \sum_m \alpha_m \int d^2 x F_{12m} = i \sum_m \alpha_m \hat{n}_m}$$

$$\int dA_{1,2,d}d\alpha \prod_{n=-\infty}^{\infty}\prod_{m,l=1}^{N}\left(\frac{2\pi i n}{\beta}+\frac{i(\alpha_m-\alpha_l)}{\beta}\right)^{\frac{1}{2}\chi s_2}\exp\left(i\int(A_{1d}\partial_3 A_{2d}+\sum_m \alpha_m F_{12m})-S_{eff}\right)$$

$$\sin x = x\prod_{n\neq 0}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

$$\int dA_{1,2,d}d\alpha\left(\prod_{l\neq m}2\sin\frac{\alpha_l-\alpha_m}{2}\right)^{\frac{1}{2}\chi s_2}\exp\left(i\int \qquad \sum_m \alpha_m F_{12m})-S_{eff}\right)$$

Integration of KK massless modes along thermal circle

By fixing the residual gauge by

$$\partial_i A^i = 0 \Rightarrow A_i = \epsilon_{ij}\partial^j\chi \quad (i=1,2)$$

Monopole,
Integer

We can see

$$\int d\chi \exp\left(i\int \chi\partial_i\partial^i\alpha\right) \Rightarrow \alpha : \text{constant on} \quad S^2\times S^1$$

$$\boxed{i\sum_m\int d^2x\alpha_m F_{12m} = i\sum_m \alpha_m\int d^2x F_{12m} = i\sum_m \alpha_m n_m}$$

$$\boxed{Z_{CS} = \sum_{\hat{n}_m} \int d\alpha_m \left(2\sin\frac{\alpha_l - \alpha_m}{2}\right)^{\frac{1}{2}\chi_{S_2}} \exp(i\sum_m \alpha_m \hat{n}_m - S_{eff}(\alpha))}$$

$$Z_{CS} = \sum_{\hat{n}_m} \int d\alpha_m \left(2\sin\frac{\alpha_l - \alpha_m}{2}\right)^{\frac{1}{2}\chi_{S_2}} \exp(i\sum_m \alpha_m \hat{n}_m - S_{eff}(\alpha))$$

Analogous

Partition function of the 2d YM on the lattice
(By Gross–Witten–Wadia)

$$Z \quad = \int d\alpha_m \left(2\sin\frac{\alpha_l - \alpha_m}{2}\right) \exp(-S_{eff}(\alpha))$$

$$Z_{CS} = \sum_{\hat{n}_m} \int d\alpha_m \left(2\sin\frac{\alpha_l - \alpha_m}{2}\right)^{\frac{1}{2}\chi_{S_2}} \exp\left(i\frac{\boxed{k}}{2\pi}\sum_m \alpha_m \hat{n}_m - S_{eff}(\alpha)\right)$$

Difference ??

(1) Additional parameter
(CS level)

(2) Sum of the monopole

$$Z = \int d\alpha_m \left(2\sin\frac{\alpha_l - \alpha_m}{2}\right) \exp(-S_{eff}(\alpha))$$

3-1-3 Effect of the monopole

$$Z_{CS} = \sum_{\hat{n}_m} \int d\alpha_m \left(2\sin\frac{\alpha_l - \alpha_m}{2}\right)^{\frac{1}{2}\chi_{S_2}} \exp(i \cdot k \sum_m \alpha_m \hat{n}_m - S_{eff}(\alpha))$$

Sum of the monopole

$$\boxed{\sum_n e^{ik\alpha n} = \sum_{m\in\mathbb{Z}} \delta(\alpha - \frac{2\pi m}{k})}$$

$$= \int \prod_{j=1}^{N} d\alpha_j \left(\prod_{m\neq l} 2\sin\left(\frac{\alpha_m(n_m) - \alpha_l(n_l)}{2}\right)\right) e^{-N\zeta v(U)} \boxed{\sum_{n\in\mathbb{Z}} \delta(k\alpha_j - 2\pi n)}$$

Delta function shows up

3-1-3 Effect of the monopole

$$Z_{CS} = \sum_{\hat{n}_m} \int d\alpha_m \left(2\sin\frac{\alpha_l - \alpha_m}{2} \right)^{\frac{1}{2}\chi_{S_2}} \exp(i \cdot k \sum_m \alpha_m \hat{n}_m - S_{eff}(\alpha))$$

Sum of the monopole

$$\sum_n e^{ik\alpha n} = \sum_{m\in\mathbb{Z}} \delta(\alpha - \frac{2\pi m}{k})$$

$$= \int \prod_{j=1}^{N} d\alpha_j \left(\prod_{m\neq l} 2\sin\left(\frac{\alpha_m(n_m) - \alpha_l(n_l)}{2}\right)\right) e^{-N\zeta v(U)} \boxed{\sum_{n\in\mathbb{Z}} \delta(k\alpha_j - 2\pi n)}$$

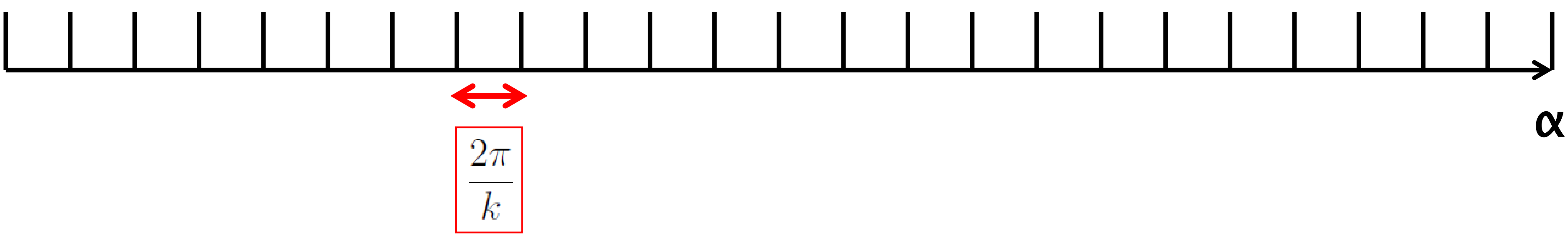
Delta function shows up

α is restricted to the ***Discretized value***

With the effect of the monopole

$$= \int \prod_{j=1}^{N} d\alpha_j \left(\prod_{m \neq l} 2 \sin \left(\frac{\alpha_m(n_m) - \alpha_l(n_l)}{2} \right) \right) e^{-N\zeta v(U)} \boxed{\sum_{n \in \mathbb{Z}} \delta(k\alpha_j - 2\pi n)}$$

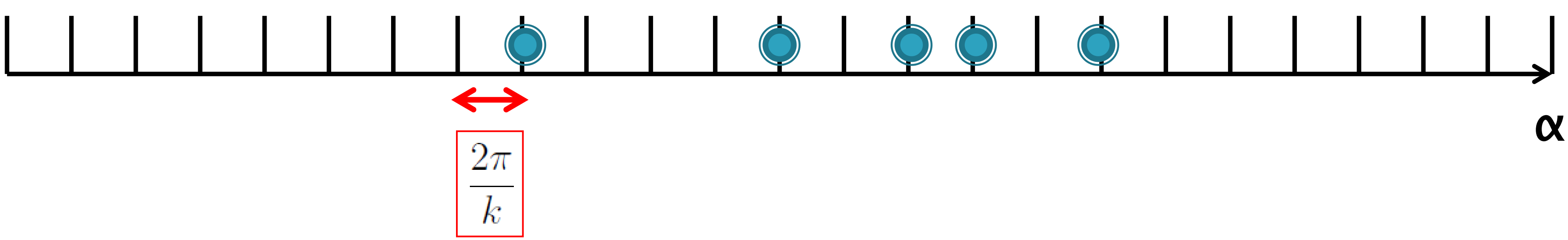
α is restricted to the ***Discretized value***



With the effect of the monopole

α is restricted to the ***Discretized value***

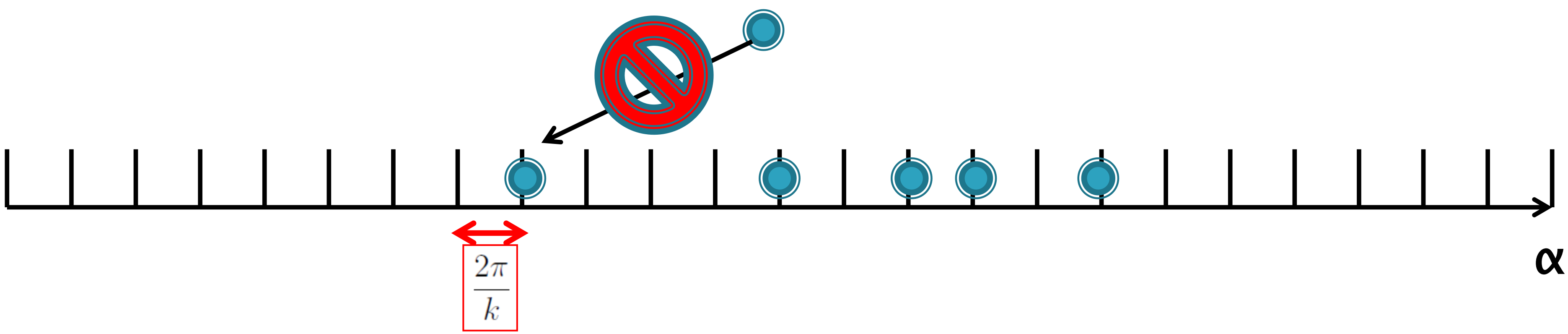
(1) Each is skewed with comb



With the effect of the monopole

α is restricted to the ***Discretized value***

(1) Each is skewed with comb



(2) Due to vandermond determinant $\prod_{m\neq l} 2\sin\left(\frac{\alpha_m(n_m)-\alpha_l(n_l)}{2}\right)$

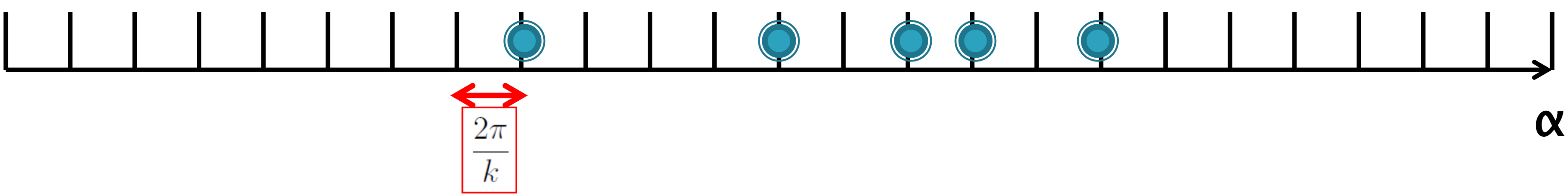
Eigenvalues cannot coincide

With the effect of the monopole

α is restricted to the ***Discretized value***

(1) Each is skewed with comb

Each steak can skew only one

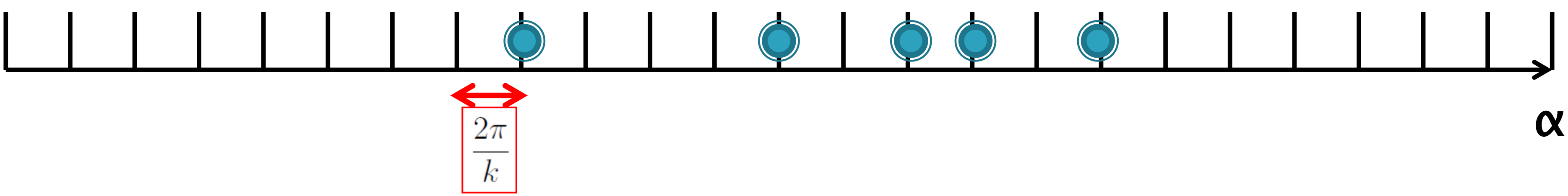


Within distance $\frac{2\pi}{k}$ only one

With the effect of the monopole

Eigenvalue density is saturated from above !

$$\boxed{\rho(\alpha) \leq \frac{k}{2\pi} \times \frac{1}{N} = \frac{1}{2\pi\lambda}}$$

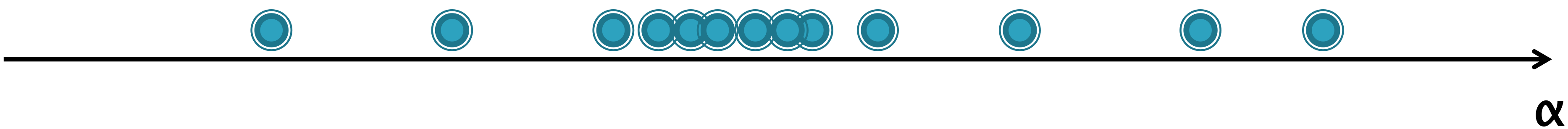


Within distance $\frac{2\pi}{k}$ only one

To see the significance,
Let us compare with the YM case without monopole effect.

Without the effect of the monopole

:Indicate the location of the eigenvalues

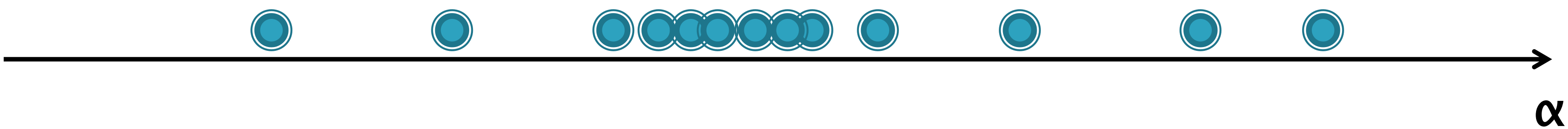


Without the effect of the monopole

:Indicate the location of the eigenvalues

(1) Due to vandermond determinant $\prod_{m\neq l} 2\sin\left(\frac{\alpha_m(n_m)-\alpha_l(n_l)}{2}\right)$

Eigenvalues cannot coincides

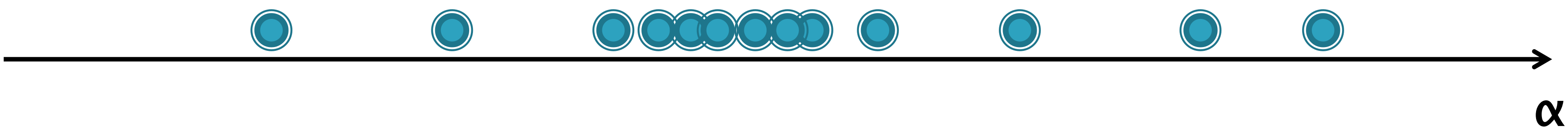


Without the effect of the monopole

:Indicate the location of the eigenvalues

(1) Due to vandermond determinant $\prod_{m\neq l} 2\sin\left(\frac{\alpha_m(n_m)-\alpha_l(n_l)}{2}\right)$

Eigenvalues cannot coincides

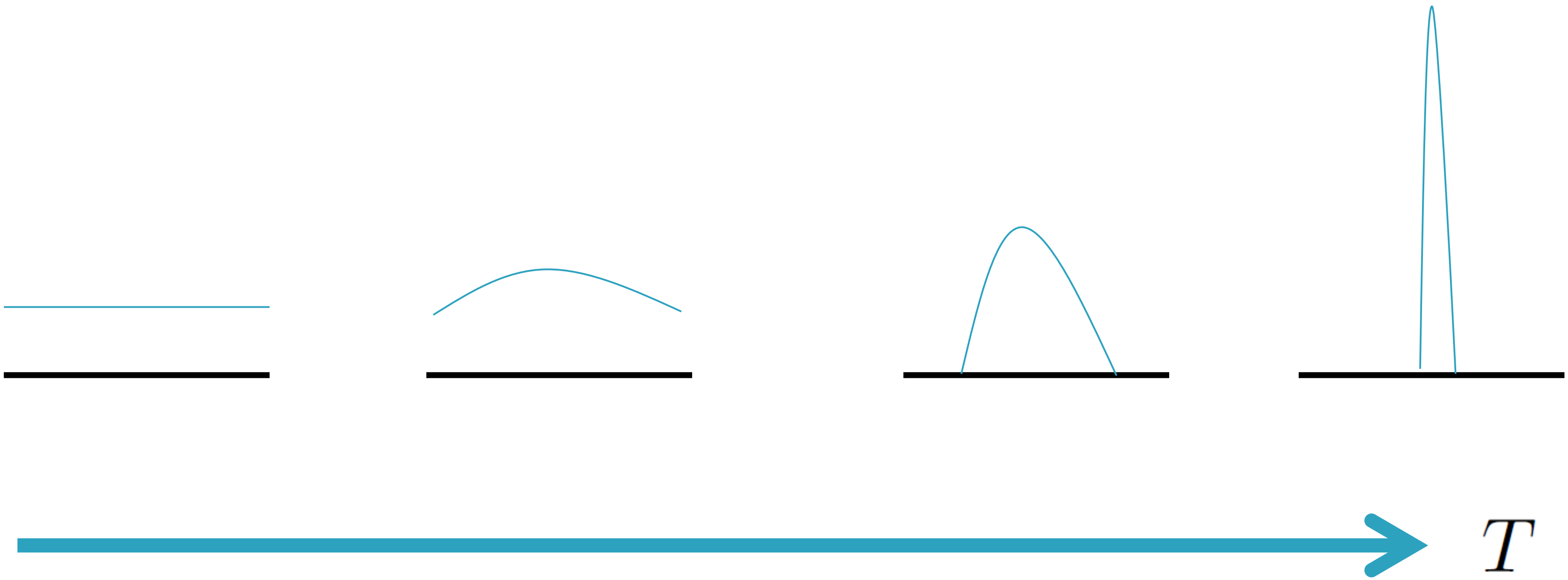


(2) But two of can close to each other as much as possible

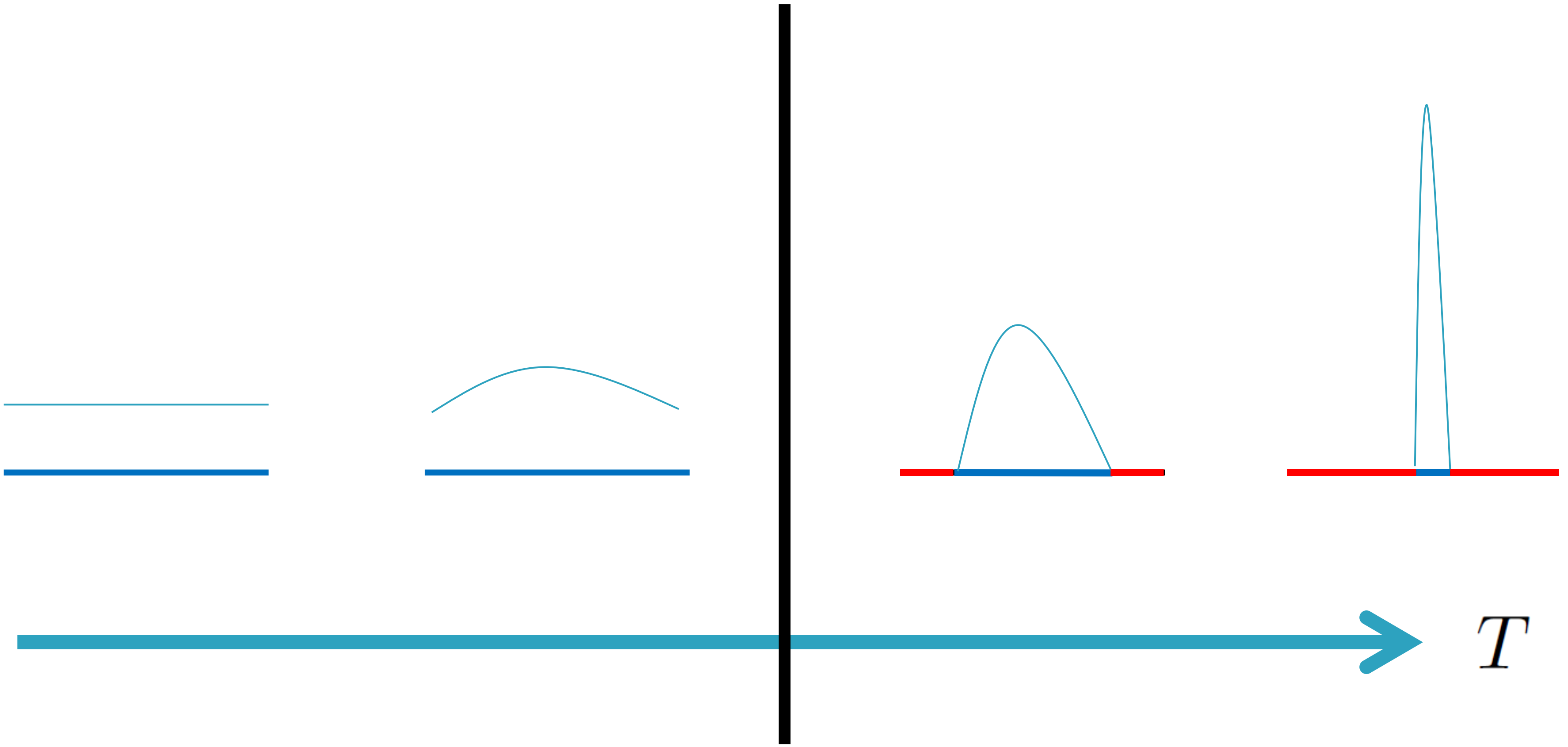
No saturation of the eigenvalue density from above

Behavior of eigenvalue density in CS matter theory

Behavior of eigenvalue density $\rho(\alpha)$

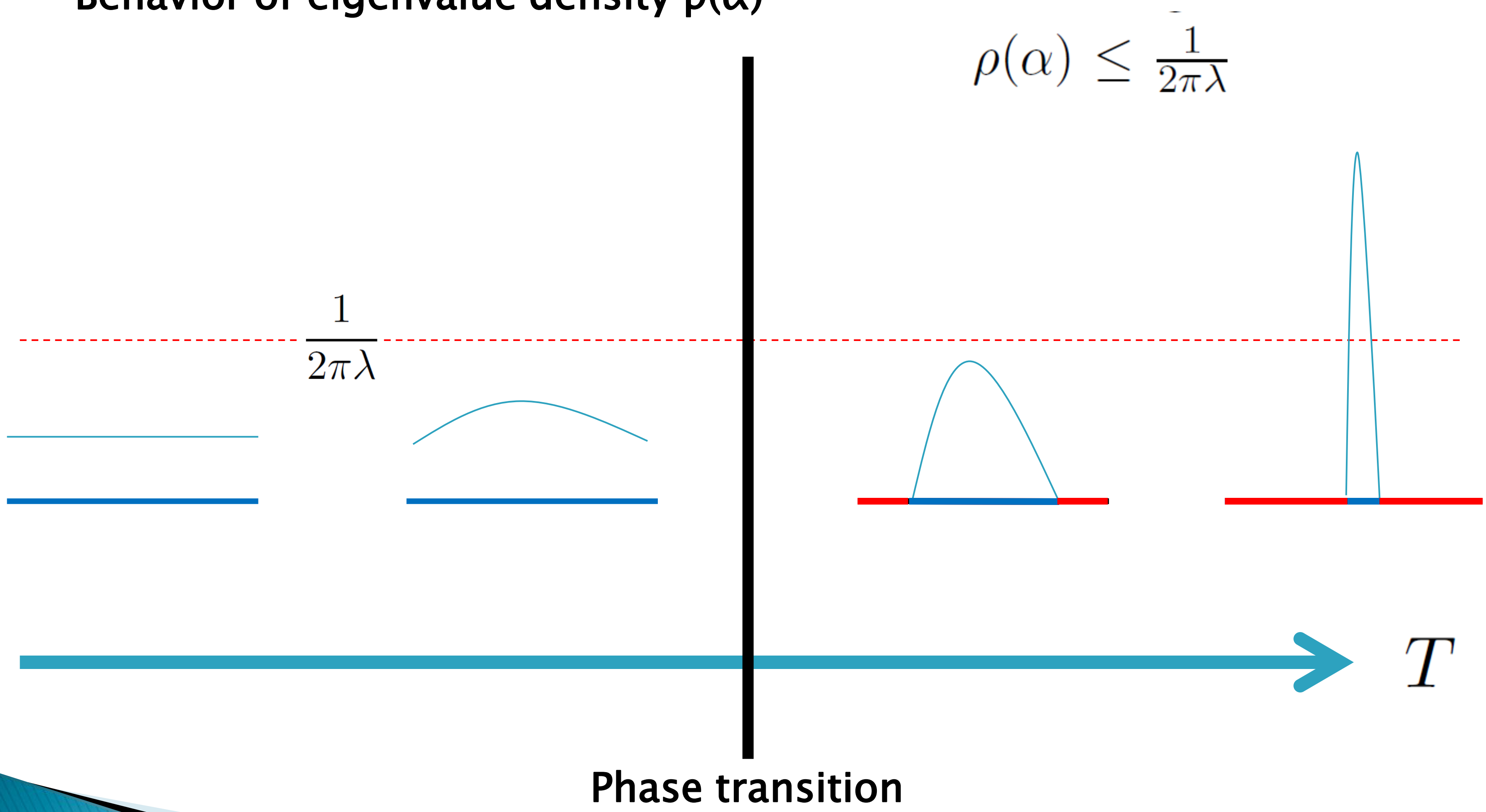


Behavior of eigenvalue density ρ(α)

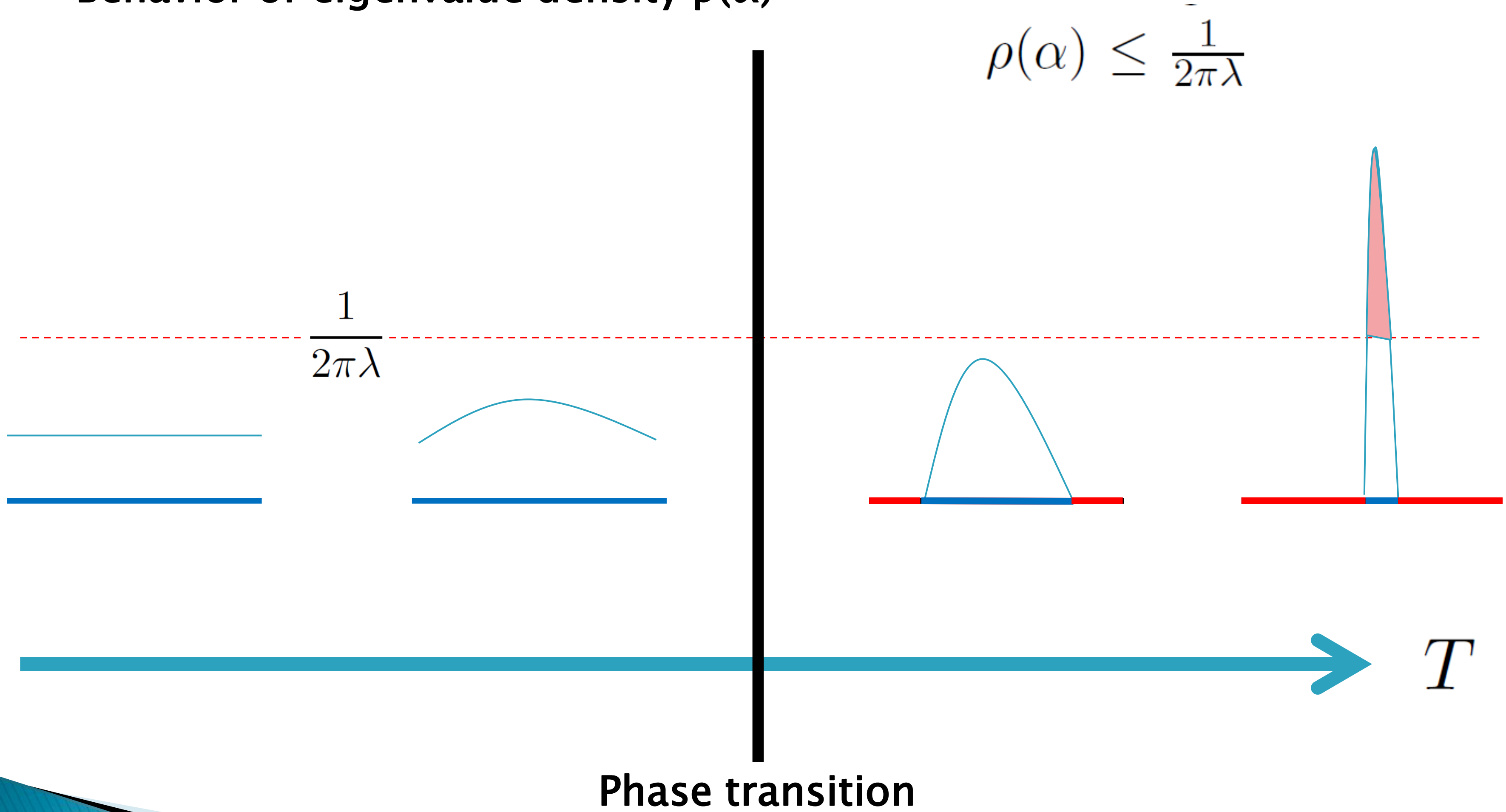


Phase transition

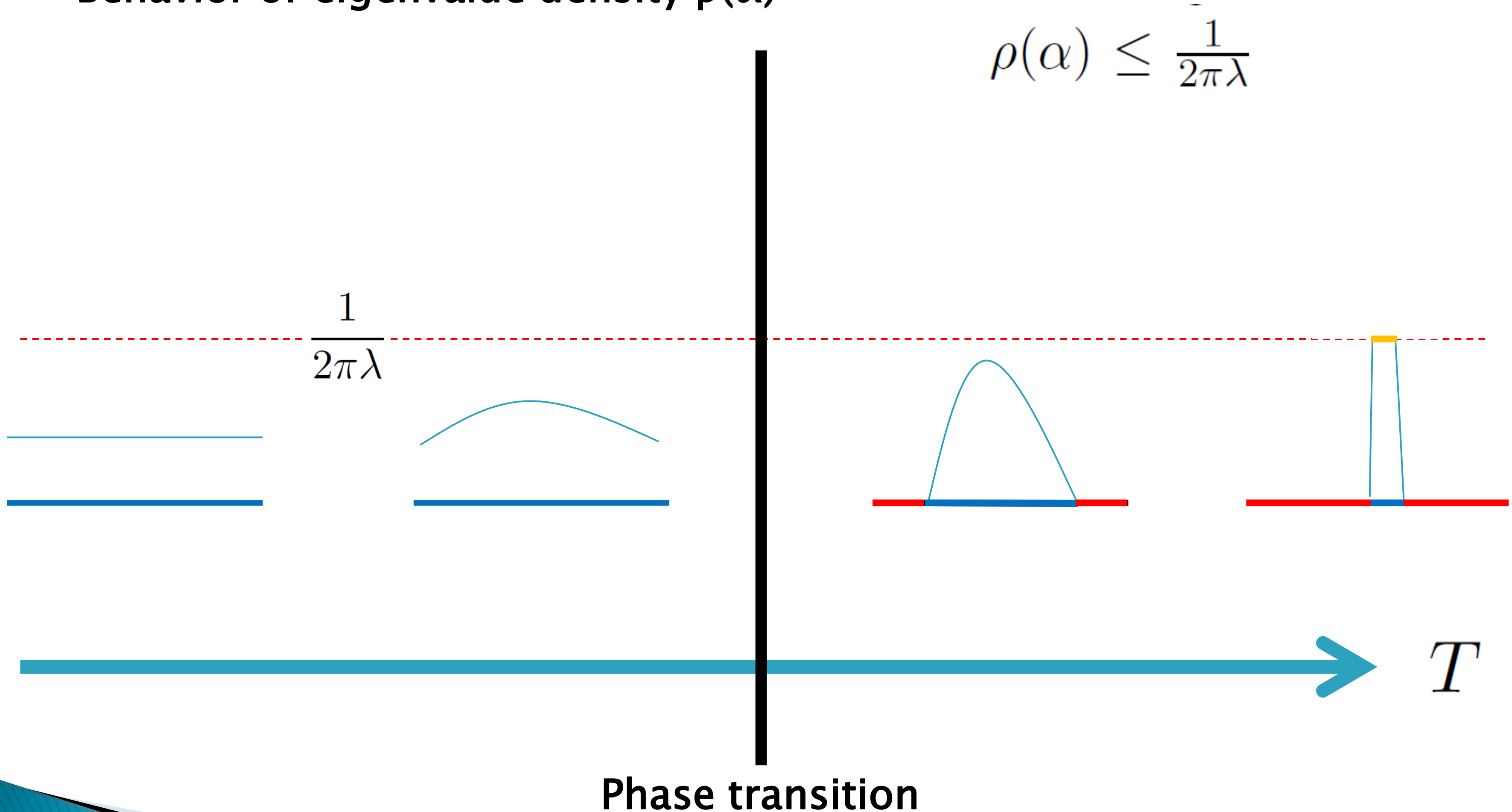
Behavior of eigenvalue density ρ(α)



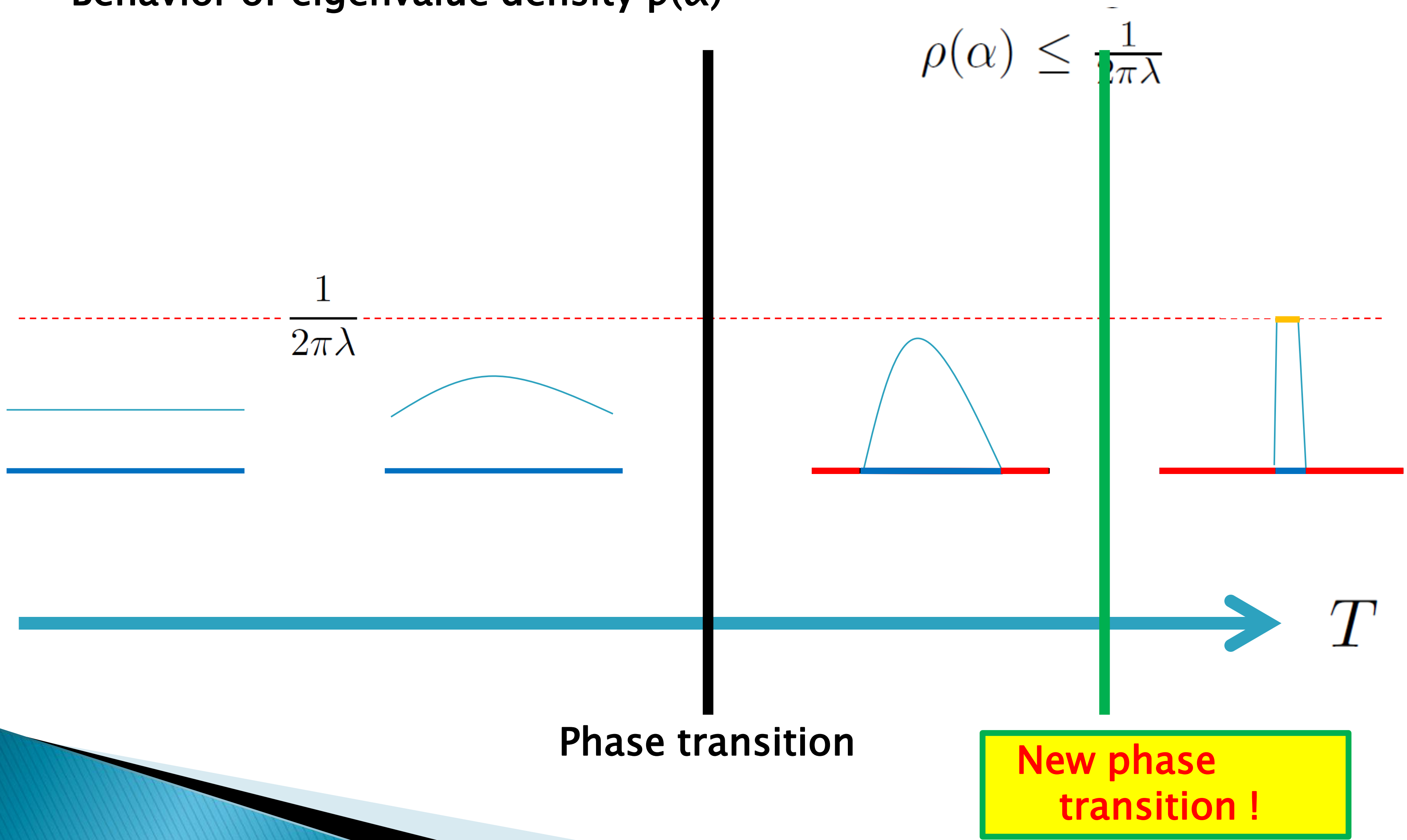
Behavior of eigenvalue density ρ(α)



Behavior of eigenvalue density ρ(α)

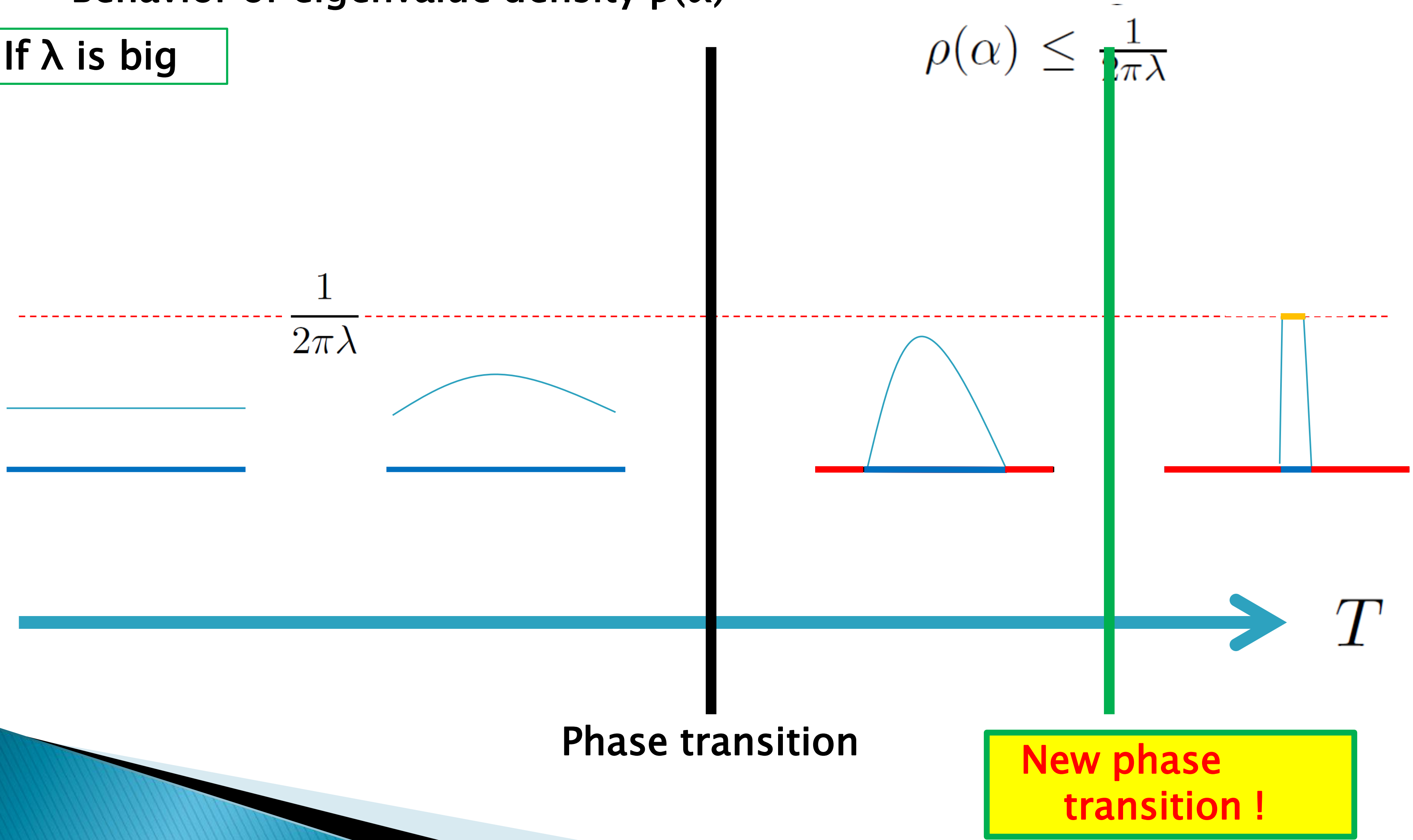


Behavior of eigenvalue density ρ(α)



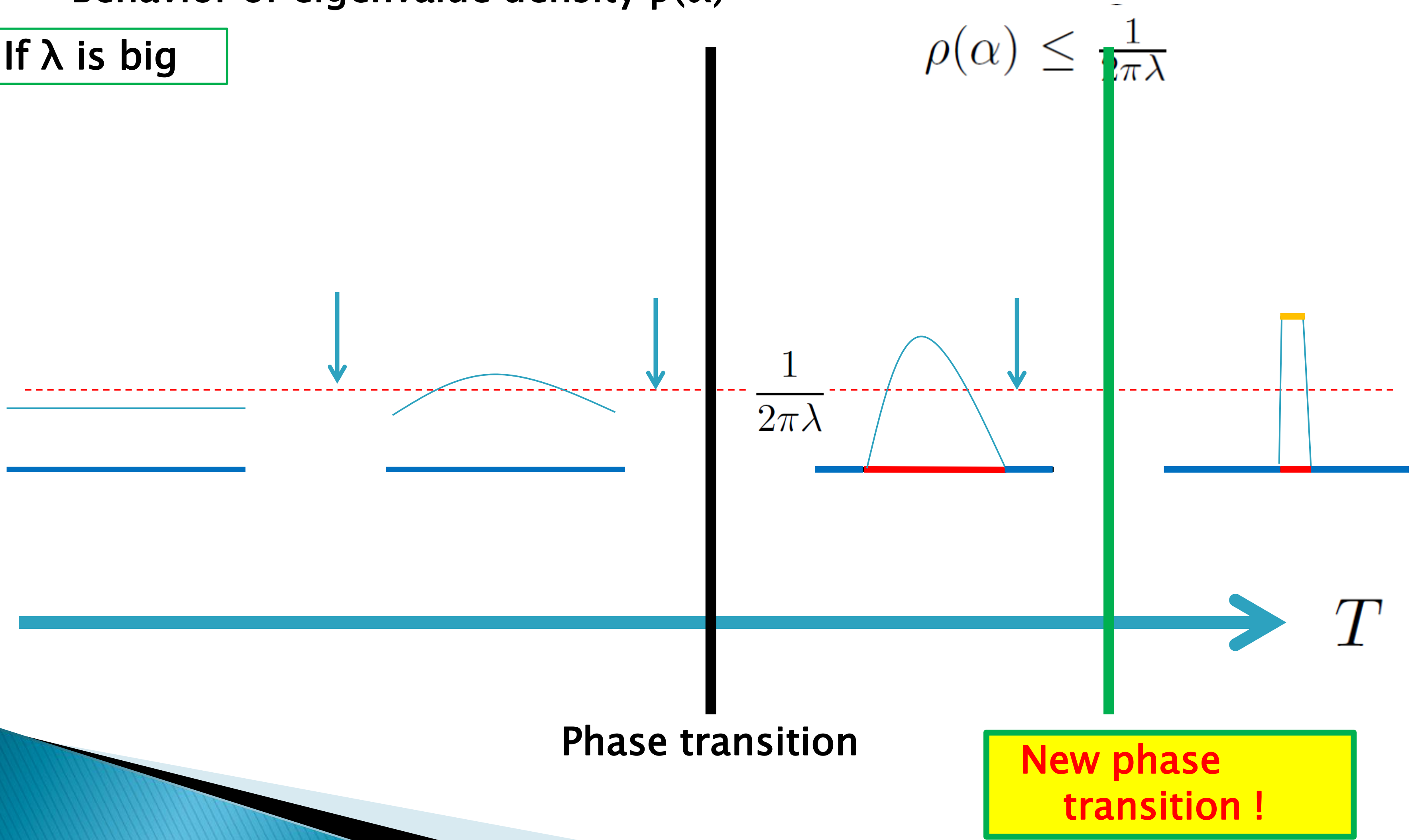
Behavior of eigenvalue density $\rho(\alpha)$

If λ is big

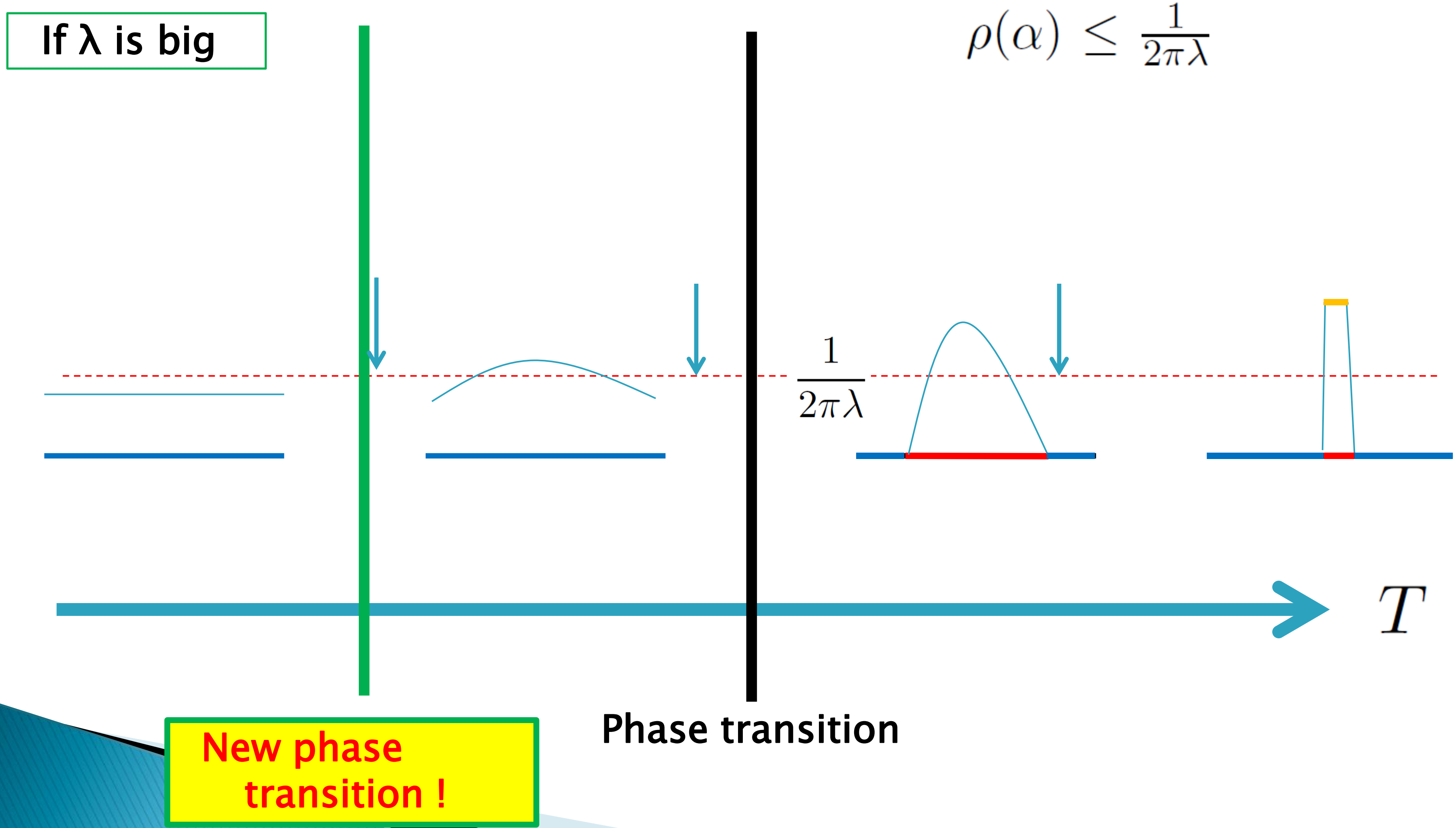


Behavior of eigenvalue density ρ(α)

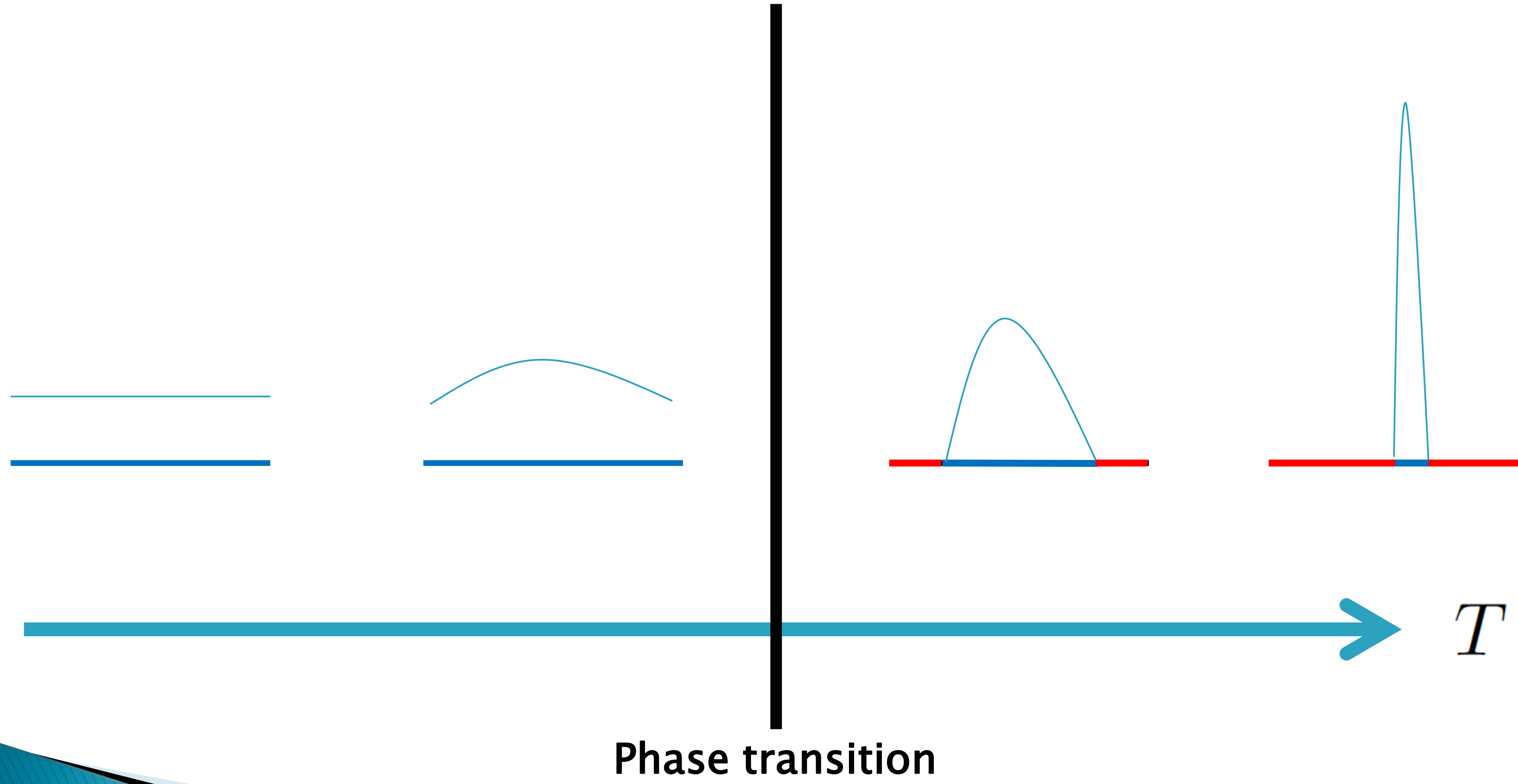
If λ is big



Behavior of eigenvalue density ρ(α)

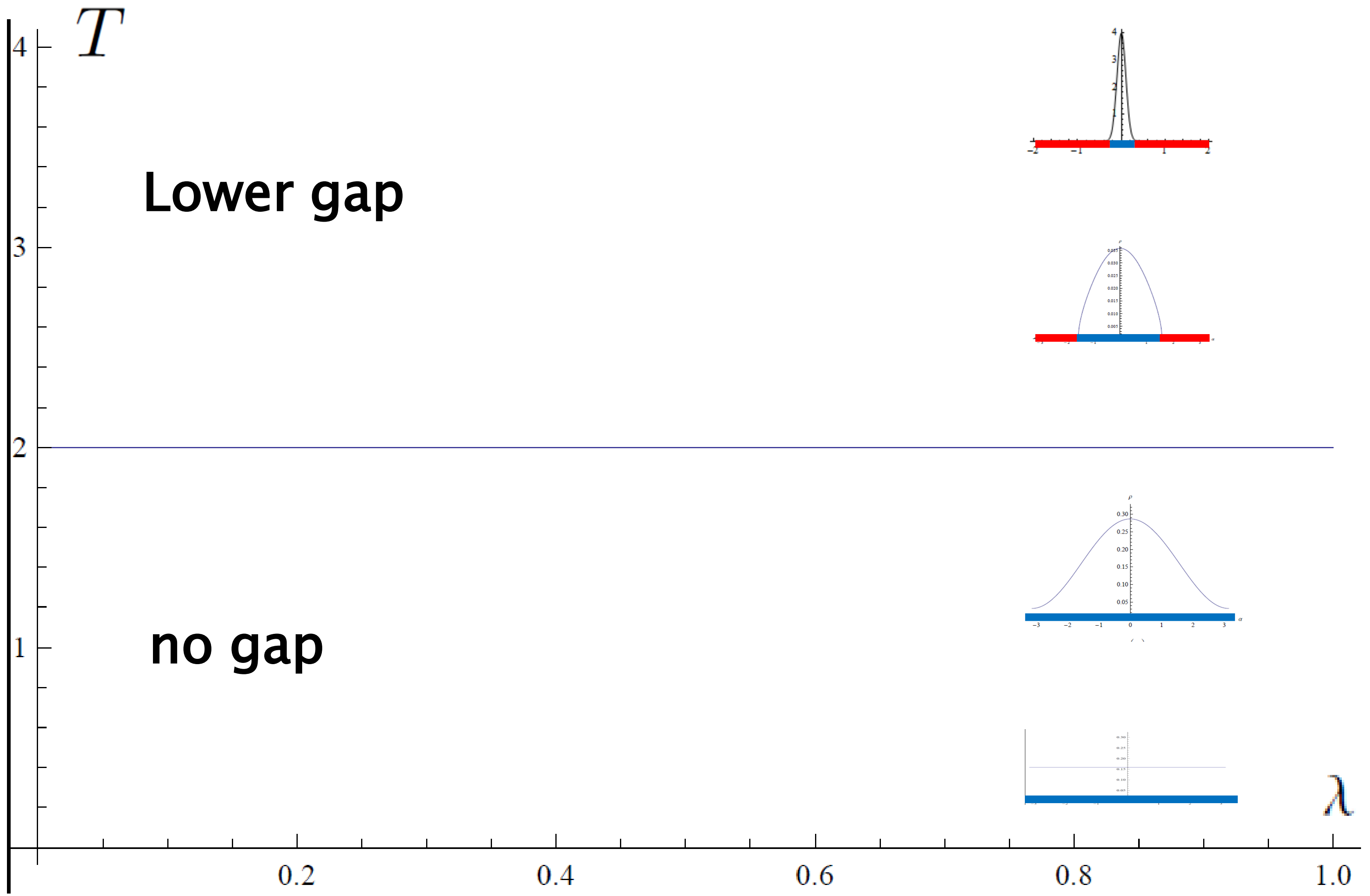


On the other hand in YM, there is no such saturation.

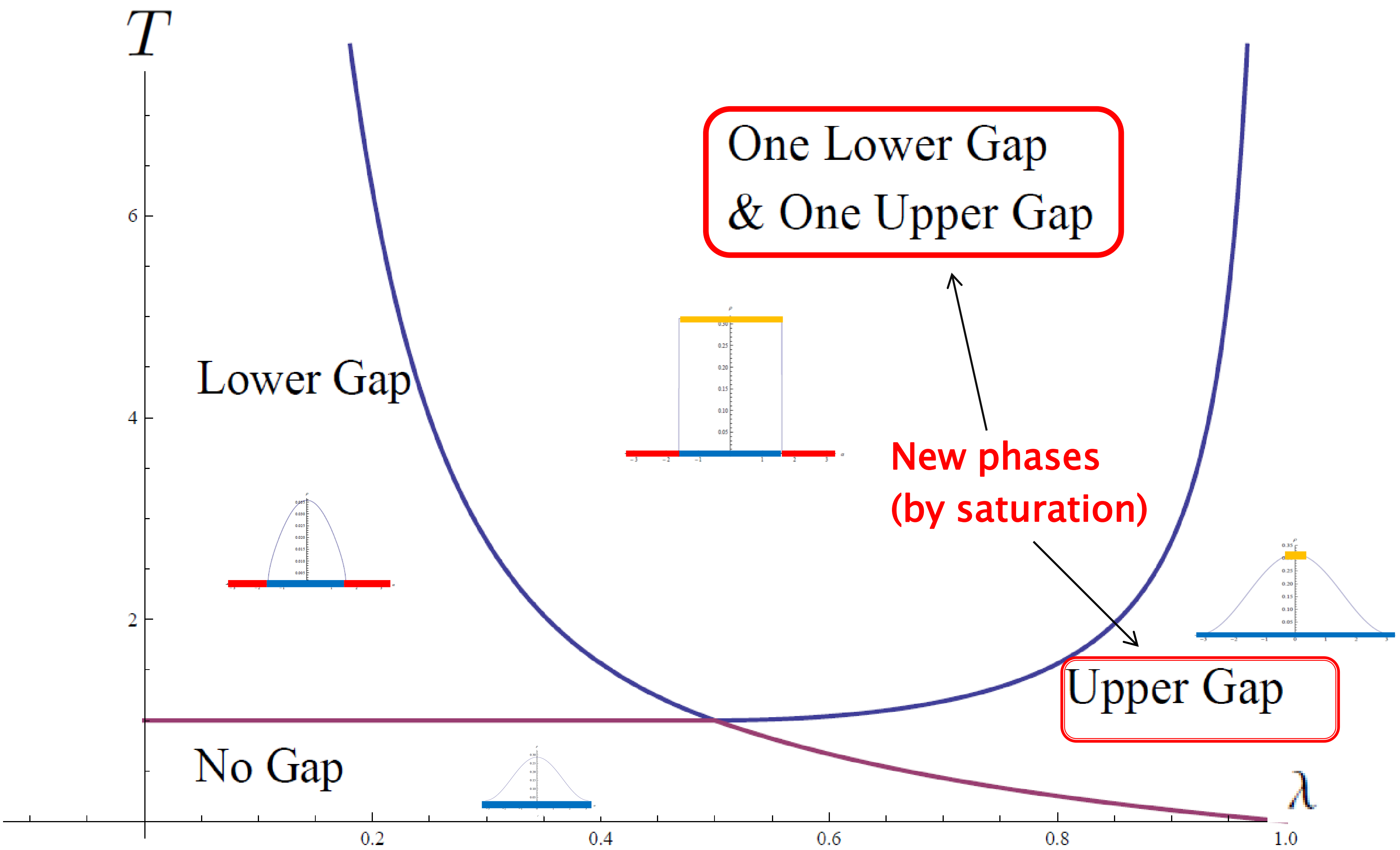


Phase structure

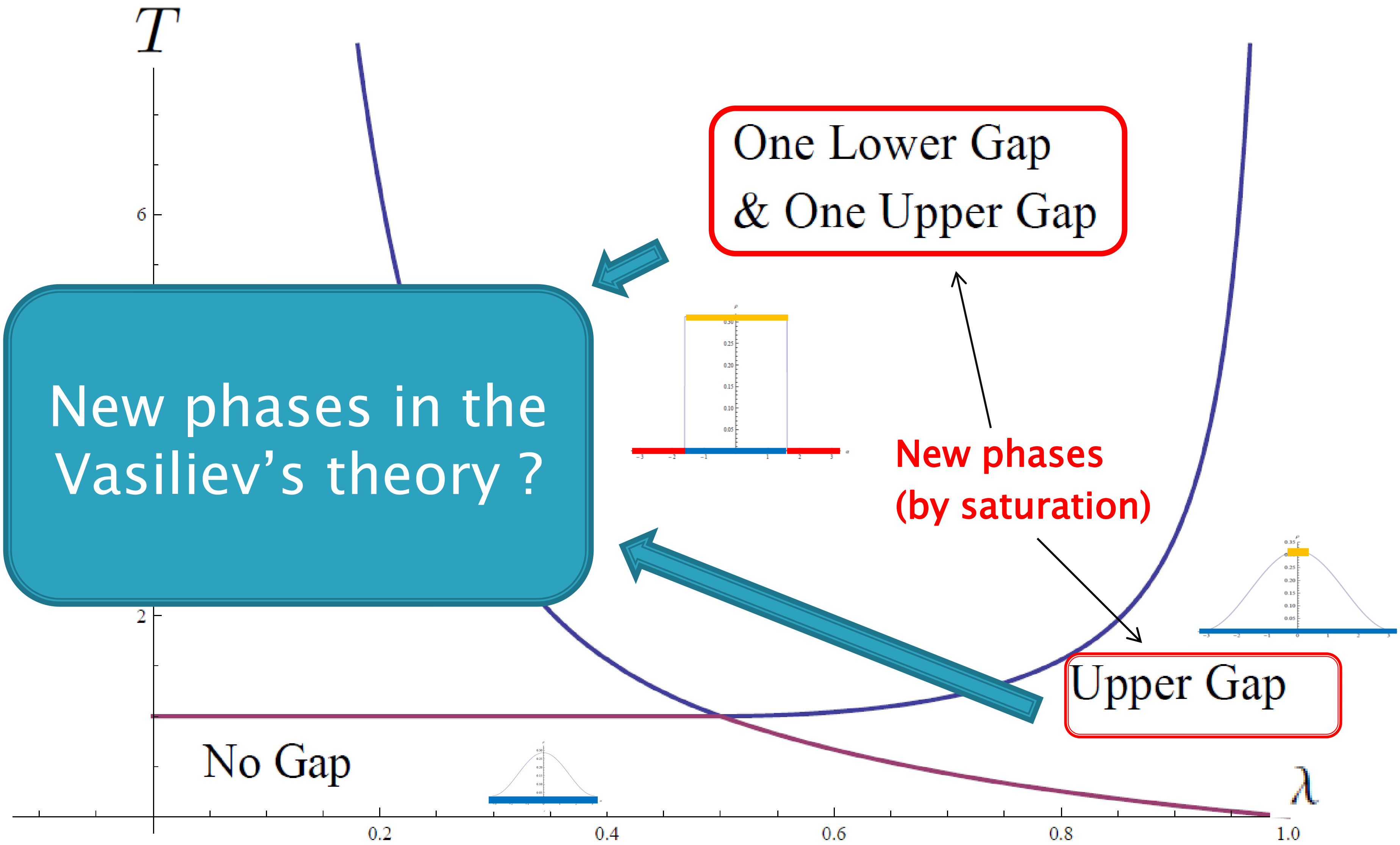
YM phase structure



CS phase structure



CS phase structure



By using this prescription to calculate the eigenvalue density, let us calculate the one in the actual CS matter theory and see the phase structure

3-2. Actual CS matter theories

~~Parity~~ Vasiliev's gravity theory

Gravity side

CS theory coupled to regular fermions

CS theory coupled to critical bosons

Chern-Simons side

3-3.Phase structure of the regular fermion CS theory

- [T.T 2013]

3-3-1. Action of the RF theory

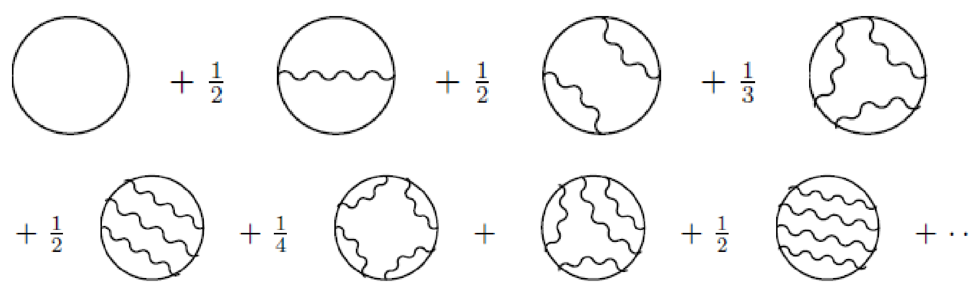
$$\text{Action}\begin{cases} S = S_{CS} + \displaystyle\int d^3x \ \ \bar{\psi}\gamma^{\mu} D_{\mu}\psi \\ S_{CS} = \dfrac{ik}{4\pi}\displaystyle\int \mathrm{Tr}\left(A \wedge dA + \dfrac{2}{3} A \wedge A \wedge A\right) \end{cases}$$

Regular → There are no coupling other than gauge coupling

3-3-1. Action of the RF theory

$$\text{Action}\begin{cases} S = S_{CS} + \displaystyle\int d^3x \;\; \bar{\psi}\gamma^\mu D_\mu \psi \\ S_{CS} = \dfrac{ik}{4\pi}\displaystyle\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \end{cases}$$

Integrate the matter fields,
Summing over the diagram including fermion,



[Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin, 2011]

3-3-1. Action of the RF theory

$$\text{Action}\begin{cases} S = S_{CS} + \int d^3x \;\; \bar{\psi}\gamma^{\mu} D_{\mu}\psi \\ S_{CS} = \frac{ik}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \end{cases}$$

$\Downarrow$

Integrate the matter fields,

Summing over the diagram including fermion,

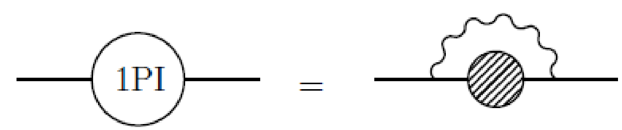
$$V(U) = -\frac{N^2\zeta}{6\pi}\left(\frac{\tilde{c}^3}{\lambda} - \tilde{c}^3 + 3\int_{-\pi}^{\pi} d\alpha\rho(\alpha)\int_{\tilde{c}}^{\infty} dy\; y(\ln(1+e^{-y-i\alpha}) + \ln(1+e^{-y+i\alpha}))\right)$$

$$\equiv V^{r.f}[\rho, N; \tilde{c}, \zeta],$$

Effective potential

3-3-1. Action of the RF theory

Action
$$\begin{cases} S = S_{CS} + \int d^3x \;\; \bar{\psi}\gamma^\mu D_\mu \psi \\ S_{CS} = \dfrac{ik}{4\pi}\displaystyle\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \end{cases}$$



Thermal mass $\Sigma_T = \tilde{c}^2 T^2$

$$V(U) = -\frac{N^2\zeta}{6\pi}\left(\frac{\tilde{c}^3}{\lambda} - \tilde{c}^3 + 3\int_{-\pi}^{\pi} d\alpha\rho(\alpha)\int_{\tilde{c}}^{\infty} dy\; y(\ln(1+e^{-y-i\alpha}) + \ln(1+e^{-y+i\alpha}))\right)$$
$$\equiv V^{r.f}[\rho, N; \tilde{c}, \zeta],$$

3-3-1. Action of the RF theory

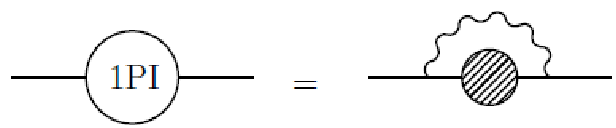
Equation determining the $\tilde{c}$

$$\tilde{c} = \lambda \int_{-\pi}^{\pi} d\alpha \ \rho(\alpha) \left(\ln 2 \cosh(\frac{\tilde{c} + i\alpha}{2}) + \ln 2 \cosh(\frac{\tilde{c} - i\alpha}{2}) \right) .$$

Gap equation

Derived by extremizing $V(U)$ *w.r.t.* $\tilde{c}$

Derived also from



3-3-2. Calculation of Free energy

$$
\begin{aligned}
F^{N}_{r.f} =& V^{r.f}[\rho, N] - N^2 \mathcal{P} \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \ \rho(\alpha)\rho(\beta) \log \left| 2 \sin \frac{\alpha - \beta}{2} \right| \\
=& V^{r.f}[\rho, N] + F_2[\rho, N].
\end{aligned}
$$

<u>Free energy density</u>

3-3-2. Calculation of Free energy

$$
\begin{aligned}
F^{N}_{r.f} =& V^{r.f}[\rho, N] - N^2 \mathcal{P} \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \;\rho(\alpha)\rho(\beta) \log \left| 2 \sin \frac{\alpha - \beta}{2} \right| \\
=& V^{r.f}[\rho, N] + F_2[\rho, N].
\end{aligned}
$$

Free energy density

3-3-2. Calculation of Free energy

$$F_{r.f}^{N} = V^{r.f}[\rho, N] - N^2 \mathcal{P} \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \ \rho(\alpha)\rho(\beta) \log \left| 2 \sin \frac{\alpha - \beta}{2} \right|$$
$$= V^{r.f}[\rho, N] + F_2[\rho, N].$$

Free energy density

In large N, the free energy is obtained by the extremizing the above (the saddle point equation.)

$$V'(\alpha_m) = \sum_{m \neq l} \cot \frac{\alpha_m - \alpha_l}{2}.$$

$$\Longleftrightarrow \quad V'(\alpha_0) = N\mathcal{P} \int d\alpha \cot \frac{\alpha_0 - \alpha}{2} \rho(\alpha)$$

3-3-2. Calculation of Free energy

$$V'(\alpha_0) = N\mathcal{P}\int d\alpha \cot\frac{\alpha_0-\alpha}{2}\rho(\alpha)$$

$$\tilde{c} = \lambda\int_{-\pi}^{\pi} d\alpha\ \rho(\alpha)\left(\ln 2\cosh(\frac{\tilde{c}+i\alpha}{2}) + \ln 2\cosh(\frac{\tilde{c}-i\alpha}{2})\right).$$

$$V(U) = -\frac{N^2\zeta}{6\pi}\left(\frac{\tilde{c}^3}{\lambda} - \tilde{c}^3 + 3\int_{-\pi}^{\pi} d\alpha\rho(\alpha)\int_{\tilde{c}}^{\infty} dy\ y(\ln(1+e^{-y-i\alpha}) + \ln(1+e^{-y+i\alpha}))\right)$$

$$0 \le \rho(\alpha) \le \frac{1}{2\pi\lambda}$$

By solving these equations we obtain the Eigenvalue density and we can see the phase structure.

3-3-3 Eigenvalue densities

- In No gap phase

$$\rho(\alpha) = \frac{1}{2\pi} - \frac{V_2 T^2}{2\pi^2 N} \sum_{m=1}^{\infty} (-1)^m \cos m\alpha \frac{1 + m\ \tilde{c}}{m^2} e^{-m\tilde{c}},$$

- In lower gap phase

$$\begin{aligned}\rho(\alpha) =& \frac{\zeta}{\sqrt{2}\pi^2} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}} \int_{\tilde{c}}^{\infty} dy \, \frac{y \cos \frac{\alpha}{2} \cosh \frac{y}{2}}{(\cosh y + \cos \alpha) \sqrt{(\cosh y + \cos b)}} \\ \equiv& \rho_{lg}^{r.f}(\zeta, \lambda; \tilde{c}, b; \alpha).\end{aligned}$$

3-3-3 Eigenvalue densities

- Upper gap phase

$$\rho(\alpha) = \frac{1}{2\pi\lambda} - \frac{\zeta}{\sqrt{2}\pi^2}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{a}{2}}\int_{\tilde{c}}^{\infty} dy \ \frac{y|\sin\frac{\alpha}{2}|\sinh\frac{y}{2}}{\sqrt{\cosh y + \cos a}(\cos\alpha + \cosh y)}$$
$$\equiv \rho_{ug}^{r.f}(\zeta, \lambda; \tilde{c}, a; \alpha).$$

- Two gap phase

$$\rho(\alpha) = \rho_{tg}^{r.f}(\alpha) = \rho_{1,tg}^{r.f}(\zeta, a, b, \tilde{c}; \alpha) + \rho_{2,tg}(\lambda, a, b; \alpha), \qquad \text{where}$$

$$\rho_{1,tg}^{r.f}(\zeta, a, b, \tilde{c}; \alpha) \equiv \frac{\zeta}{\pi^2}\mathcal{F}(a, b; \alpha)\int_{\tilde{c}}^{\infty} dy \frac{ye^{-y}}{\nu_{r.f}(a, b; y)}\left(\frac{|\sin\alpha|}{\cos\alpha + \cosh y}\right),$$

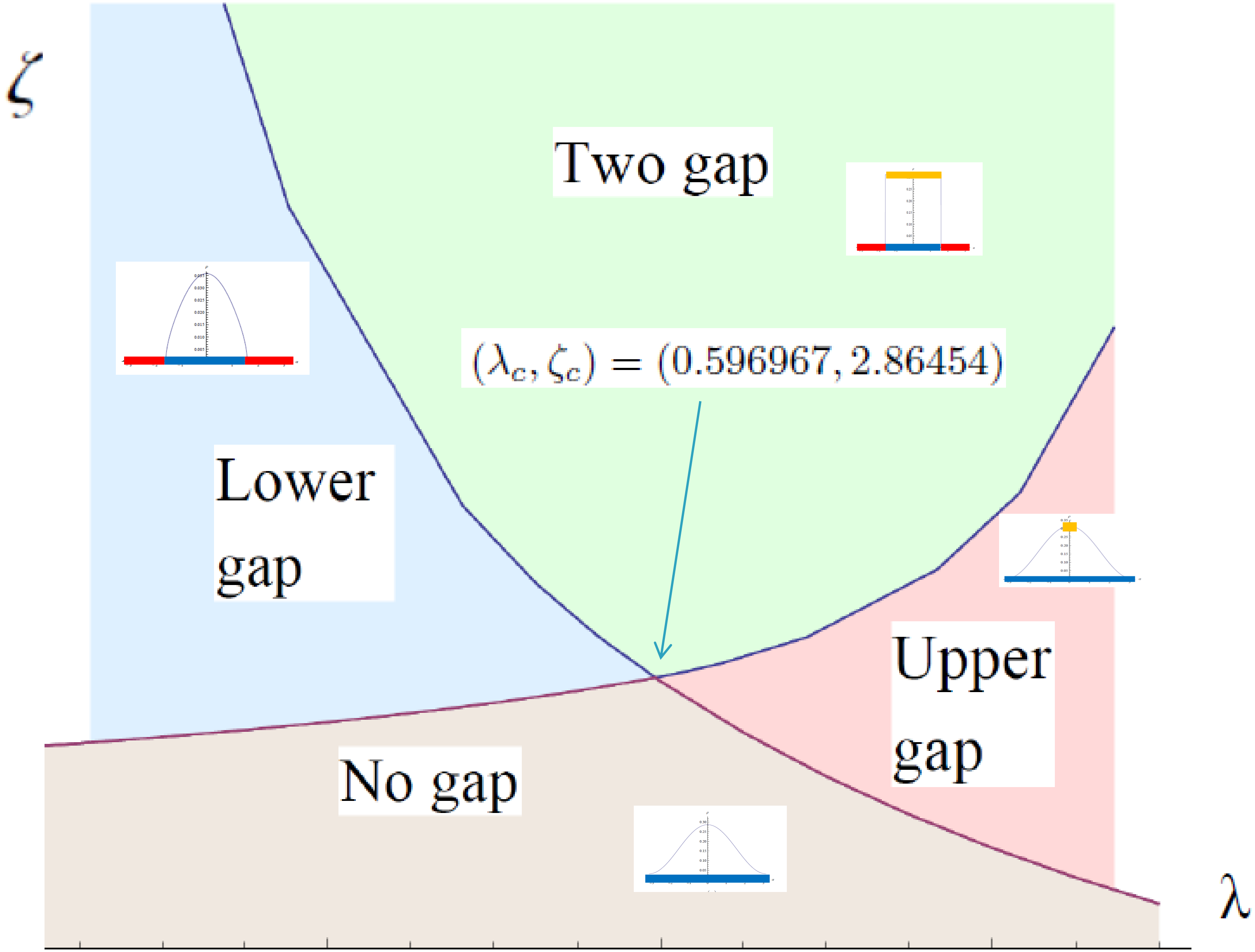
$$\rho_{2,tg}(\lambda, a, b; \alpha) \equiv \frac{|\sin\alpha|}{4\pi^2\lambda}\mathcal{F}(a, b; \alpha)\ I_1(a, b, \alpha),$$

$$\mathcal{F}(a, b, \alpha) \equiv \sqrt{(\sin^2\frac{\alpha}{2} - \sin^2\frac{a}{2})(\sin^2\frac{b}{2} - \sin^2\frac{\alpha}{2})},$$

$$\nu_{r.f}(a, b; y) \equiv \sqrt{(1 + 2e^{-y}\cos a + e^{-2y})(1 + 2e^{-y}\cos b + e^{-2y})},$$

$$I_1(a, b; \alpha) \equiv \int_{-a}^{a} \frac{d\theta}{(\cos\theta - \cos\alpha)\sqrt{(\sin^2\frac{a}{2} - \sin^2\frac{\theta}{2})\ (\sin^2\frac{b}{2} - \sin^2\frac{\theta}{2})}}.$$

3-3-4. Phase structure of RF theory



3-4.Phase structure of the Critical Boson CS theory

- [T.T 2013]

3-4-1. Action of the CB theory

$$\text{Action}\ \begin{cases} S = S_{CS} + \int d^3x\left(D_\mu\bar{\phi}D^\mu\phi + C\bar{\phi}\phi\right) \\ S_{CS} = \dfrac{ik}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \dfrac{2}{3}A\wedge A\wedge A\right) \end{cases}$$

3-4-1. Action of the CB theory

Action
$$\begin{cases} S = S_{CS} + \int d^3x \left(D_\mu \bar{\phi} D^\mu \phi + C\bar{\phi}\phi \right) \\ S_{CS} = \dfrac{ik}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \dfrac{2}{3} A \wedge A \wedge A \right) \end{cases}$$

CS gauged version of the U(N) Wilson Fisher theory.

\`C" is a field dynamical field variable.
(Source field with respect to bilinear $\bar{\phi}\phi$)
(Here after obtaining the free energy in terms of "C", and we will
Integrate the C at last and we obtain the form of the free energy.

3-4-1. Action of the CB theory

$$\text{Action}\ \begin{cases} S = S_{CS} + \displaystyle\int d^3x \left(D_\mu \bar{\phi} D^\mu \phi + C \bar{\phi}\phi\right) \\ S_{CS} = \dfrac{ik}{4\pi} \displaystyle\int \mathrm{Tr}\left(A \wedge dA + \dfrac{2}{3} A \wedge A \wedge A\right) \end{cases}$$

<u>Integrate the matter fields,</u>
<u>Summing over the diagram including scalar boson</u>

[Jain, Trivedi, Wadia, Yokoyama, 2012]
[Aharony, Giombi, Gur-Ari, Maldacena, Yacoby, 2012]

3-4-1. Action of the CB theory

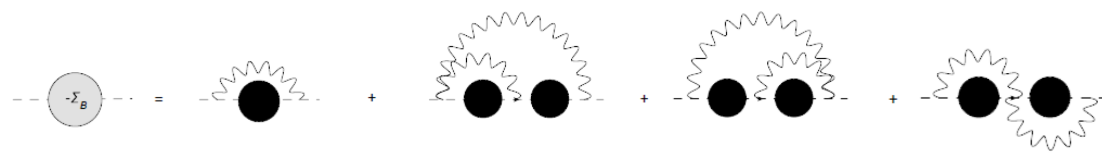
$$\text{Action} \begin{cases} S = S_{CS} + \int d^3x \left(D_\mu \bar{\phi} D^\mu \phi + C \bar{\phi}\phi \right) \\ S_{CS} = \frac{ik}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \end{cases}$$

$$V(U) = -\frac{N^2\zeta}{6\pi}\sigma^3 + \frac{N^2\zeta}{2\pi}\int_\sigma^\infty dy \int_{-\pi}^{\pi} d\alpha \; y\rho(\alpha) \left(\ln(1-e^{-y+i\alpha}) + \ln(1-e^{-y-i\alpha}) \right)$$
$$\equiv V^{c.b}[\rho, N],$$

Effective potential

3-4-1. Action of the CB theory

$$\text{Action}\begin{cases} S = S_{CS} + \int d^3x \left(D_\mu \bar{\phi} D^\mu \phi + C \bar{\phi}\phi\right) \\ S_{CS} = \dfrac{ik}{4\pi}\displaystyle\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \end{cases}$$



thermal mass as $\sigma^2 T^2$

$$V(U) = -\frac{N^2\zeta}{6\pi}\sigma^3 + \frac{N^2\zeta}{2\pi}\int_\sigma^\infty dy \int_{-\pi}^{\pi} d\alpha\; y\rho(\alpha)\left(\ln(1-e^{-y+i\alpha}) + \ln(1-e^{-y-i\alpha})\right)$$

$$\equiv V^{c.b}[\rho, N],$$

3-4-1. Action of the CB theory

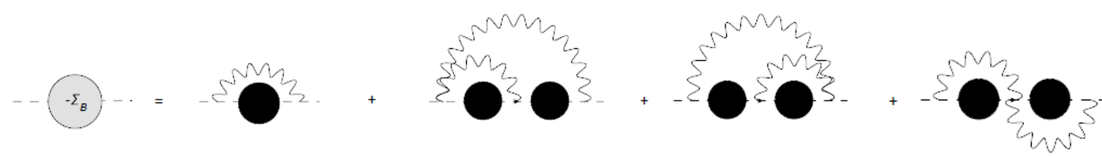
Equation determining the σ

$$\int_{-\pi}^{\pi} \rho(\alpha) \left(\ln 2 \sinh(\frac{\sigma - i\alpha}{2}) + \ln 2 \sinh(\frac{\sigma + i\alpha}{2}) \right) = 0.$$

Gap equation

Derived by extremizing $V(U)$ *w.r.t.* σ

Obtained also by



3-4-2. Calculation of Free energy

$$F^N_{c.b} = V^{c.b}[\rho, N] - N^2 \mathcal{P} \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \; \rho(\alpha)\rho(\beta) \log \left| 2 \sin \frac{\alpha - \beta}{2} \right|$$

$$= V^{c.b}[\rho, N] + F_2[\rho, N].$$

Free energy density

In large N, the free energy is obtained by the extremizing the above (the saddle point equation.)

$$V'(\alpha_m) = \sum_{m \neq l} \cot \frac{\alpha_m - \alpha_l}{2}.$$

$$\Longleftrightarrow \quad V'(\alpha_0) = N \mathcal{P} \int d\alpha \cot \frac{\alpha_0 - \alpha}{2} \rho(\alpha)$$

3-4-2. Calculation of Free energy

$$V'(\alpha_0) = N\mathcal{P}\int d\alpha \cot\frac{\alpha_0-\alpha}{2}\rho(\alpha)$$

$$\int_{-\pi}^{\pi}\rho(\alpha)\left(\ln 2\sinh(\frac{\sigma-i\alpha}{2}) + \ln 2\sinh(\frac{\sigma+i\alpha}{2})\right) = 0.$$

$$V(U) = -\frac{N^2\zeta}{6\pi}\sigma^3 + \frac{N^2\zeta}{2\pi}\int_{\sigma}^{\infty}dy\int_{-\pi}^{\pi}d\alpha\ y\rho(\alpha)\left(\ln(1-e^{-y+i\alpha}) + \ln(1-e^{-y-i\alpha})\right)$$

$$0 \le \rho(\alpha) \le \frac{1}{2\pi\lambda}$$

By solving these equations we obtain the
Eigenvalue density and we can discuss the phase transition.

3-4-3 Eigenvalue densities

- In No gap phase

$$\rho(\alpha) = \frac{1}{2\pi} + \frac{T^2 V_2}{2N\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\alpha) e^{-n\sigma} (1 + n\sigma).$$

- In Lower gap phase

$$\rho(\alpha) = \frac{\zeta}{\sqrt{2}\pi^2} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}} \int_{\sigma}^{\infty} dy \frac{y \sinh \frac{y}{2} \cos \frac{\alpha}{2}}{\sqrt{\cosh y - \cos b}(\cosh y - \cos \alpha)}$$
$$\equiv \rho_{lg}^{c.b}(\zeta, \lambda; \sigma, b; \alpha).$$

3-4-3 Eigenvalue densities

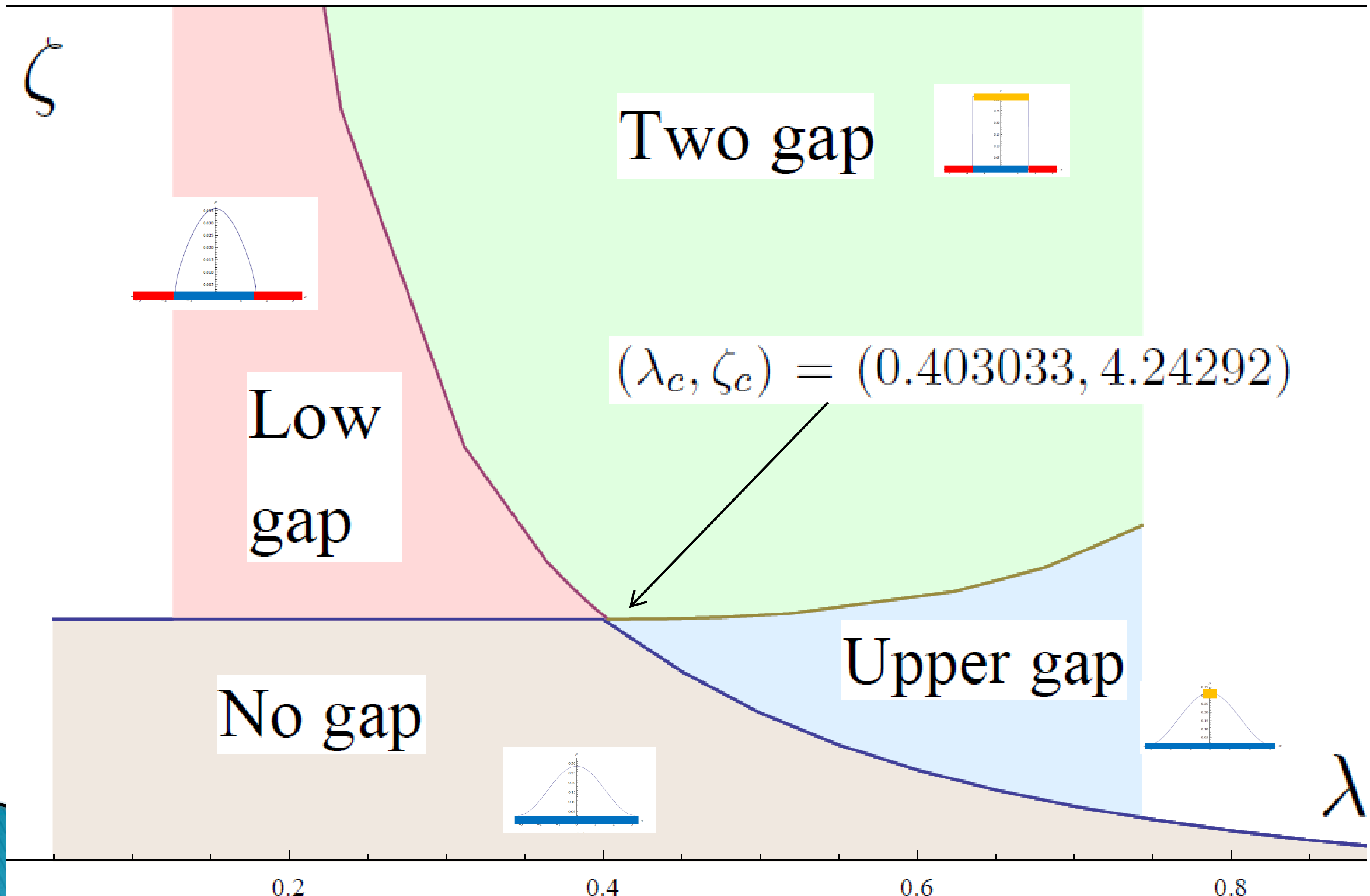
- Upper gap phase

$$\rho(\alpha) = \frac{1}{2\pi\lambda} - \frac{\zeta}{\sqrt{2}\pi^2}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{a}{2}}\int_\sigma^\infty dy\ \frac{y|\sin\frac{\alpha}{2}|\cosh\frac{y}{2}}{\sqrt{\cosh y - \cos a}(\cosh y - \cos\alpha)}$$
$$\equiv \rho_{ug}^{c.b}(\zeta, \lambda; \sigma, a; \alpha).$$

- Two gap phase

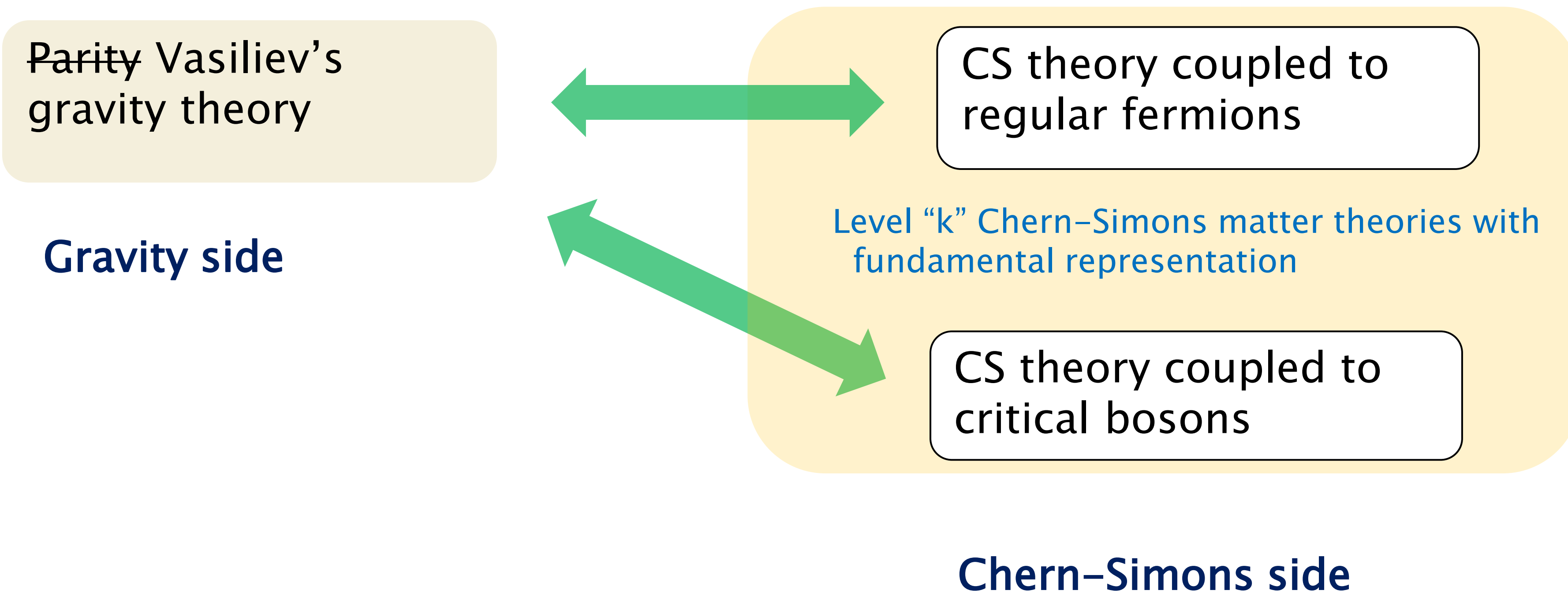
$$\rho(\alpha) = \rho_{tg}^{c.b}(\alpha) = \rho_{1,tg}^{c.b}(\zeta, a, b, \tilde{c}; \alpha) + \rho_{2,tg}(\lambda, a, b; \alpha), \qquad \text{where}$$
$$\rho_{1,tg}^{c.b}(\zeta, a, b, \tilde{c}; \alpha) \equiv -\frac{\zeta}{\pi^2}\mathcal{F}(a, b; \alpha), \int_{\tilde{c}}^\infty dy \frac{ye^{-y}}{\nu_{c.b}(a, b; y)}\left(\frac{|\sin\alpha|}{\cosh y - \cos\alpha}\right)$$
$$\nu_{c.b}(a, b; y) \equiv \sqrt{(e^{-2y} - 2e^{-y}\cos a + 1)(e^{-2y} - 2e^{-y}\cos b + 1)}.$$

3-4-4 Phase structure of CB theory



4.AdS-CFT-CFT triality and the Level-rank duality in the CS theory

4-1 AdS-CFT-CFT triality



4-1 AdS-CFT-CFT triality

~~Parity~~ Vasiliev's gravity theory

Gravity side

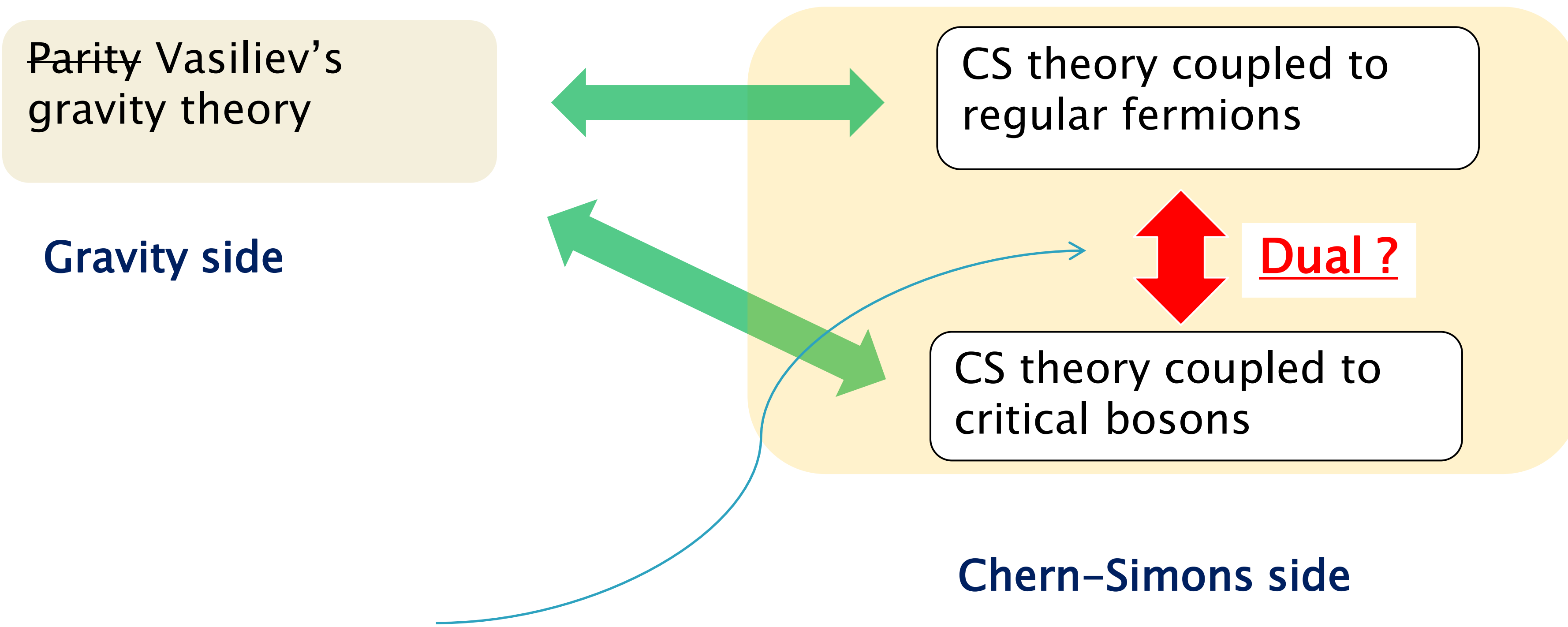
CS theory coupled to regular fermions

Dual ?

CS theory coupled to critical bosons

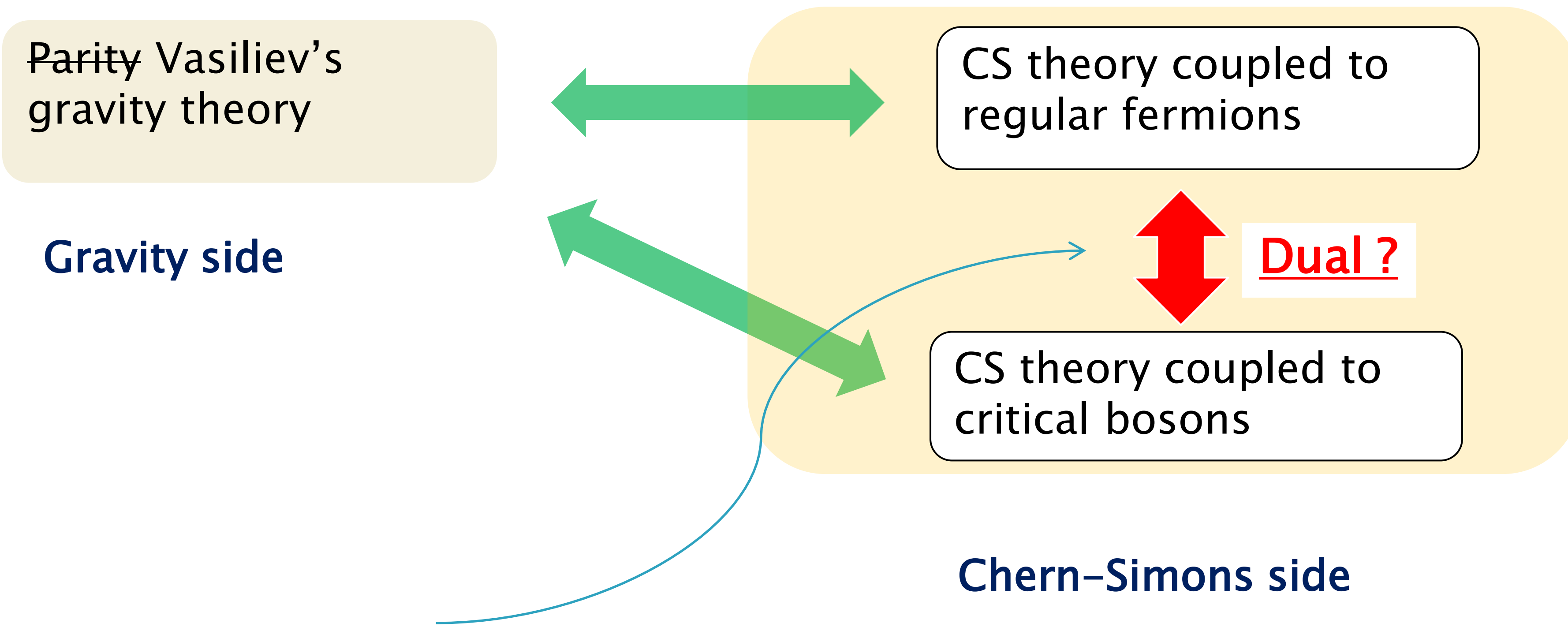
Chern-Simons side

4-1 AdS-CFT-CFT triality



We need to establish this duality.

4-1 AdS-CFT-CFT triality



There are former works which tried to show it but they did not succeed

4-1 AdS-CFT-CFT triality

~~Parity~~ Vasiliev's gravity theory

Gravity side

CS theory coupled to regular fermions

Dual ?

CS theory coupled to critical bosons

Chern-Simons side

There are former works which tried to show it but they did not succeed

Because they neglect the new phase caused by the holonomy and the linear coupling to magnetic fields in S2

4-1 AdS-CFT-CFT triality

~~Parity~~ Vasiliev's gravity theory

CS theory coupled to regular fermions

Dual ?

Gravity side

CS theory coupled to critical bosons

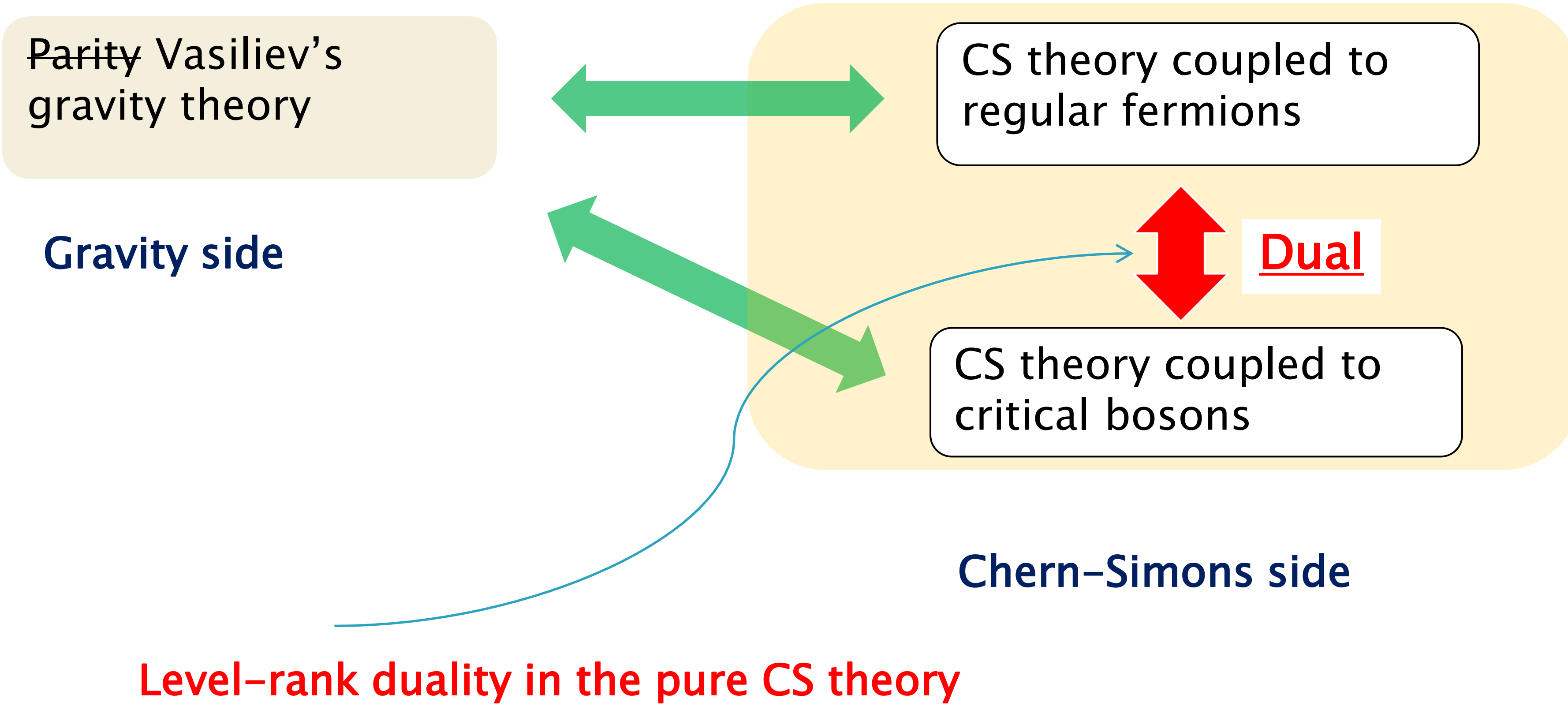
We have taken into account it, and succeeded

Chern-Simons side

There are former works which tried to show it but they did not succeed

Because they neglect the contribution of new phases caused by the holonomy and the linear coupling to magnetic fields in S2

4-1 AdS-CFT-CFT triality



4-2 Free energy of CS matter theory in terms of pure CS theory.

$$Z_{\mathrm{CS}} = \int DAe^{i\frac{k}{4\pi}\mathrm{Tr}\int\left(AdA+\frac{2}{3}A^3\right)-S_{eff}(U)}$$

$$= \int DAe^{i\frac{k}{4\pi}\mathrm{Tr}\int\left(AdA+\frac{2}{3}A^3\right)-T^2\int d^2x\sqrt{g}\ v(U)}$$

$$= \boxed{\langle e^{-T^2\int d^2x\sqrt{g}\ v(U(x))}\rangle_{N,k}}$$

Expectation value in the pure U(N) level k Chern-Simons theory.

Any expectation value $\langle\Psi\rangle_{N,k}$ in the pure U(N) level k Chern–Simons theory

⇓

written by polynomial of tr(U) (trace in fundamental rep.)

through the character expansion

$$\langle\Psi\rangle_{N,k} = \sum_Y c_Y \chi_Y(U)$$

with Schur polynomial

$$\langle\Psi\rangle_{N,k} = \boxed{\chi_Y(U) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_Y(\sigma)\left(\prod_{m=1}^{n}(\mathrm{Tr}U^m)^{k_m}\right)}$$

4-3. Level-rank duality in CS

Level k $U(N)$ pure CS theory

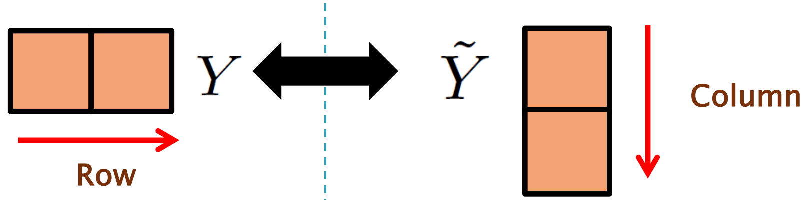
Level k $U(k-N)$ pure CS theory

4-3. Level-rank duality in CS

Level k U(N) pure CS theory

Level k U(k-N) pure CS theory

$$\langle \sum_Y c_Y \chi_Y(U) \rangle_{k,N} \Longleftrightarrow \langle \sum_Y c_Y \chi_{\tilde{Y}}(U) \rangle_{k,k-N}$$



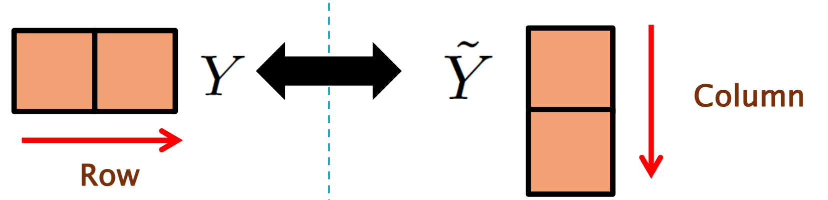
In young table, row and column is flipped

4-3. Level-rank duality in CS

Level k U(N) pure CS theory

Level k U(k-N) pure CS theory

$$\langle \sum_Y c_Y \chi_Y(U) \rangle_{k,N} \longleftrightarrow \langle \sum_Y c_Y \chi_{\tilde{Y}}(U) \rangle_{k,k-N}$$



In young table, row and column is flipped

$$\boxed{\chi_Y(U) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_Y(\sigma) \left(\prod_{m=1}^{n} (\mathrm{Tr} U^m)^{k_m} \right)} \longleftrightarrow \boxed{\chi_{\tilde{Y}}(U) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_Y(\sigma) \left(\prod_{m=1}^{n} ((-1)^{m+1} \mathrm{Tr} U^m)^{k_m} \right).}$$

Schur polynomial

$$\boxed{\mathrm{Tr} U^n} \leftrightarrow \boxed{(-1)^{n+1} \mathrm{Tr} U^n.}$$

4-3. Level-rank duality in CS

Level k U(N) pure CS theory

Level k U(k-N) pure CS theory

In Current CS matter theory,

4-3. Level-rank duality in CS

Level k U(N) pure CS theory

Level k U(k-N) pure CS theory

In Current CS matter theory,

$$\boxed{\langle e^{V(\mathrm{Tr}U^n)}\rangle_{N,k}} = \boxed{\langle e^{V\left((-1)^{n+1}\mathrm{Tr}U^n\right)}\rangle_{k-N,k}}$$

Z_{cs} in critical boson

Z_{cs} in regular fermion

4-3. Level-rank duality in CS

Level k U(N) pure CS theory ⟺ Level k U(k-N) pure CS theory

In Current CS matter theory,

$$\boxed{\langle e^{V(\mathrm{Tr}U^n)}\rangle_{N,k}} = \boxed{\langle e^{V((-1)^{n+1}\mathrm{Tr}U^n)}\rangle_{k-N,k}}$$

Z_{CS} in critical boson ‖ Z_{CS} in regular fermion

CFT-CFT duality in CS side

Realized by the interpolation

$$\boxed{\mathrm{Tr}U^n} \leftrightarrow \boxed{(-1)^{n+1}\mathrm{Tr}U^n}.$$

Level k U(N) pure CS theory

Level k U(k−N) pure CS theory

$$\boxed{\mathrm{Tr}U^n} \leftrightarrow \boxed{(-1)^{n+1}\mathrm{Tr}U^n.}$$

Level k U(N) pure CS theory

Level k U(k−N) pure CS theory

$$\boxed{\mathrm{Tr} U^n} \leftrightarrow \boxed{(-1)^{n+1}\mathrm{Tr} U^n.}$$

$$\boxed{\mathrm{Tr}_{U(N)} U^n = N \int d\alpha \rho(\alpha) e^{in\alpha} = N \rho_{-n}}$$

=

$$\begin{aligned} (-1)^n \mathrm{Tr}_{U(k-N)} U^n &= (k-N)(-1)^n \int d\alpha \tilde{\rho}(\alpha) e^{in\alpha} \\ &= (-1)^n (k-N) \tilde{\rho}_{-n} \end{aligned}$$

Level k U(N) pure CS theory

Level k U(k−N) pure CS theory

$$\boxed{\mathrm{Tr}U^n} \leftrightarrow \boxed{(-1)^{n+1}\mathrm{Tr}U^n}.$$

$$\mathrm{Tr}_{U(N)}U^n = N\int d\alpha\,\rho(\alpha)e^{in\alpha} = N\rho_{-n}$$

$$=$$

$$(-1)^n\mathrm{Tr}_{U(k-N)}U^n = (k-N)(-1)^n\int d\alpha\,\tilde{\rho}(\alpha)e^{in\alpha} = (-1)^n(k-N)\tilde{\rho}_{-n}$$

$$\tilde{\rho}(\alpha) = \sum_n \frac{\tilde{\rho}_n}{2\pi}e^{in\alpha} = \frac{1}{2\pi(1-\lambda)} - \frac{\lambda}{1-\lambda}\rho(\alpha+\pi)$$

Duality relationship in terms of eigenvalue density

4-4.Discussion on the level-rank duality

- [T.T 2013]

4-4-1 duality relationship

Level k U(k-N)
RF theory

Level k U(N)
CB theory

Relationship between eigenvalue density

$$\boxed{\rho^{r.f}(\alpha) = \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\left(\frac{1}{2\pi\lambda_{c.b}} - \rho^{c.b}(\alpha+\pi)\right),}$$

With $\tilde{c} = \sigma.$

$$\lambda_{r.f} = 1 - \lambda_{c.b}, \qquad \zeta_{r.f} = \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\zeta_{c.b}, \qquad \left(\frac{N}{k} = \lambda_{c.b}, \quad \frac{k-N}{k} = \lambda_{r.f}\right).$$

4-4-1 duality relationship

Let us confirm

$$\boxed{\rho^{r.f}(\alpha) = \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\left(\frac{1}{2\pi\lambda_{c.b}} - \rho^{c.b}(\alpha+\pi)\right),}$$

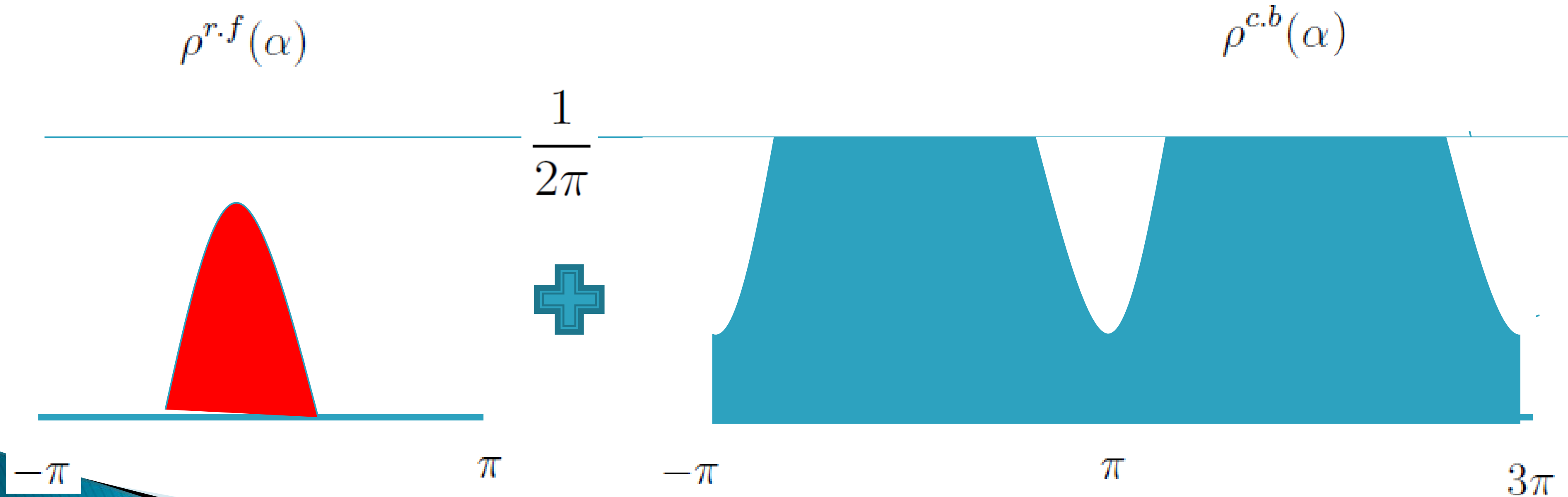
Equivalent to

$$\boxed{\lambda_{r.f}\rho_{r.f}(\alpha) + \lambda_{c.b}\rho_{c.b}(\pi+\alpha) = \frac{1}{2\pi}}$$

4-4-1 duality relationship

Let us confirm

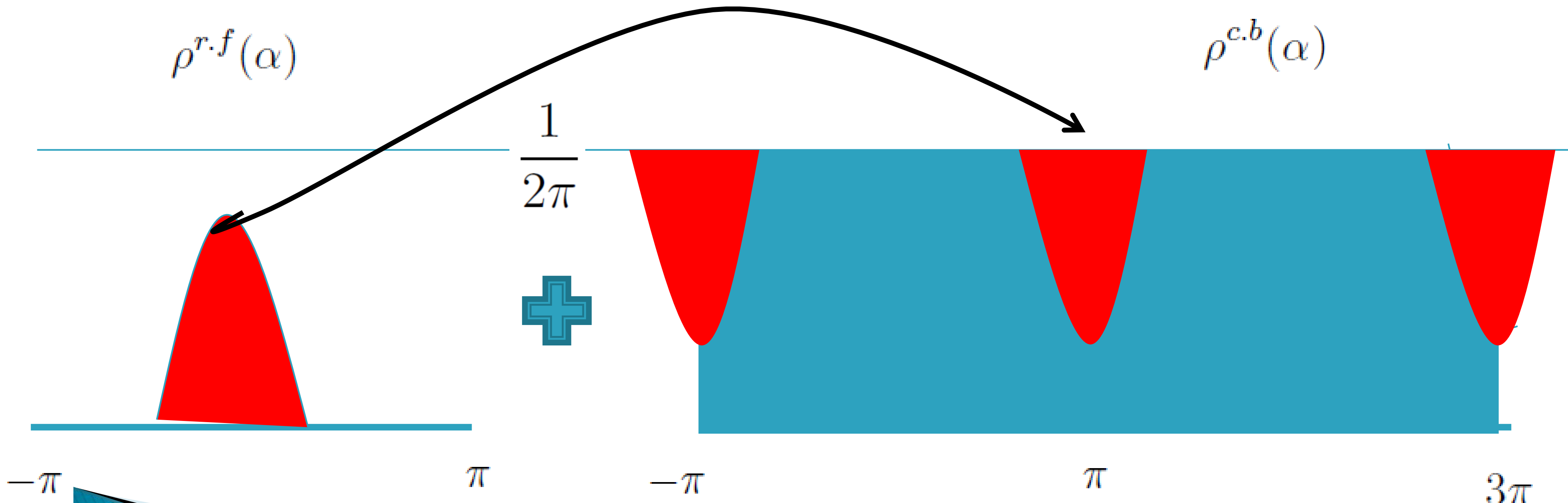
$$\boxed{\lambda_{r.f}\rho_{r.f}(\alpha)+\lambda_{c.b}\rho_{c.b}(\pi+\alpha)=\frac{1}{2\pi}}$$



4-4-1 duality relationship

Let us confirm

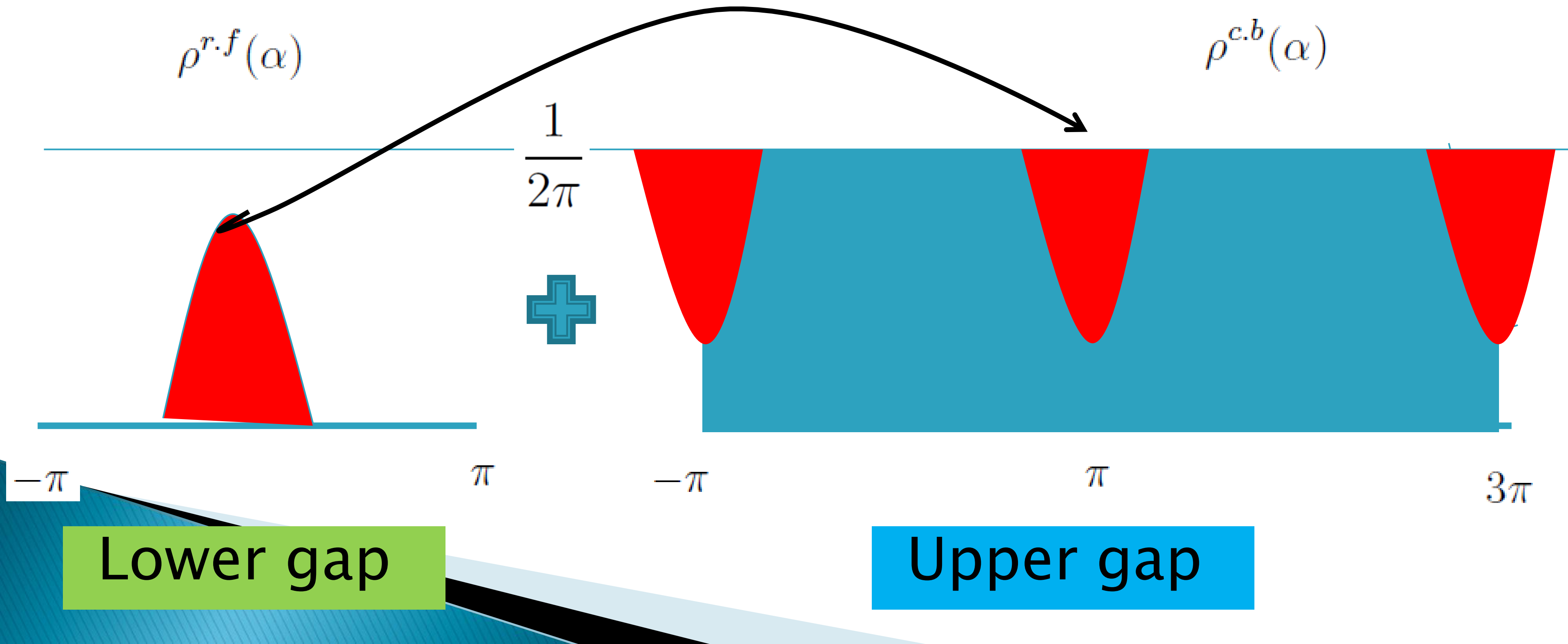
$$\boxed{\lambda_{r.f}\rho_{r.f}(\alpha)+\lambda_{c.b}\rho_{c.b}(\pi+\alpha)=\frac{1}{2\pi}}$$



4-4-1 duality relationship

Let us confirm

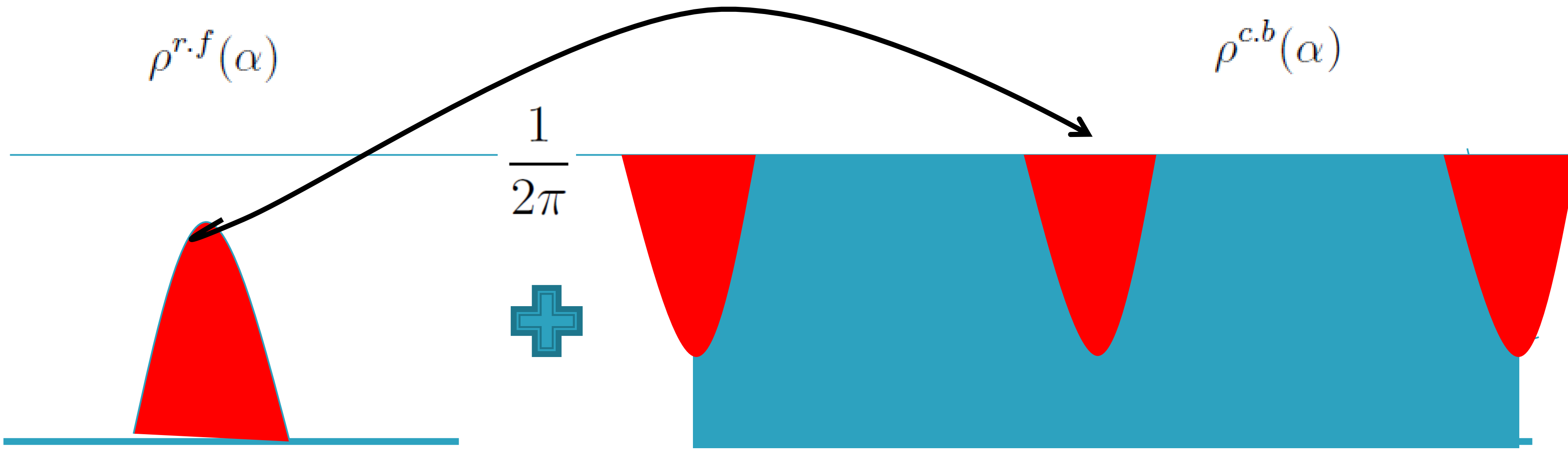
$$\boxed{\lambda_{r.f}\rho_{r.f}(\alpha)+\lambda_{c.b}\rho_{c.b}(\pi+\alpha)=\frac{1}{2\pi}}$$



4-4-1 duality relationship

Let us confirm

$$\boxed{\lambda_{r.f}\rho_{r.f}(\alpha) + \lambda_{c.b}\rho_{c.b}(\pi+\alpha) = \frac{1}{2\pi}}$$



Presence of the upper limit plays crucial role for the duality !!

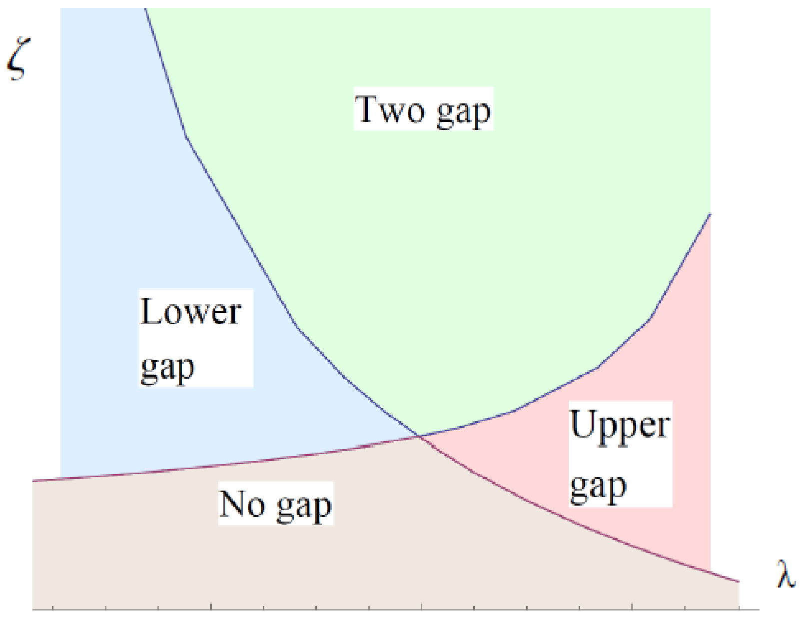
4-4-1 duality relationship

Let us confirm this phase by phase

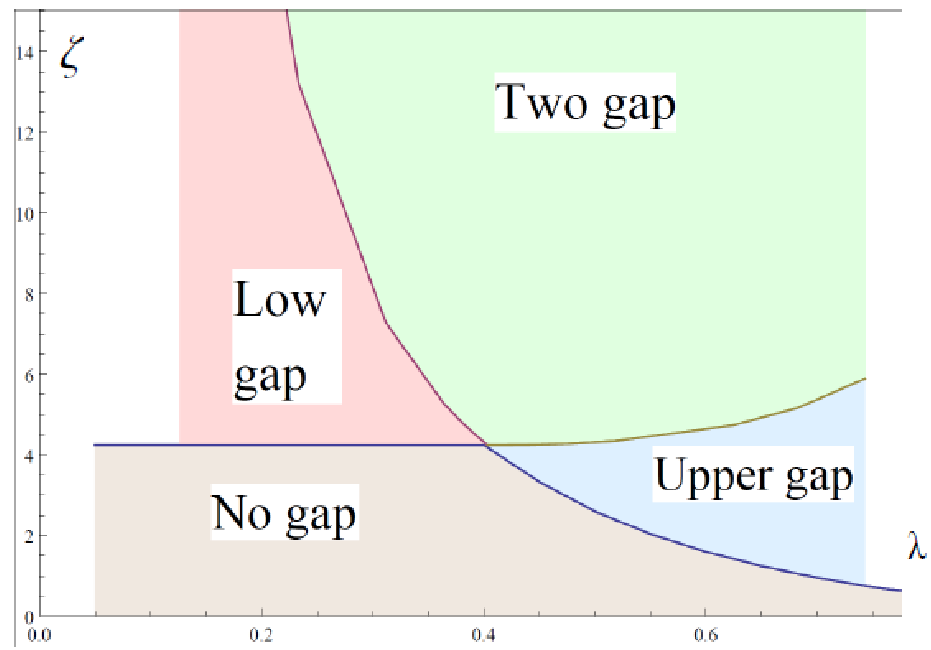
$$\boxed{\rho^{r.f}(\alpha) = \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\left(\frac{1}{2\pi\lambda_{c.b}} - \rho^{c.b}(\alpha+\pi)\right),}$$

4-2 Relationships between phases

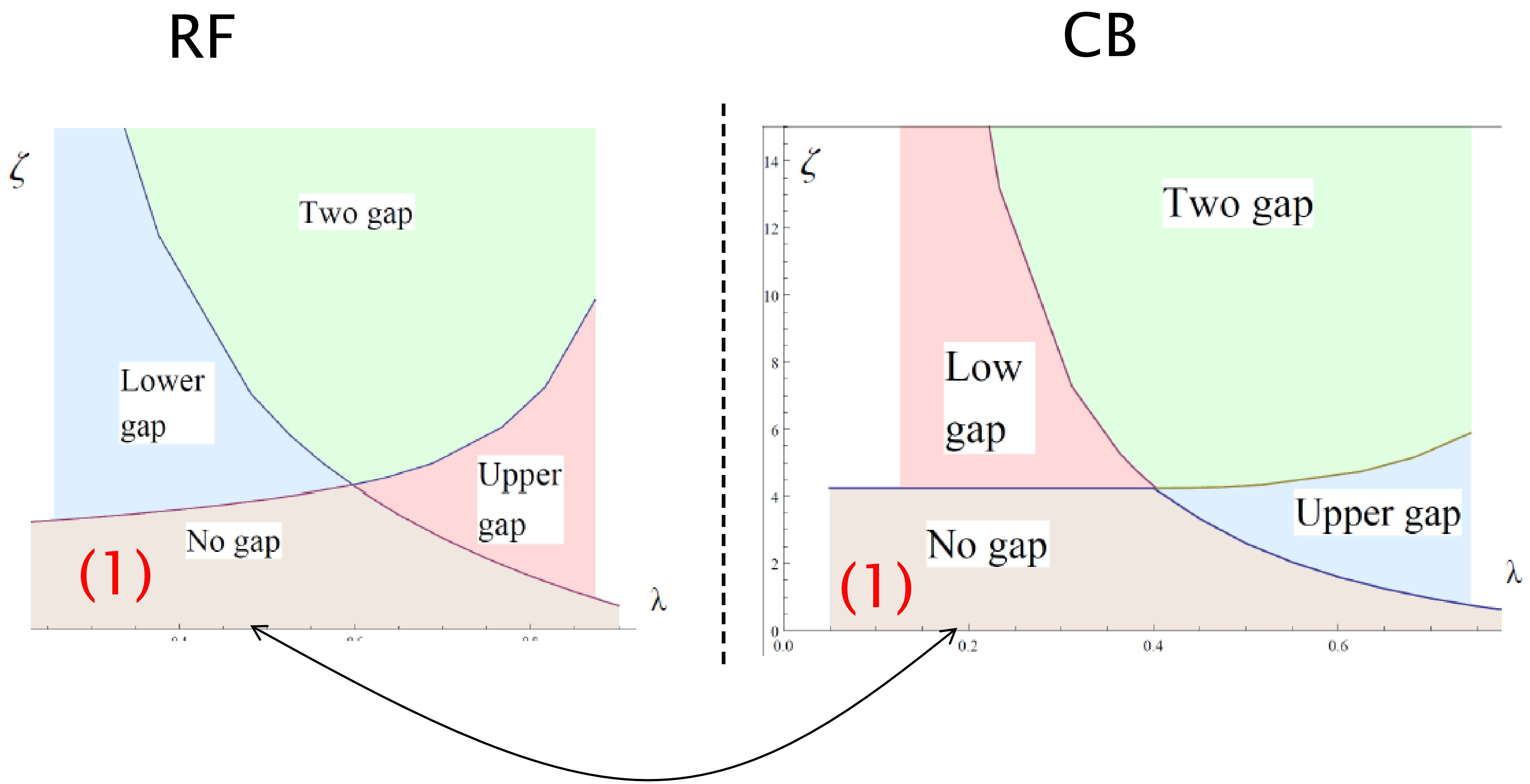
RF



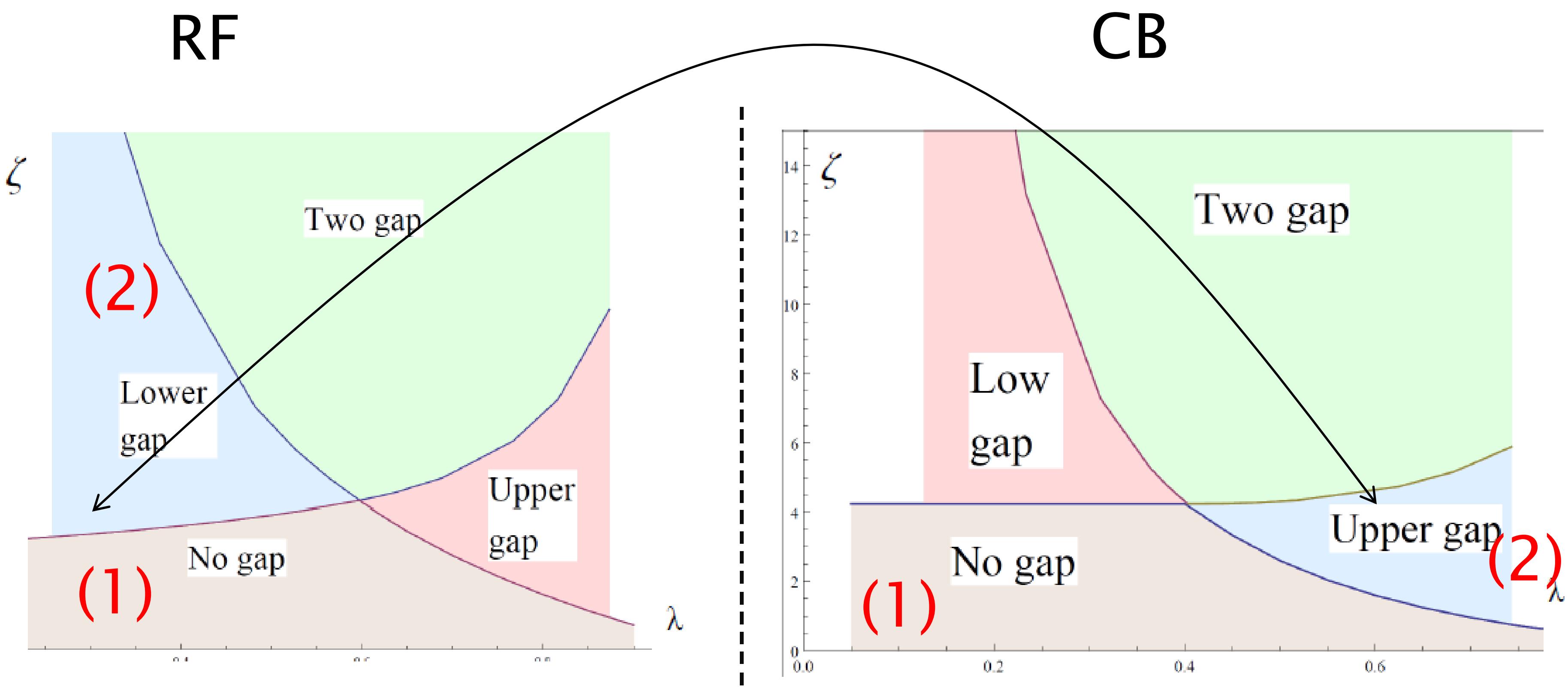
CB



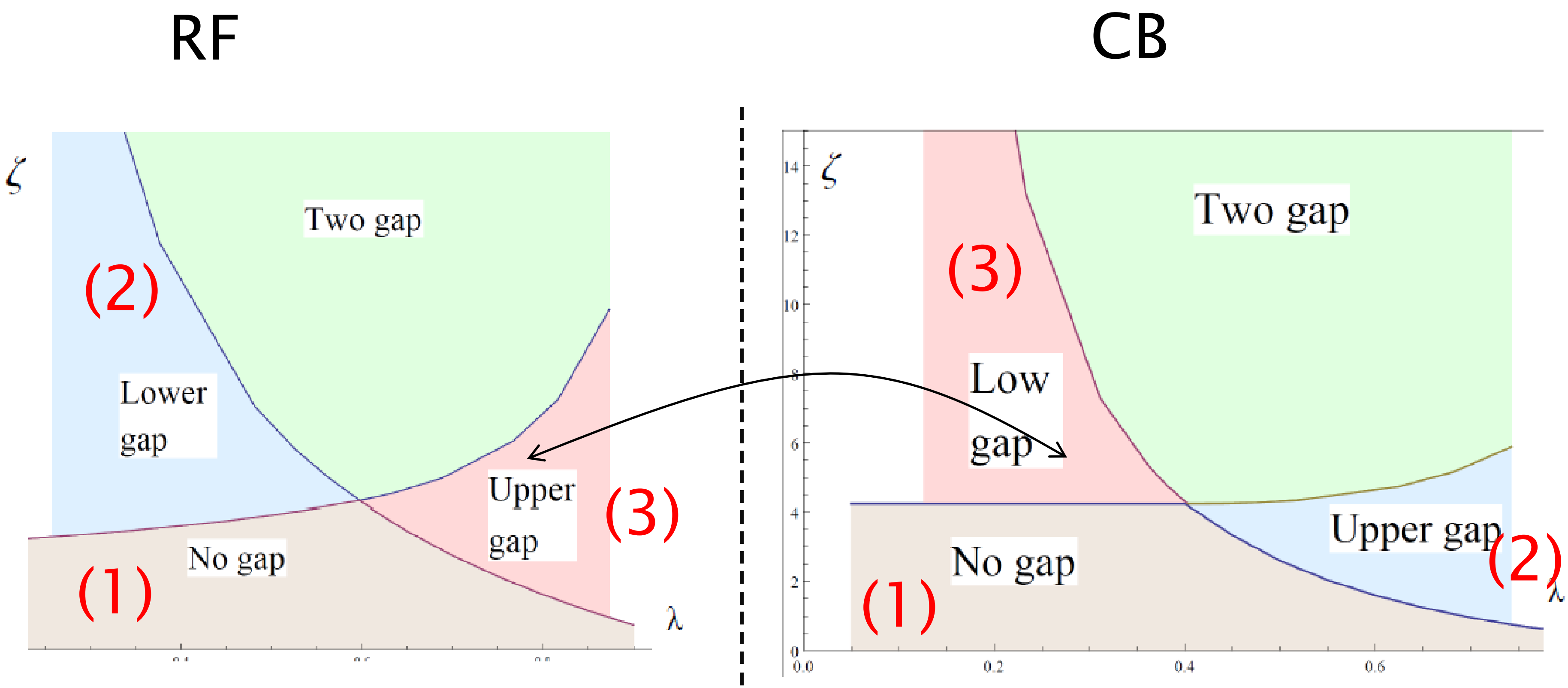
4-2 Relationships between phases



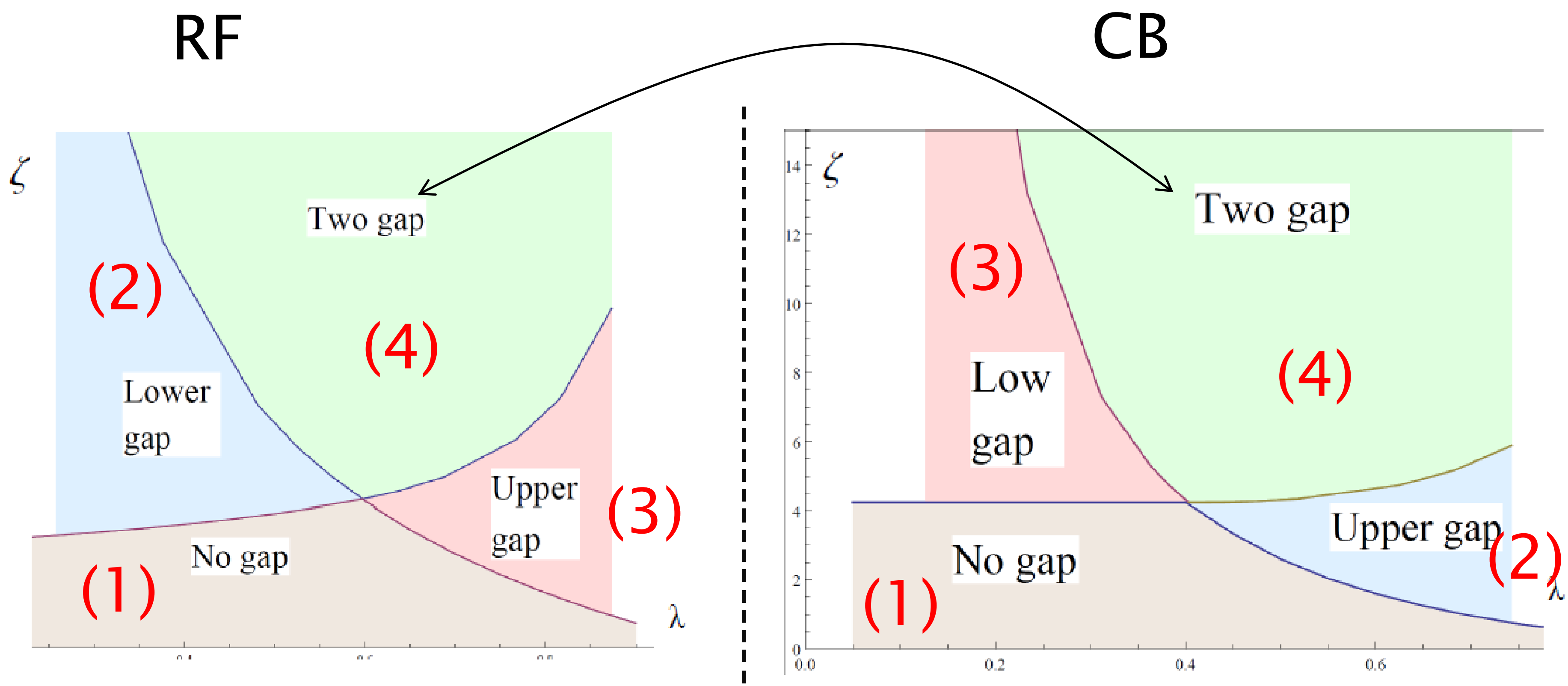
4-2 Relationships between phases



4-2 Relationships between phases



4-4-2 Relationships between phases



(1) Duality between no gap phases

$$
\boxed{\rho^{r.f}(\alpha)} = \frac{1}{2\pi} - \frac{\zeta_{r.f}}{2\pi^2}\sum_{m=1}^{\infty}(-1)^m \cos m\alpha \frac{1+m\tilde{c}}{m^2}e^{-m\tilde{c}}
$$

$$
= \frac{1}{2\pi} - \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\frac{\zeta_{c.b}}{2\pi^2}\sum_{m=1}^{\infty}\cos m(\alpha+\pi)\frac{1+m\sigma}{m^2}e^{-m\sigma}
$$

$$
= \boxed{\frac{\lambda_{c.b}}{1-\lambda_{c.b}}\left(\frac{1}{2\pi\lambda_{c.b}} - \rho^{c.b}(\alpha+\pi)\right)}
$$

Eiganvalue densities

Dual!

$$
\boxed{0 = \int_{-\pi}^{\pi} d\alpha \rho^{c.b}(\alpha)\left(\log\left(2\sinh\left(\frac{\sigma - i\alpha}{2}\right)\right) + c.c\right)}
$$

$$
\Leftrightarrow 0 = \int_{2\pi}^{0} d\tilde{\alpha}\left(\frac{1}{2\pi\lambda_{r.f}} - \rho^{r.f}(\tilde{\alpha})\right)\left(\log\left(2\cosh\left(\frac{\tilde{c}+i\tilde{\alpha}}{2}\right)\right) + c.c\right)
$$

$$
\Leftrightarrow \boxed{\tilde{c} = \lambda_{r.f}\int_{-\pi}^{\pi} d\alpha \rho_{r.f}(\alpha)\left(\log\left(2\cosh\left(\frac{\tilde{c}+i\alpha}{2}\right)\right) + c.c\right)}.
$$

gap equation

Dual!

(1) Duality between no gap phases

$$\boxed{\rho^{r.f}(\alpha)} = \frac{1}{2\pi} - \frac{\zeta_{r.f}}{2\pi^2}\sum_{m=1}^{\infty}(-1)^m \cos m\alpha \frac{1+m\tilde{c}}{m^2}e^{-m\tilde{c}}$$

$$= \frac{1}{2\pi} - \frac{\lambda_{c.b}}{1-\lambda}\frac{\zeta_{c.b}}{2\pi^2}\sum^{\infty}\cos m(\alpha+\pi)\frac{1+m\sigma}{m^2}e^{-m\sigma}$$

Eiganvalue densities

Dual!

Duality is confirmed !!

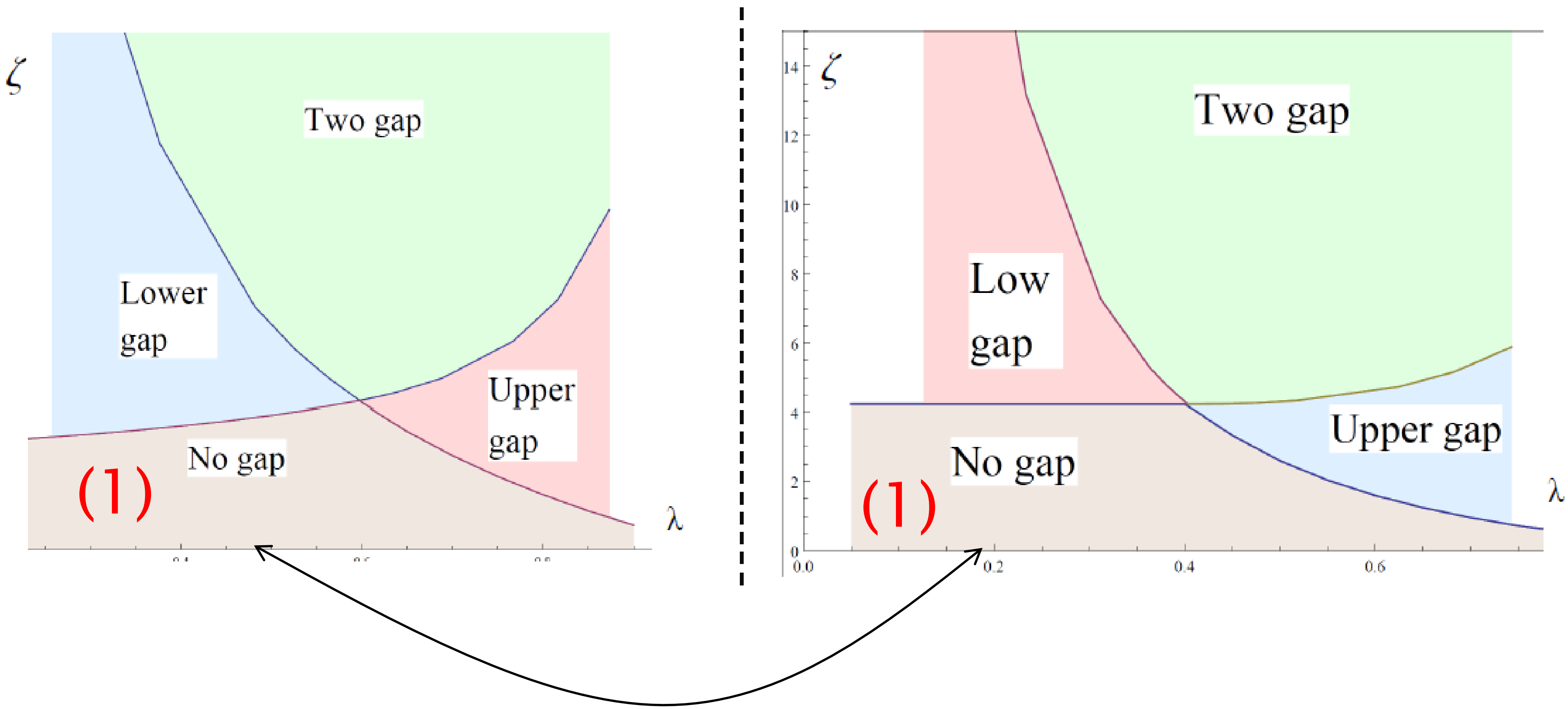
$$\Leftrightarrow 0 = \int_{2\pi} d\tilde{\alpha}\left(\frac{1}{2\pi\lambda_{r.f}} - \rho^{r.f}(\tilde{\alpha})\right)\left(\log\left(2\cosh\left(\frac{\tilde{c}+i\tilde{\alpha}}{2}\right)\right)+c.c\right)$$

$$\Leftrightarrow \boxed{\tilde{c} = \lambda_{r.f}\int_{-\pi}^{\pi} d\alpha\rho_{r.f}(\alpha)\left(\log\left(2\cosh\left(\frac{\tilde{c}+i\alpha}{2}\right)\right)+c.c\right)}.$$

Dual!

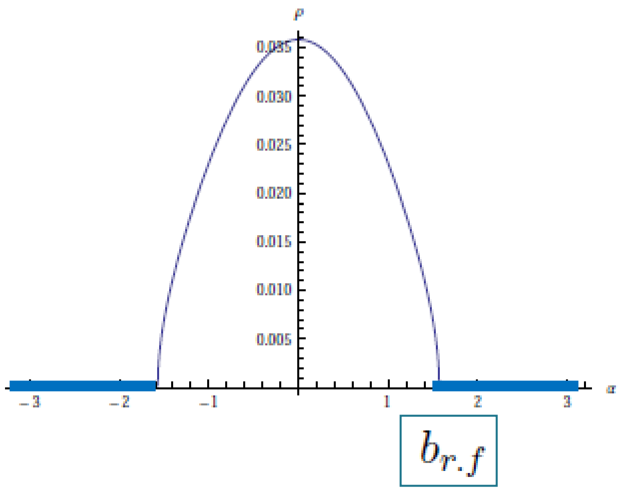
RF

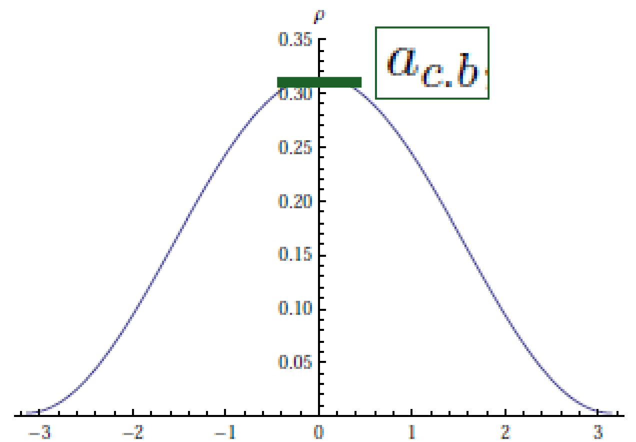
CB


ζ
Two gap
Lower gap
Upper gap
No gap
(1)
λ
ζ
14
12
10
8
6
4
2
0
Two gap
Low gap
Upper gap
No gap
(1)
λ
0.0
0.2
0.4
0.6

OK!

(2) Duality between lower gap phase of RF theory and the upper gap phase of CB theory





In this case, not only the saddle point and gap equation, there is also a relationship between the the domain of the lower gap (zero point of the eigenvalue in the lower gap phase in RF) and the domain of the saturated plate (Upper gap in CB)

$$\boxed{b_{r.f}} = \boxed{\pi - a_{c.b},}$$

(2) Duality between lower gap phase of RF theory and the upper gap phase of CB theory

There is another equation relating to the

$$\boxed{b_{r.f}} = \boxed{\pi - a_{c.b},}$$

For RF theory in the lower gap phase,

$$\tilde{M}_{lg}^{r.f}(\zeta, \tilde{c}, b) \equiv \frac{\zeta}{2\pi} \int_0^{e^{-\tilde{c}}} dx \ \left(\frac{\log x}{x} - \frac{(1+x)}{x} \frac{\log x}{\sqrt{x^2 + 2x\cos b + 1}} \right) = 1.$$

For CB theory in the upper gap phase,

$$\tilde{M}_{ug}^{c.b}(\zeta, \sigma, a) \equiv -\frac{\zeta}{2\pi} \int_0^{e^{-\sigma}} dx \ \left(\frac{\log x}{x} - \frac{1+x}{x} \frac{\log x}{\sqrt{x^2 - 2x\cos a + 1}} \right) = 1 - \frac{1}{\lambda}.$$

We will call these as the domain equation.

(2) Duality between lower gap phase of RF theory and the upper gap phase of CB theory

For RF theory in the lower gap phase,

$$\tilde{M}^{r.f}_{lg}(\zeta,\tilde{c},b) \equiv \frac{\zeta}{2\pi}\int_0^{e^{-\tilde{c}}} dx \,\left(\frac{\log x}{x} - \frac{(1+x)}{x}\frac{\log x}{\sqrt{x^2+2x\cos b+1}}\right) = 1.$$

$b_{r.f} \longrightarrow \pi$ ⇨ Condition of the phase transition from no gap to lower gap

For CB theory in the upper gap phase,

$$\tilde{M}^{c.b}_{ug}(\zeta,\sigma,a) \equiv -\frac{\zeta}{2\pi}\int_0^{e^{-\sigma}} dx \,\left(\frac{\log x}{x} - \frac{1+x}{x}\frac{\log x}{\sqrt{x^2-2x\cos a+1}}\right) = 1-\frac{1}{\lambda}.$$

$a_{c.b},\longrightarrow 0$ ⇨ Condition of the phase transition from no gap to upper gap

Let us check the duality of domain equations

$$\boxed{\frac{\zeta_{r.f}}{2\pi}\left(\int_0^{e^{-\tilde{c}}} dx \left(\frac{\log x}{x}\left(1-\frac{1}{\sqrt{x^2+2x\cos b_{r.f}+1}}\right)\right)-\log x\left(\frac{1}{\sqrt{x^2+2x\cos b_{r.f}+1}}\right)\right)=1.}$$

$$\Leftrightarrow \quad \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\frac{\zeta_{c.b}}{2\pi}\left(\int_0^{e^{-\sigma}} dx \left(\frac{\log x}{x}\left(1-\frac{1}{\sqrt{x^2-2x\cos a_{c.b}+1}}\right)\right)-\log x\left(\frac{1}{\sqrt{x^2-2x\cos a_{c.b}+1}}\right)\right)=1.$$

$$\Leftrightarrow \quad \boxed{-\frac{\zeta_{c.b}}{2\pi}\left(\int_0^{e^{-\sigma}} dx \left(\frac{\log x}{x}\left(1-\frac{1}{\sqrt{x^2-2x\cos a_{c.b}+1}}\right)\right)-\log x\left(\frac{1}{\sqrt{x^2-2x\cos a_{c.b}+1}}\right)\right)=1-\frac{1}{\lambda_{c.b}}.}$$

Dual!

$$
\begin{aligned}
& \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\left(\frac{1}{2\pi\lambda_{c.b}}-\rho_{c.b}(\alpha_{c.b})\right) \\
= \; & \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\frac{\zeta_{c.b}}{\sqrt{2}\pi^2}\sqrt{\sin^2\frac{\alpha_{c.b}}{2}-\sin^2\frac{a_{c.b}}{2}}\int_{\tilde{c}}^{\infty}dy\,\frac{y\sin\frac{\alpha_{c.b}}{2}\cosh\frac{y}{2}}{\sqrt{\cosh y-\cos a_{c.b}}(\cosh y-\cos\alpha_{c.b})} \\
= \; & \frac{\zeta_{r.f}}{\sqrt{2}\pi^2}\sqrt{\sin^2\frac{b_{r.f}}{2}-\sin^2\frac{\alpha_{r.f}}{2}}\int_{\tilde{c}}^{\infty}dy\,\frac{y\cos\frac{\alpha_{r.f}}{2}\cosh\frac{y}{2}}{\sqrt{\cosh y+\cos b_{r.f}}(\cosh y+\cos\alpha_{r.f})} \\
= \; & \rho_{r.f}(\alpha_{r.f})
\end{aligned}
$$

Eigenvalue density

Dual!

$$
\begin{aligned}
0 &= \int_{-\pi}^{\pi}d\alpha_{c.b}\rho_{c.b}(\alpha_{c.b})\left(\log\left(2\sinh\left(\frac{\sigma-i\alpha_{c.b}}{2}\right)\right)+\log\left(2\sinh\left(\frac{\sigma+i\alpha_{c.b}}{2}\right)\right)\right) \\
\Leftrightarrow 0 &= \int_{2\pi}^{0}d\alpha_{r.f}\left(\frac{1}{2\pi\lambda_{r.f}}-\rho_{r.f}(\alpha_{r.f})\right)\left(\log\left(2\sinh\left(\frac{\sigma+i\alpha_{r.f}-i\pi}{2}\right)\right)+\log\left(2\sinh\left(\frac{\sigma-i\alpha_{r.f}+i\pi}{2}\right)\right)\right) \\
\Leftrightarrow 0 &= \int_{0}^{2\pi}d\alpha_{r.f}\left(\frac{1}{2\pi\lambda_{r.f}}-\rho_{r.f}(\alpha_{r.f})\right)\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) \\
\Leftrightarrow \; & 2\pi\lambda_{r.f}\int_{0}^{2\pi}d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) \\
&= \int_{0}^{2\pi}d\alpha_{r.f}\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) \\
\Leftrightarrow \; & 2\pi\lambda_{r.f}\int_{0}^{2\pi}d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right)=2\pi\sigma \\
\Leftrightarrow \; & \lambda_{r.f}\int_{0}^{2\pi}d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right)=\sigma \qquad (61)
\end{aligned}
$$

gap equation

Dual!

$$
\begin{aligned}
& \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\left(\frac{1}{2\pi\lambda_{c.b}}-\rho_{c.b}(\alpha_{c.b})\right) \\
= \; & \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\frac{\zeta_{c.b}}{\sqrt{2}\pi^2}\sqrt{\sin^2\frac{\alpha_{c.b}}{2}-\sin^2\frac{a_{c.b}}{2}}\int_{\tilde{c}}^{\infty}dy\;\frac{y\sin\frac{\alpha_{c.b}}{2}\cosh\frac{y}{2}}{\sqrt{\cosh y-\cos a_{c.b}}(\cosh y-\cos\alpha_{c.b})} \\
= \; & \frac{\zeta_{r.f}}{\sqrt{2}\pi^2}\sqrt{\sin^2\frac{b_{r.f}}{2}-\sin^2\frac{\alpha_{r.f}}{2}}\int_{\tilde{c}}^{\infty}dy\;\frac{y\cos\frac{\alpha_{r.f}}{2}\cosh\frac{y}{2}}{\sqrt{\cosh y+\cos b_{r.f}}(\cosh y+\cos\alpha_{r.f})} \\
= \; & \rho_{r.f}(\alpha_{r.f})
\end{aligned}
$$

Eigenvalue density

Dual!

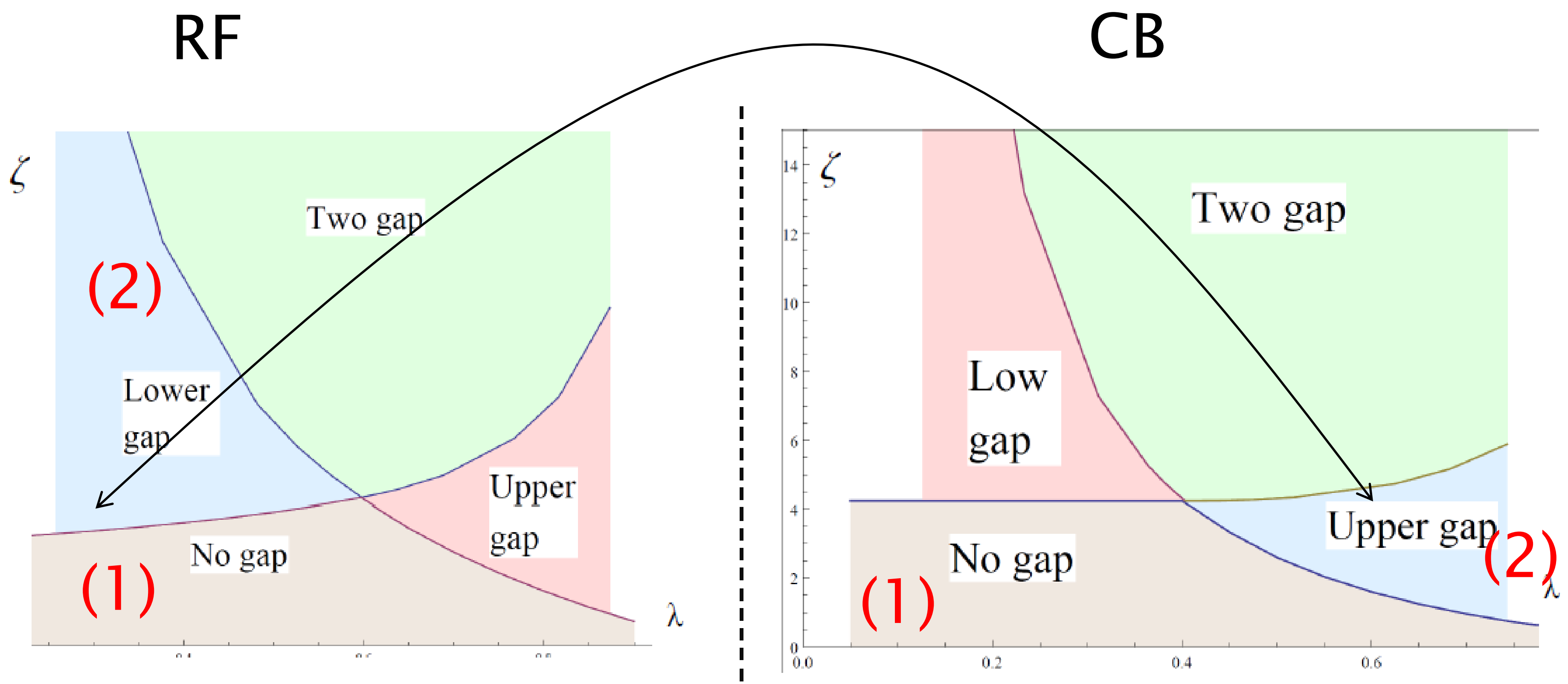
Duality is confirmed !!

$$
\begin{aligned}
0 \; & = \\
\Leftrightarrow 0 \; & = \\
\Leftrightarrow 0 \; & =
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \quad & 2\pi\lambda_{r.f}\int_0 d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{}{2}\right)\right)+\log\left(2\cosh\left(\frac{}{2}\right)\right)\right) \\
= \; & \int_0^{2\pi}d\alpha_{r.f}\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) \\
\Leftrightarrow \quad & 2\pi\lambda_{r.f}\int_0^{2\pi}d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right)=2\pi\sigma \\
\Leftrightarrow \quad & \lambda_{r.f}\int_0^{2\pi}d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right)=\sigma \qquad (61)
\end{aligned}
$$

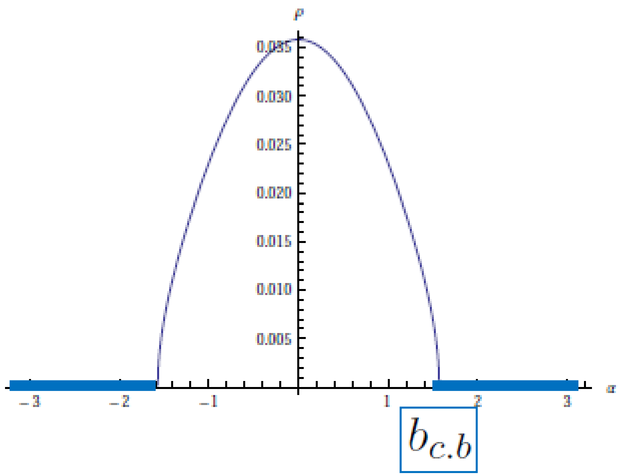
Dual!

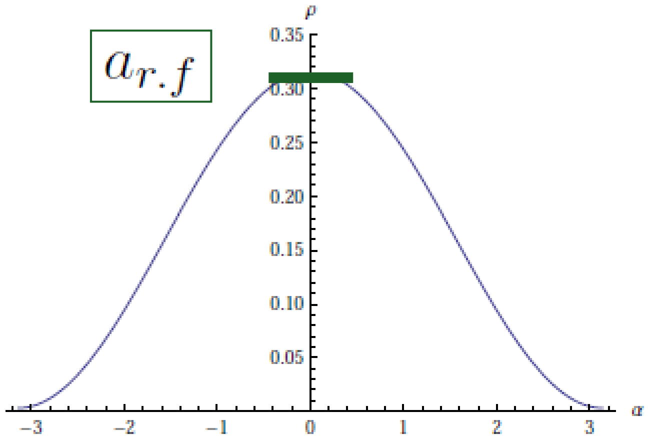
Dual!



RF
CB
ζ
Two gap
(2)
Lower gap
Upper gap
No gap
(1)
λ
ζ
Two gap
Low gap
Upper gap
No gap
(1)
(2)
λ
14
12
10
8
6
4
2
0
0.0
0.2
0.4
0.6

(3) Duality between lower gap phase of CB theory and the upper gap phase of RF theory





In this case, not only the saddle point and gap equation, there is also a relationship between the the domain of the lower gap (zero point of the eigenvalue in the lower gap phase in CB) and the domain of the saturated plate (Upper gap in RF)

$$\boxed{a_{r.f}} = \boxed{\pi - b_{c.b},}$$

(3) Duality between lower gap phase of CB theory and the upper gap phase of RF theory

The domain equations w.r.t

For CB theory in the lower gap phase, w.r.t. $a_{r.f}$

$$\tilde{M}_{lg}^{r.f}(\zeta,\tilde{c},b) \equiv \frac{\zeta}{2\pi}\int_0^{e^{-\tilde{c}}} dx \; \left(\frac{\log x}{x} - \frac{(1+x)}{x}\frac{\log x}{\sqrt{x^2+2x\cos b+1}}\right) = 1.$$

For RF theory in the upper gap phase, w.r.t. $b_{c.b},$

$$\tilde{M}_{ug}^{c.b}(\zeta,\sigma,a) \equiv -\frac{\zeta}{2\pi}\int_0^{e^{-\sigma}} dx \; \left(\frac{\log x}{x} - \frac{1+x}{x}\frac{\log x}{\sqrt{x^2-2x\cos a+1}}\right) = 1 - \frac{1}{\lambda}.$$

(3) Duality between lower gap phase of CB theory and the upper gap phase of RF theory

The domain equations w.r.t

For CB theory in the lower gap phase, w.r.t. $a_{r.f}$

$$\tilde{M}_{lg}^{r.f}(\zeta,\tilde{c},b) \equiv \frac{\zeta}{2\pi}\int_0^{e^{-\tilde{c}}} dx \,\left(\frac{\log x}{x} - \frac{(1+x)}{x}\frac{\log x}{\sqrt{x^2+2x\cos b+1}}\right) = 1.$$

For RF theory in the upper gap phase, w.r.t. $b_{c.b},$

$$\tilde{M}_{ug}^{c.b}(\zeta,\sigma,a) \equiv -\frac{\zeta}{2\pi}\int_0^{e^{-\sigma}} dx \,\left(\frac{\log x}{x} - \frac{1+x}{x}\frac{\log x}{\sqrt{x^2-2x\cos a+1}}\right) = 1-\frac{1}{\lambda}.$$

In a certain limit these becomes the
Conditions of the phase transition from the no gap phase to lower or upper gap phase

Let us check the duality of domain equations

$$\boxed{-\frac{\zeta_{c.b}}{2\pi}\left(\int_0^{e^{-\sigma}} dx\ \left(\frac{\log x}{x}\left(1-\frac{1}{\sqrt{x^2-2x\cos b_{c.b}+1}}\right)\right)+\log x\left(\frac{1}{\sqrt{x^2-2x\cos b_{c.b}+1}}\right)\right)=1}$$

$$\Leftrightarrow \quad -\frac{\lambda_{r.f}}{1-\lambda_{r.f}}\frac{\zeta_{r.f}}{2\pi}\left(\int_0^{e^{-\tilde{c}}} dx\ \left(\frac{\log x}{x}\left(1-\frac{1}{\sqrt{x^2+2x\cos a_{r.f}+1}}\right)\right)+\log x\left(\frac{1}{\sqrt{x^2+2x\cos a_{r.f}+1}}\right)\right)=1$$

$$\Leftrightarrow \quad \boxed{\frac{\zeta_{r.f}}{2\pi}\left(\int_0^{e^{-\tilde{c}}} dx\ \left(\frac{\log x}{x}\left(1-\frac{1}{\sqrt{x^2+2x\cos a_{r.f}+1}}\right)\right)+\log x\left(\frac{1}{\sqrt{x^2+2x\cos a_{r.f}+1}}\right)\right)=1-\frac{1}{\lambda_{r.f}}}$$

Dual!

$$
\begin{aligned}
& \rho_{r.f}(\alpha) - \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\left(\frac{1}{2\pi\lambda_{c.b}} - \rho_{c.b}(\pi-\alpha)\right) \\
= \ & \frac{i}{2\pi^2\lambda_{r.f}}\int_{upg} d\omega\, \frac{1}{h(\omega)(\omega-u(\alpha))}\sqrt{(u-e^{i a_{r.f}})(u-e^{-i a_{r.f}})} - \frac{1}{2\pi\lambda_{r.f}} \\
= \ & \frac{1}{2\pi\lambda_{r.f}} - \frac{1}{2\pi\lambda_{r.f}} \\
= \ & 0
\end{aligned}
$$

Eigenvalue density

Dual!

$$
\begin{aligned}
0 &= \int_{-\pi}^{\pi} d\alpha_{c.b}\rho_{c.b}(\alpha_{c.b})\left(\log\left(2\sinh\left(\frac{\sigma-i\alpha_{c.b}}{2}\right)\right)+\log\left(2\sinh\left(\frac{\sigma+i\alpha_{c.b}}{2}\right)\right)\right) \\
\Leftrightarrow 0 &= \int_{2\pi}^{0} d\alpha_{r.f}\left(\frac{1}{2\pi\lambda_{r.f}}-\rho_{r.f}(\alpha_{r.f})\right)\left(\log\left(2\sinh\left(\frac{\sigma+i\alpha_{r.f}-i\pi}{2}\right)\right)+\log\left(2\sinh\left(\frac{\sigma-i\alpha_{r.f}+i\pi}{2}\right)\right)\right) \\
\Leftrightarrow 0 &= \int_{0}^{2\pi} d\alpha_{r.f}\left(\frac{1}{2\pi\lambda_{r.f}}-\rho_{r.f}(\alpha_{r.f})\right)\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) \\
\Leftrightarrow &\quad 2\pi\lambda_{r.f}\int_{0}^{2\pi} d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) \\
&= \int_{0}^{2\pi} d\alpha_{r.f}\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) \\
\Leftrightarrow &\quad 2\pi\lambda_{r.f}\int_{0}^{2\pi} d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) = 2\pi\sigma \\
\Leftrightarrow &\quad \lambda_{r.f}\int_{0}^{2\pi} d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) = \sigma \qquad (61)
\end{aligned}
$$

gap equation

Dual!

$$
\begin{aligned}
& \rho_{r.f}(\alpha) - \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\left(\frac{1}{2\pi\lambda_{c.b}} - \rho_{c.b}(\pi-\alpha)\right) \\
= \; & \frac{i}{2\pi^2\lambda_{r.f}}\int_{upg} d\omega\, \frac{1}{h(\omega)(\omega-u(\alpha))}\sqrt{(u-e^{i\alpha_{r.f}})(u-e^{-i\alpha_{r.f}})} - \frac{1}{2\pi\lambda_{r.f}} \\
= \; & \frac{1}{2\pi\lambda_{r.f}} - \frac{1}{2\pi\lambda_{r.f}} \\
= \; & 0
\end{aligned}
$$

Eigenvalue density

Dual!

Duality is confirmed !!

0

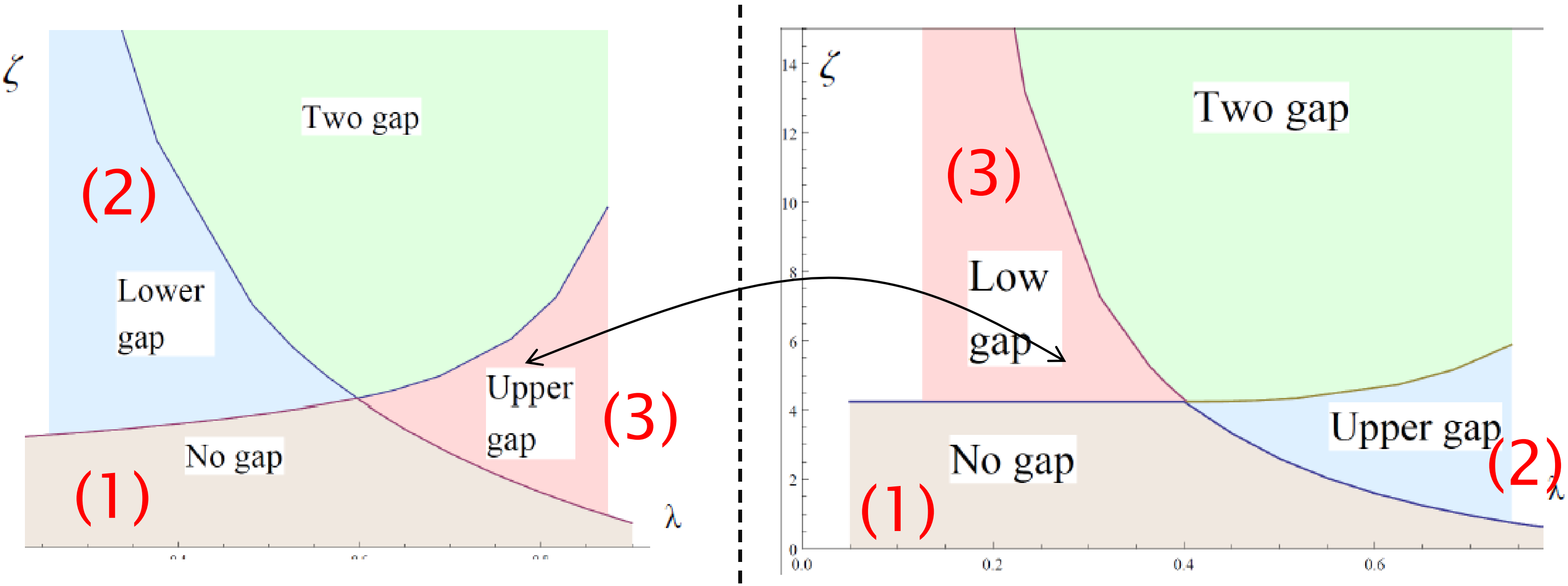
$\Leftrightarrow 0$

$\Leftrightarrow 0$

$$
\begin{aligned}
\Leftrightarrow \quad & 2\pi\lambda_{r.f}\int_0^{2\pi} d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) \\
= \quad & \int_0^{2\pi} d\alpha_{r.f}\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) \\
\Leftrightarrow \quad & 2\pi\lambda_{r.f}\int_0^{2\pi} d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) = 2\pi\sigma \\
\Leftrightarrow \quad & \lambda_{r.f}\int_0^{2\pi} d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma+i\alpha_{r.f}}{2}\right)\right)+\log\left(2\cosh\left(\frac{\sigma-i\alpha_{r.f}}{2}\right)\right)\right) = \sigma \qquad (61)
\end{aligned}
$$

Dual!

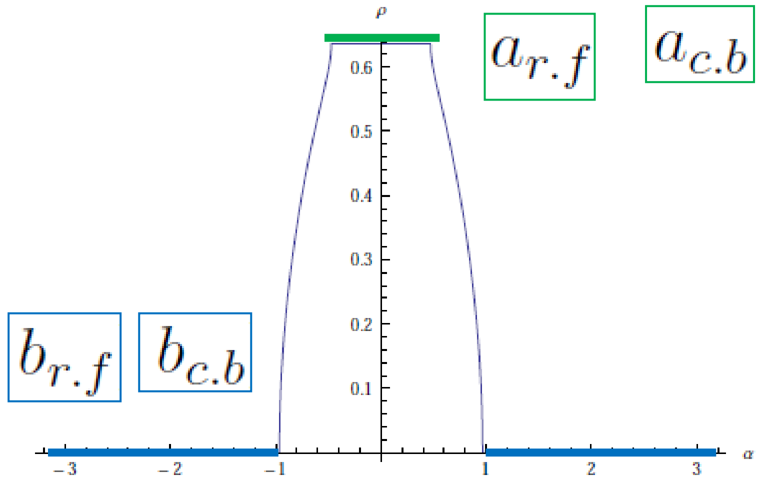
RF CB



ζ
Two gap
(2)
Lower gap
Upper gap
(3)
No gap
(1)
λ
ζ
Two gap
(3)
Low gap
Upper gap
(2)
No gap
(1)
λ
0.0
0.2
0.4
0.6
0
2
4
6
8
10
12
14

Dual !

(4) Duality between Two gap phases



In this case, there are two domain equations for both regular fermion and the critical boson, since there are two gap region in both theories, $a_{r.f}, b_{r.f}$ $a_{c.b}, b_{c.b}$

(4) Duality between Two gap phases

For RF theory in the two gap phase, w.r.t. $a_{r.f}, b_{r.f}$

$$\boxed{\frac{1}{4\pi\lambda}\int_{-a}^{a} d\alpha \frac{1}{\sqrt{\sin^2\frac{a}{2}-\sin^2\frac{\alpha}{2}}\sqrt{\sin^2\frac{b}{2}-\sin^2\frac{\alpha}{2}}} = \frac{\zeta}{2\pi}\int_{\tilde{c}}^{\infty} dy \frac{y}{\sqrt{(\cosh y+\cos a)(\cosh y+\cos b)}}}$$

$b_{r.f} \longrightarrow \pi$ ⇒ Condition of the phase transition from upper gap to two gap

$a_{r.f} \longrightarrow 0$ ⇒ Condition of the phase transition from lower gap to two gap

$$\boxed{\begin{aligned} 1 \;=\; & \frac{\zeta}{4\pi}\int_{\tilde{c}}^{\infty} dy \frac{ye^{-y}}{\sqrt{(\cosh y+\cos a)(\cosh y+\cos b)}} \\ & +\frac{\zeta}{4\pi}\int_{\tilde{c}}^{\infty} dy\, y\left(\frac{e^{y}}{\sqrt{(\cosh y+\cos a)(\cosh y+\cos b)}}-2\right) \\ & +\frac{1}{4\pi\lambda}\int_{-a}^{a} d\alpha \frac{\cos\alpha}{\sqrt{\sin^2\frac{a}{2}-\sin^2\frac{\alpha}{2}}\sqrt{\sin^2\frac{b}{2}-\sin^2\frac{\alpha}{2}}} \end{aligned}}$$

$b_{r.f} \longrightarrow \pi$

Domain equation in upper gap

$a_{r.f} \longrightarrow 0$

Domain equation in lower gap

(4) Duality between Two gap phases

For CB theory in the two gap phase, w.r.t. $a_{c.b}, b_{c.b}$

$$0 = \frac{1}{4\pi\lambda}\int_{-a}^{a} d\alpha \, \frac{1}{\sqrt{\sin^2\frac{a}{2}-\sin^2\frac{\alpha}{2}}\sqrt{\sin^2\frac{b}{2}-\sin^2\frac{\alpha}{2}}} - \frac{\zeta}{2\pi}\int_{\sigma}^{\infty} dy \, \frac{y}{\sqrt{(\cosh y-\cos a)(\cosh y-\cos b)}}$$

$b_{c.b} \longrightarrow \pi$

Condition of the phase transition from upper to two

$a_{c.b} \longrightarrow 0$

Condition of the phase transition from lower to two

$$1 = -\frac{\zeta}{4\pi}\int_{\sigma}^{\infty} dy \, \frac{ye^{-y}}{\sqrt{(\cosh y-\cos a)(\cosh y-\cos b)}} - \frac{\zeta}{4\pi}\int_{\sigma}^{\infty} dy \, y\left(\frac{e^{y}}{\sqrt{(\cosh y-\cos a)(\cosh y-\cos b)}}-2\right) + \frac{1}{4\pi\lambda}\int_{-a}^{a} d\alpha \, \frac{\cos\alpha}{\sqrt{\sin^2\frac{a}{2}-\sin^2\frac{\alpha}{2}}\sqrt{\sin^2\frac{b}{2}-\sin^2\frac{\alpha}{2}}}$$

$b_{c.b} \longrightarrow \pi$

Domain equation in upper gap

$a_{c.b} \longrightarrow 0$

Domain equation in lower gap

Duality of domain equations

R.F

$$\frac{1}{4\pi\lambda}\int_{-a}^{a} d\alpha \, \frac{1}{\sqrt{\sin^2\frac{a}{2}-\sin^2\frac{\alpha}{2}}\sqrt{\sin^2\frac{b}{2}-\sin^2\frac{\alpha}{2}}} = \frac{\zeta}{2\pi}\int_{\tilde{c}}^{\infty} dy \, \frac{y}{\sqrt{(\cosh y+\cos a)(\cosh y+\cos b)}}$$

Dual !

C.B

$$\begin{aligned} 0 \;=\; & \frac{1}{4\pi\lambda}\int_{-a}^{a} d\alpha \, \frac{1}{\sqrt{\sin^2\frac{a}{2}-\sin^2\frac{\alpha}{2}}\sqrt{\sin^2\frac{b}{2}-\sin^2\frac{\alpha}{2}}} \\ & -\frac{\zeta}{2\pi}\int_{\sigma}^{\infty} dy \, \frac{y}{\sqrt{(\cosh y-\cos a)(\cosh y-\cos b)}} \end{aligned}$$

Duality of domain equations

$$
\begin{aligned}
1 \;=\; & \frac{\zeta}{4\pi}\int_{\tilde{c}}^{\infty} dy\, \frac{y e^{-y}}{\sqrt{(\cosh y + \cos a)(\cosh y + \cos b)}} \\
& + \frac{\zeta}{4\pi}\int_{\tilde{c}}^{\infty} dy\; y \left(\frac{e^{y}}{\sqrt{(\cosh y + \cos a)(\cosh y + \cos b)}} - 2 \right) \\
& + \frac{1}{4\pi\lambda}\int_{-a}^{a} d\alpha\, \frac{\cos\alpha}{\sqrt{\sin^2\frac{a}{2} - \sin^2\frac{\alpha}{2}}\sqrt{\sin^2\frac{b}{2} - \sin^2\frac{\alpha}{2}}}
\end{aligned}
$$

R.F

Dual !

$$
\begin{aligned}
1 \;=\; & -\frac{\zeta}{4\pi}\int_{\sigma}^{\infty} dy\, \frac{y e^{-y}}{\sqrt{(\cosh y - \cos a)(\cosh y - \cos b)}} \\
& - \frac{\zeta}{4\pi}\int_{\sigma}^{\infty} dy\; y \left(\frac{e^{y}}{\sqrt{(\cosh y - \cos a)(\cosh y - \cos b)}} - 2 \right) \\
& + \frac{1}{4\pi\lambda}\int_{-a}^{a} d\alpha\, \frac{\cos\alpha}{\sqrt{\sin^2\frac{a}{2} - \sin^2\frac{\alpha}{2}}\sqrt{\sin^2\frac{b}{2} - \sin^2\frac{\alpha}{2}}}
\end{aligned}
$$

C.B

Duality of eigenvalue density is confirmed directly as

$$
\begin{aligned}
\boxed{\rho_{1b}(\alpha)} &= -\frac{\zeta_{c.b}}{\pi^2}\sqrt{(\sin^2\frac{\alpha_{c.b}}{2}-\sin^2\frac{a_{c.b}}{2})(\sin^2\frac{b_{c.b}}{2}-\sin^2\frac{\alpha_{c.b}}{2})} \\
&\times\int_{\tilde{c}}^{\infty}dy\frac{ye^{-y}}{\sqrt{(e^{-2y}-2e^{-y}\cos a_{c.b}+1)(e^{-2y}-2e^{-y}\cos b_{c.b}+1)}}\left(\frac{\sin\alpha_{c.b}}{\cosh y-\cos\alpha_{c.b}}\right) \\
&= -\frac{\lambda_{r.f}}{1-\lambda_{r.f}}\frac{\zeta_{r.f}}{\pi^2}\sqrt{(\cos^2\frac{\alpha_{r.f}}{2}-\cos^2\frac{a_{r.f}}{2})(\cos^2\frac{b_{r.f}}{2}-\cos^2\frac{\alpha_{r.f}}{2})} \\
&\times\int_{\tilde{c}}^{\infty}dy\frac{ye^{-y}}{\sqrt{(e^{-2y}+2e^{-y}\cos a_{r.f}+1)(e^{-2y}+2e^{-y}\cos b_{r.f}+1)}}\left(\frac{\sin\alpha_{r.f}}{\cosh y+\cos\alpha_{r.f}}\right) \\
&= -\frac{\lambda_{r.f}}{1-\lambda_{r.f}}\frac{\zeta_{r.f}}{\pi^2}\sqrt{(\sin^2\frac{\alpha_{r.f}}{2}-\sin^2\frac{a_{r.f}}{2})(\sin^2\frac{b_{r.f}}{2}-\sin^2\frac{\alpha_{r.f}}{2})} \\
&\times\int_{\tilde{c}}^{\infty}dy\frac{ye^{-y}}{\sqrt{(e^{-2y}+2e^{-y}\cos a_{r.f}+1)(e^{-2y}+2e^{-y}\cos b_{r.f}+1)}}\left(\frac{\sin\alpha_{r.f}}{\cosh y+\cos\alpha_{r.f}}\right) \\
&= \boxed{-\frac{\lambda_{r.f}}{1-\lambda_{r.f}}\rho_{1f}(\alpha_{r.f})} \qquad (266)
\end{aligned}
$$

and

$$\boxed{\rho_{2b}(\alpha_{c.b}) + \frac{\lambda_{r.f}}{1-\lambda_{r.f}}\rho_{2f}(\alpha_{r.f})}$$

$$= \frac{\lambda_{r.f}}{1-\lambda_{r.f}} \frac{|\sin\alpha_{r.f}|}{4\pi^2\lambda_{r.f}} \sqrt{\left(\sin^2\frac{\alpha_{r.f}}{2} - \sin^2\frac{a_{r.f}}{2}\right)\left(\sin^2\frac{b_{r.f}}{2} - \sin^2\frac{\alpha_{r.f}}{2}\right)}$$

$$\times\left(-2\int_{b_{r.f}}^{\pi} \frac{d\theta}{(\cos\theta - \cos\alpha_{r.f})\sqrt{\left(\sin^2\frac{\theta}{2} - \sin^2\frac{b_{r.f}}{2}\right)\left(\sin^2\frac{\theta}{2} - \sin^2\frac{a_{r.f}}{2}\right)}}\right.$$

$$\left. + \int_{-a_{r.f}}^{a_{r.f}} \frac{d\theta}{(\cos\theta - \cos\alpha_{r.f})\sqrt{\left(\sin^2\frac{b_{r.f}}{2} - \sin^2\frac{\theta}{2}\right)\left(\sin^2\frac{a_{r.f}}{2} - \sin^2\frac{\theta}{2}\right)}}\right)$$

$$= \frac{ih^{+}(u_{r.f})}{4\pi^2\lambda_{c.b}} \int_{upg} d\omega \frac{1}{h(\omega)}\left(\frac{2}{(\omega - u_{r.f})} + \frac{1}{u_{r.f}}\right)$$

$$+ \frac{ih^{+}(u_{r.f})}{4\pi^2\lambda_{c.b}} \int_{lowg} d\omega \frac{1}{h(\omega)}\left(\frac{2}{(\omega - u_{r.f})} + \frac{1}{u_{r.f}}\right)$$

$$= \boxed{\frac{1}{2\pi\lambda_{c.b}}}$$

Duality of gap equation

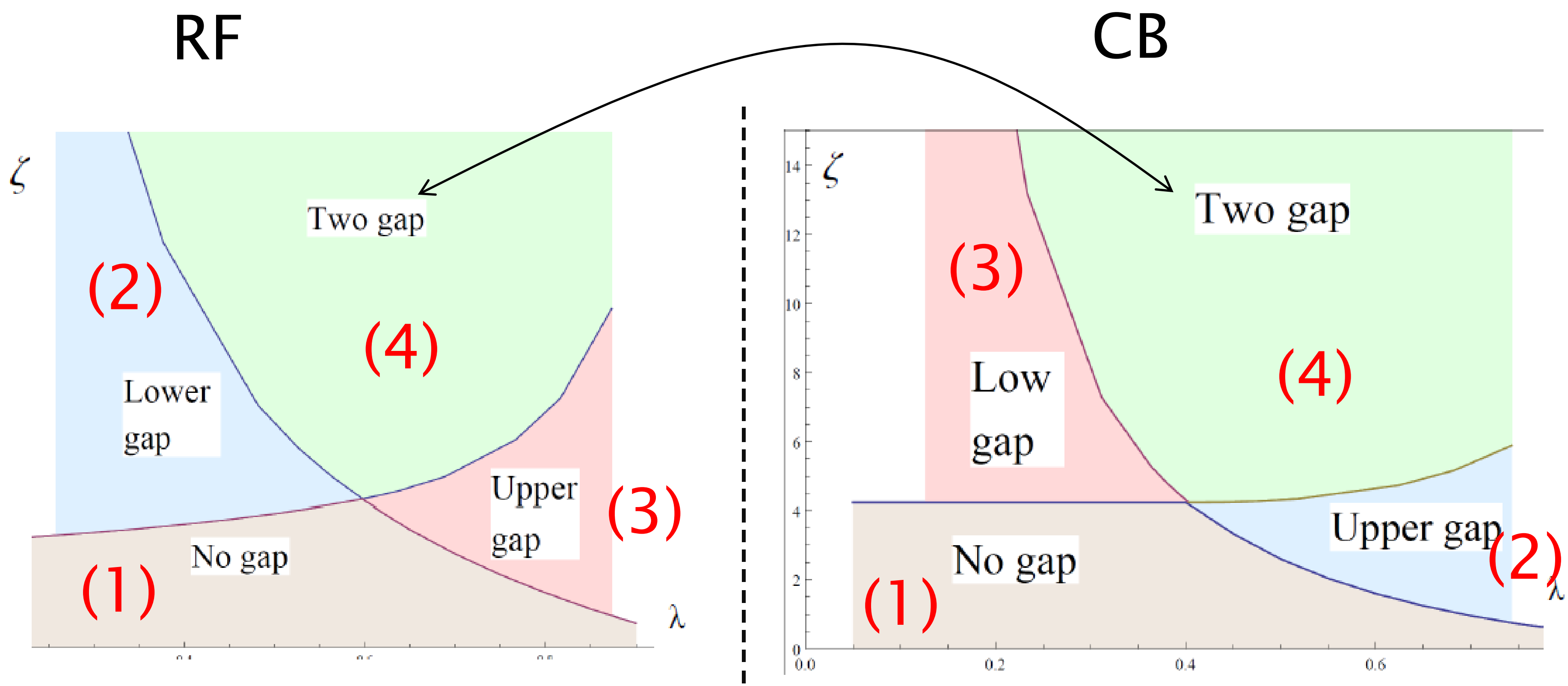
$$
\begin{aligned}
0 &= \int_{-\pi}^{\pi} d\alpha_{c.b}\rho_{c.b}(\alpha_{c.b}) \left(\log\left(2\sinh\left(\frac{\sigma - i\alpha_{c.b}}{2}\right)\right) + \log\left(2\sinh\left(\frac{\sigma + i\alpha_{c.b}}{2}\right)\right)\right) \\
\Leftrightarrow 0 &= \int_{2\pi}^{0} d\alpha_{r.f}\left(\frac{1}{2\pi\lambda_{r.f}} - \rho_{r.f}(\alpha_{r.f})\right)\left(\log\left(2\sinh\left(\frac{\sigma + i\alpha_{r.f} - i\pi}{2}\right)\right) + \log\left(2\sinh\left(\frac{\sigma - i\alpha_{r.f} + i\pi}{2}\right)\right)\right) \\
\Leftrightarrow 0 &= \int_{0}^{2\pi} d\alpha_{r.f}\left(\frac{1}{2\pi\lambda_{r.f}} - \rho_{r.f}(\alpha_{r.f})\right)\left(\log\left(2\cosh\left(\frac{\sigma + i\alpha_{r.f}}{2}\right)\right) + \log\left(2\cosh\left(\frac{\sigma - i\alpha_{r.f}}{2}\right)\right)\right) \\
\Leftrightarrow & \quad 2\pi\lambda_{r.f}\int_{0}^{2\pi} d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma + i\alpha_{r.f}}{2}\right)\right) + \log\left(2\cosh\left(\frac{\sigma - i\alpha_{r.f}}{2}\right)\right)\right) \\
&= \int_{0}^{2\pi} d\alpha_{r.f}\left(\log\left(2\cosh\left(\frac{\sigma + i\alpha_{r.f}}{2}\right)\right) + \log\left(2\cosh\left(\frac{\sigma - i\alpha_{r.f}}{2}\right)\right)\right) \\
\Leftrightarrow & \quad 2\pi\lambda_{r.f}\int_{0}^{2\pi} d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma + i\alpha_{r.f}}{2}\right)\right) + \log\left(2\cosh\left(\frac{\sigma - i\alpha_{r.f}}{2}\right)\right)\right) = 2\pi\sigma \\
\Leftrightarrow & \quad \lambda_{r.f}\int_{0}^{2\pi} d\alpha_{r.f}\rho_{r.f}(\alpha_{r.f})\left(\log\left(2\cosh\left(\frac{\sigma + i\alpha_{r.f}}{2}\right)\right) + \log\left(2\cosh\left(\frac{\sigma - i\alpha_{r.f}}{2}\right)\right)\right) = \sigma \qquad (285)
\end{aligned}
$$

Duality of gap equation

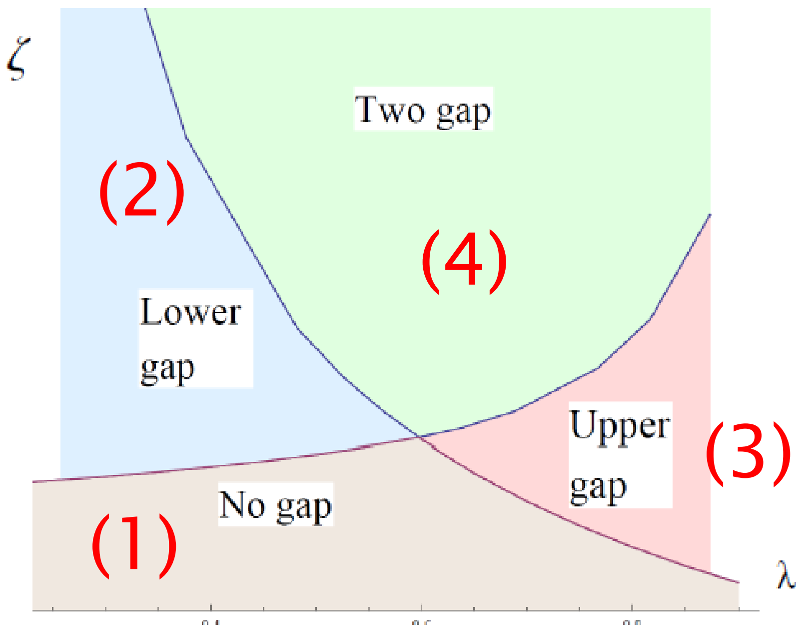
$$
\begin{aligned}
0 &= \int_{-\pi}^{\pi} d\alpha_{c.b}\rho_{c.b}(\alpha_{c.b}) \left(\log\left(2\sinh\left(\frac{\sigma - i\alpha_{c.b}}{2}\right)\right) + \log\left(2\sinh\left(\frac{\sigma + i\alpha_{c.b}}{2}\right)\right)\right) \\
\Leftrightarrow 0 &= \int_{2\pi}^{0} d\alpha_{r.f} \left(\frac{1}{2\pi\lambda_{r.f}} - \rho_{r.f}(\alpha_{r.f})\right) \left(\log\left(2\sinh\left(\frac{\sigma + i\alpha_{r.f} - i\pi}{2}\right)\right) + \log\left(2\sinh\left(\frac{\sigma - i\alpha_{r.f} + i\pi}{2}\right)\right)\right)
\end{aligned}
\tag{285}
$$

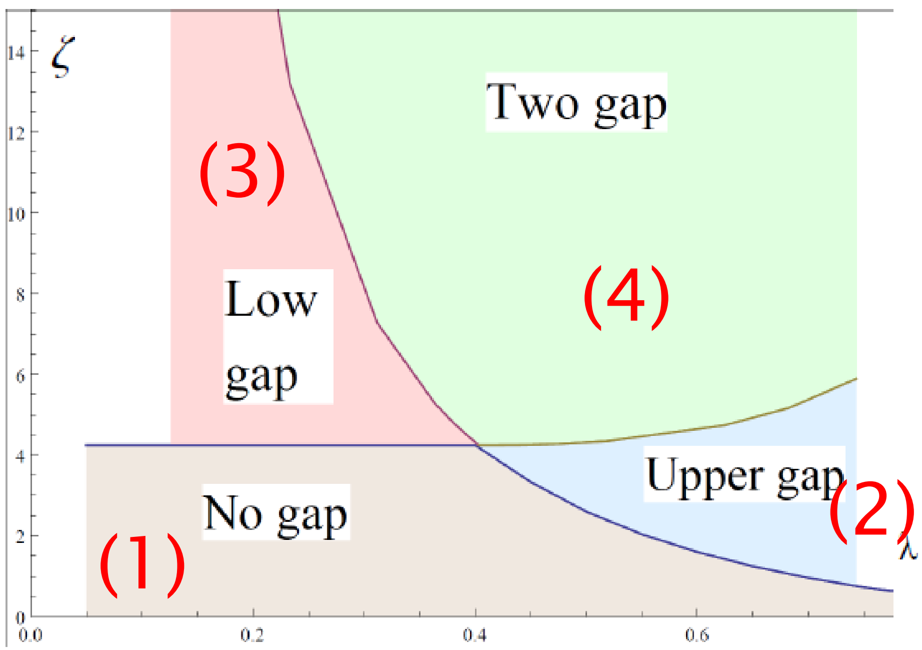
Duality is confirmed !!

Dual !


RF
CB
ζ
Two gap
(2)
(4)
Lower gap
Upper gap
(3)
No gap
(1)
λ
ζ
Two gap
(3)
Low gap
(4)
Upper gap
(2)
No gap
(1)
λ

RF CB

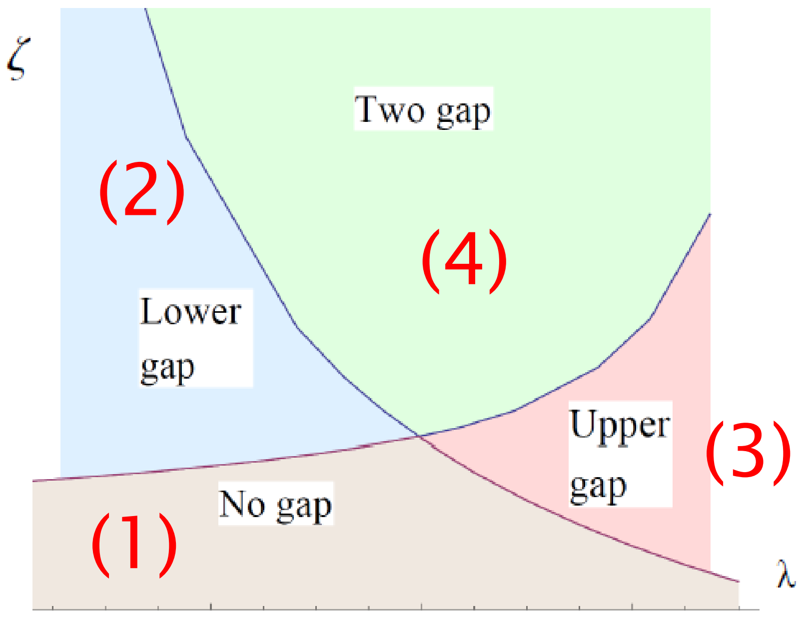




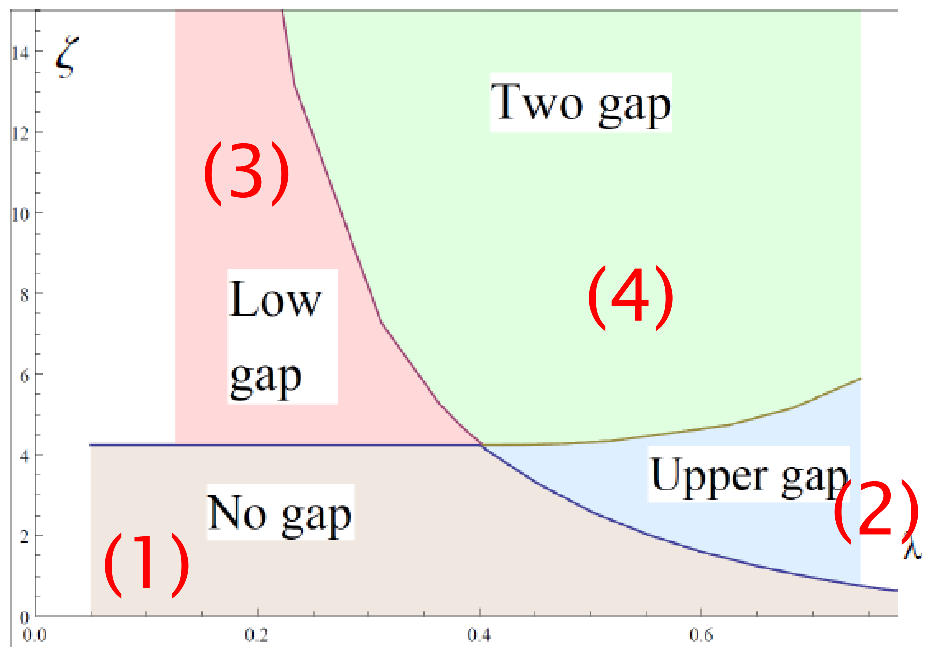
All pairs (1)~(4) are dual !

How about between boundaries ?

RF



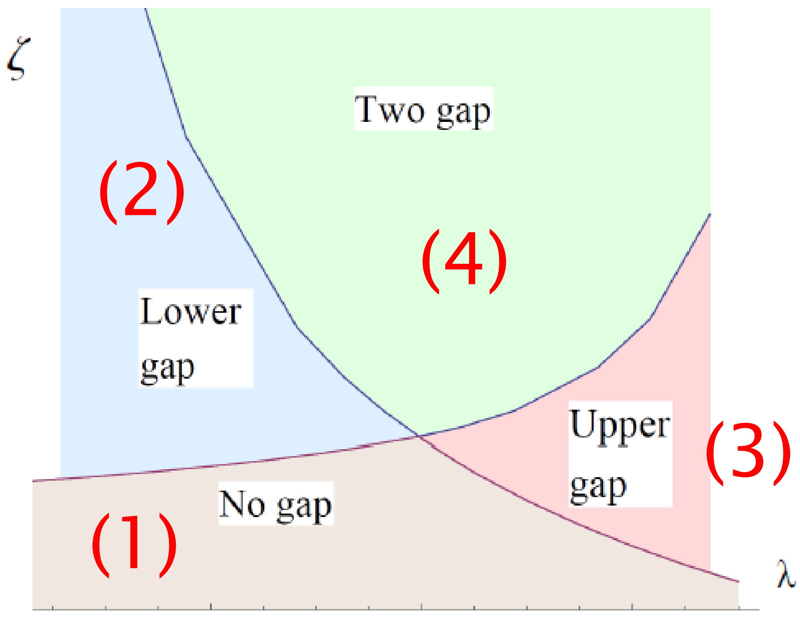
CB



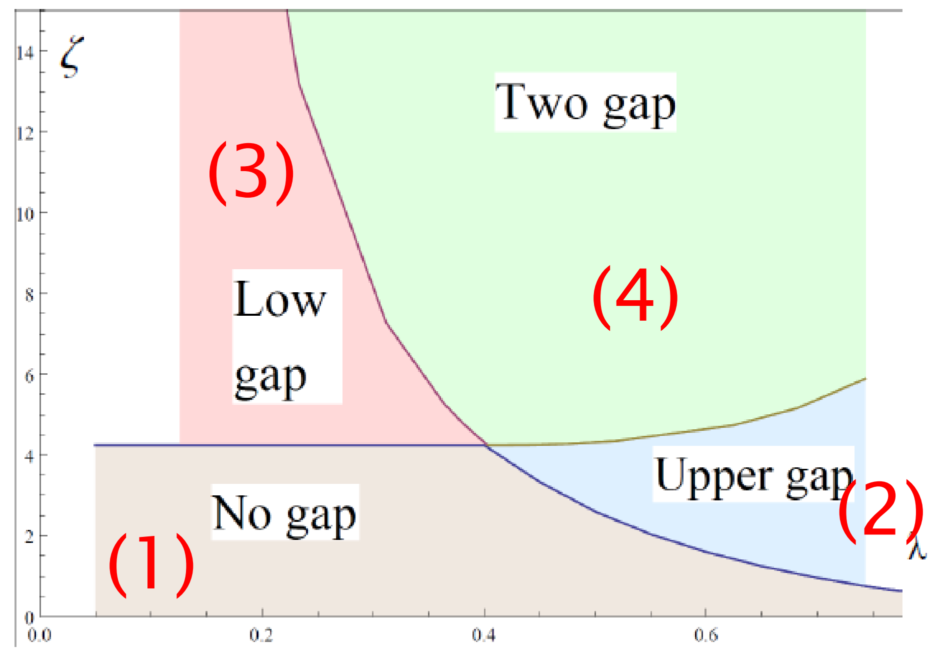
How about between boundaries ?

We have already confirmed

RF



CB



①Duality between the domain equations

②The map $b_{r.f} = \pi - a_{c.b}, \quad a_{r.f} = \pi - b_{c.b},$

③I have also checked the map between the boundary as

$$\lambda_{r.f} = 1 - \lambda_{c.b}, \qquad \zeta_{r.f} = \frac{\lambda_{c.b}}{1-\lambda_{c.b}}\zeta_{c.b},$$

Relationship between Free energy

By substituting the eigenvalue density, we can confirm the duality of the Free energy as

$$F_{c.b}^{N} = V^{c.b}[\rho^{c.b}, N] + F_2[\rho^{c.b}, N] = V^{r.f}[\rho^{r.f}, k-N] + F_2[\rho^{r.f}, k-N] = F_{r.f}^{k-N}$$

$$\Downarrow$$

Duality is completely confirmed !

Level k U(N) CB theory

Level k U(k−N) RF theory

We have confirmed the duality !

5.Self-dualtiy of N=2 3d supersymmetric CS matter theory and the Giveon-Kutasov duality and the Seiberg duality

5-1. N=2 SUSY CS matter theory

- This theory has the same number of supercharge as the N=1 4d SUSY theory.

- Hence checking the Seiberg-like duality is interesting.

5-1 N=2 SUSY CS matter theory

- The action is

$$S=\int d^3x\Big[i\varepsilon^{\mu\nu\rho}\frac{k}{4\pi}\mathrm{Tr}(A_\mu\partial_\nu A_\rho-\frac{2i}{3}A_\mu A_\nu A_\rho)+D_\mu\bar{\phi}D^\mu\phi+\bar{\psi}\gamma^\mu D_\mu\psi$$

$$+\lambda_4(\bar{\psi}\psi)(\bar{\phi}\phi)+\lambda_4'(\bar{\psi}\phi)(\bar{\phi}\psi)+\lambda_4''\left((\bar{\psi}\phi)(\bar{\psi}\phi)+(\bar{\phi}\psi)(\bar{\phi}\psi)\right)+\lambda_6(\bar{\phi}\phi)^3\Big].$$

$$\lambda_4=\frac{x_4}{\kappa},\quad \lambda_4'=\frac{x_4'}{2\kappa},\quad \lambda_4''=\frac{x_4''}{4\kappa},\quad \lambda_6=\frac{x_6}{(2\kappa)^2},\qquad \kappa=\frac{k}{4\pi}.$$

$$x_4=x_6=1$$

5-1 N=2 SUSY CS matter theory

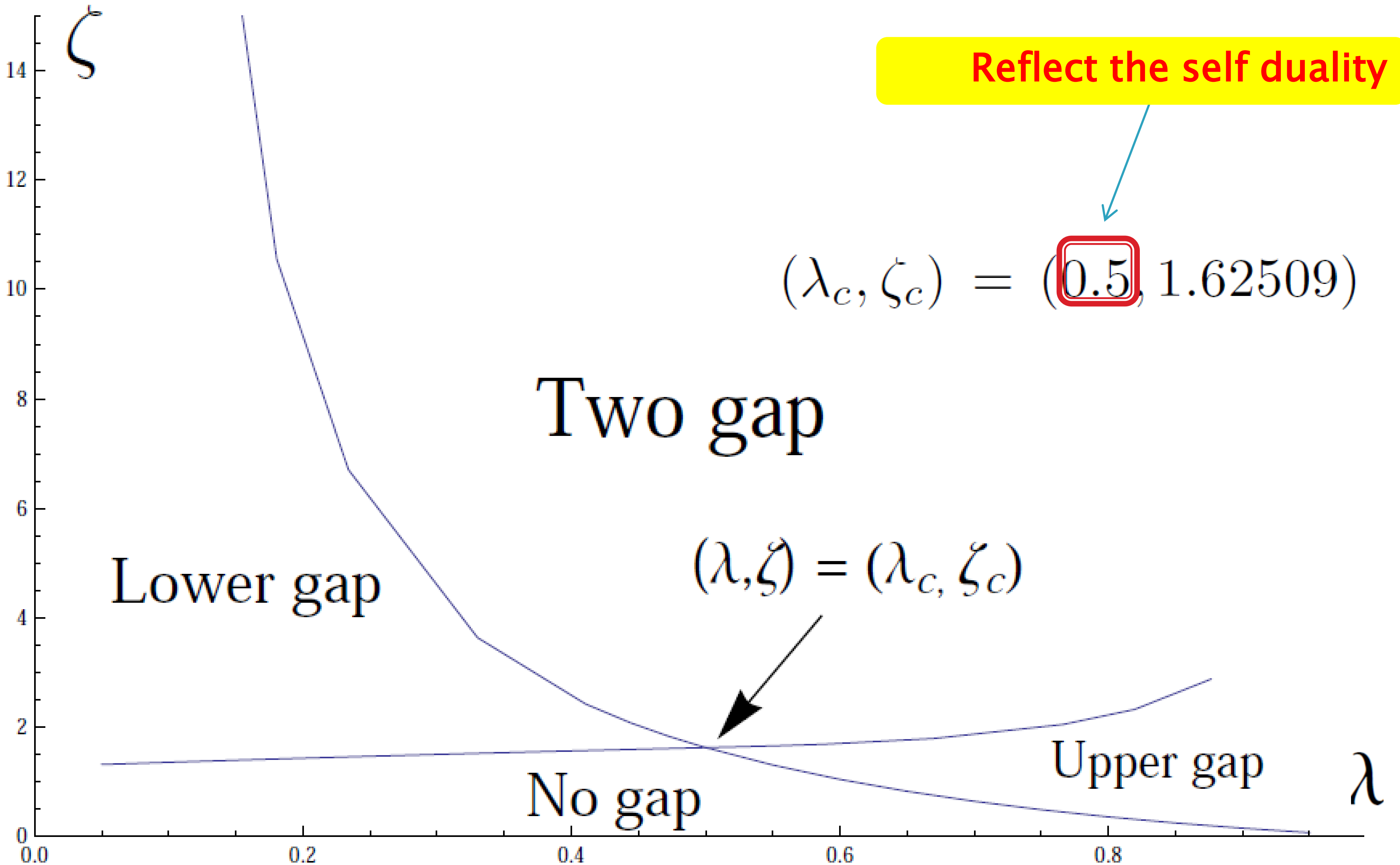
- The eigenvalue density has the form of the summation of the ones of RF and CB

(No gap, lower gap, upper gap, two gap phase eigenvalues are described by the sum)

$$\rho(\alpha) = \rho^{r.f}(\zeta, \lambda; \tilde{c}, a; \alpha) + \rho^{c.b}(\zeta, \lambda; \tilde{c}, a; \alpha)$$

- Same kind of phase transition with upper limit of the eigenvalue density.

5-1-2 Phase diagram



5-2 Self-duality under the level-rank duality (Seiberg-like)

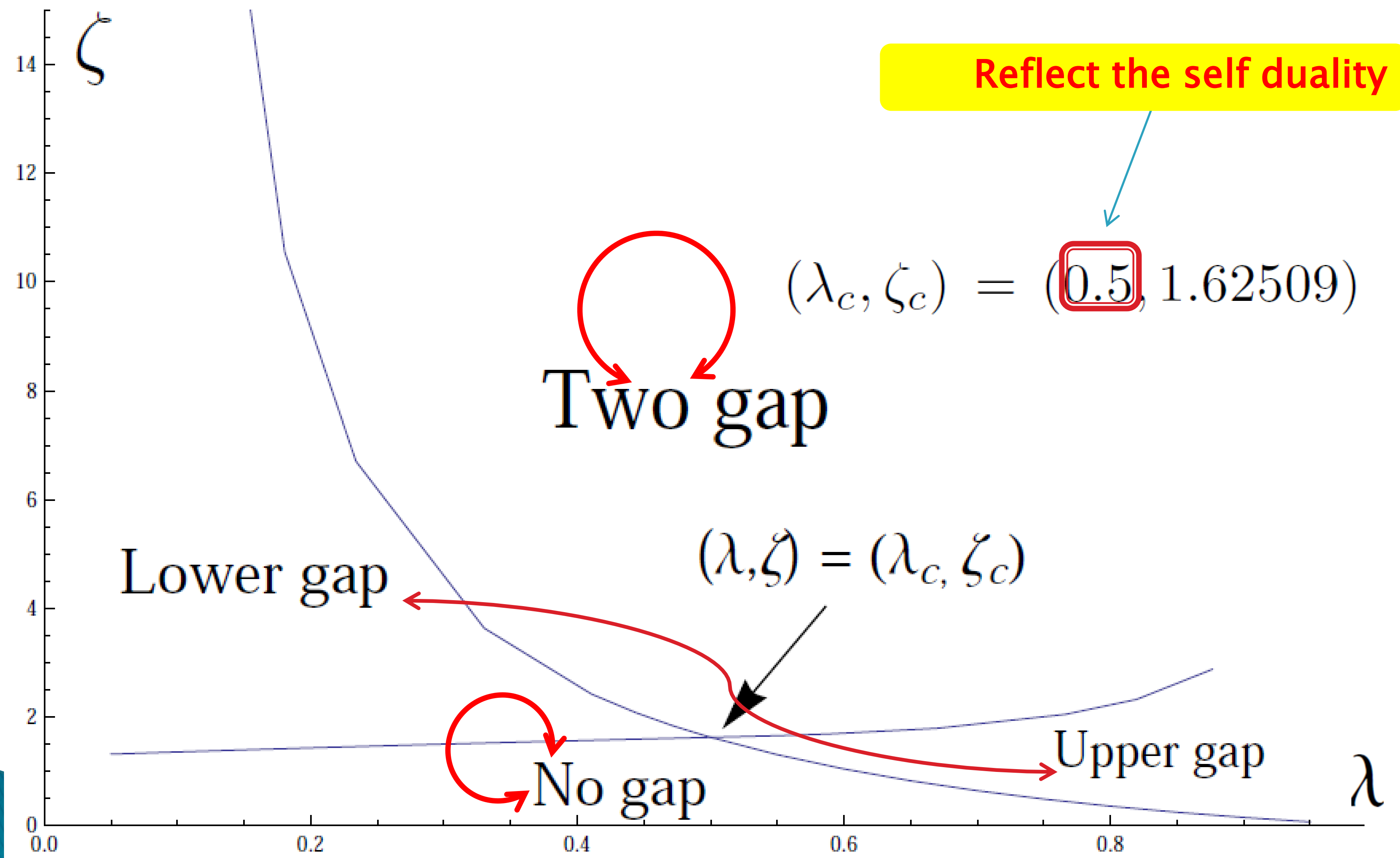
N=2 SUSY 't Hooft coupling λ with temperature ζ

$\Longleftrightarrow$

N=2 SUSY 't Hooft coupling $1-\lambda$ with temperature $\lambda\zeta/(1-\lambda)$

We have confirmed the duality (Because we have confirmed the duality between RF and CB)

5-2-1 Duality between phases



5-2-2 Giveon-Kutasov duality

[Giveon-Kutasov 2008]

N=2 SUSY Level k
U(N_c) N_f fundamental flavor

dual

N=2 SUSY Level $-k$
U($k+N_f-N_c$)
N_f fundamental flavor with
$N_f \times N_f$ meson operator $M^i{}_j$

5-2-2 Giveon-Kutasov duality

[Giveon-Kutasov 2008]

N=2 SUSY Level k
- $U(N_c)$ N_f fundamental flavor

dual

N=2 SUSY Level $-k$
- $U(k+N_f-N_c)$
- N_f fundamental flavor with
- $N_f \times N_f$ meson operator $M^i{}_j$

If we reinterpret the $N'_f = k + N_f$, it resemble to Seiberg-duality

5-2-2 Giveon-Kutasov duality

[Giveon-Kutasov 2008]

N=2 SUSY Level k
$U(N_c)$ N_f fundamental flavor

N=2 SUSY Level $-k$
$U(k+N_f-N_c)$
N_f fundamental flavor with
$N_f \times N_f$ meson operator $M^i{}_j$

In our analysis, $k >> N_f$, $N_c >> N_f$,
and based on $\lambda = N_c / k$, adding N_f is negligible in our analysis.

5-2-2 Giveon-Kutasov duality

[Giveon-Kutasov 2008]

N=2 SUSY Level k
$U(N_c)$ N_f fundamental flavor

N=2 SUSY Level $-k$
$U(k+N_f-N_c)$
N_f fundamental flavor with
$N_f \times N_f$ meson operator $M^i{}_j$

Would be ||

N=2 SUSY 't Hooft coupling λ with temperature ζ

N=2 SUSY 't Hooft coupling $1-\lambda$ with temperature $\lambda\zeta/(1-\lambda)$

5-3 Giveon-Kutasov and Seiberg-duality

- Aharony duality. [Aharony 1997]

N=2 SUSY U(N_c) N_f fundamental flavor

gauge theory without CS termmatter

⟺

N=2 SUSY U($N_f - N_c$) gauge theory with N_f fundamental flavor with

$N_f \times N_f$ meson operator $M^i{}_j$,

With additional singlet with $V_{+,-}$ in supepotential

$$W = V_+ v_- + V_- v_+ + q\tilde{q}M.$$

v_+, v_- are the monopole operators

5-3 Giveon-Kutasov and Seiberg-duality

- Giveon Kutasov can be derived from Aharony duality. [Kapustin-Willet-Yaakov 2010]

- We can understand the Aharony duality by the brane configuation composed by NS5, NS5, Nc D3-brane and Nf D5-brane, from the context of Aharony duality, if we replace the k D5-brane by (1,k) 5-brane which is the bound state with the NS5-brane, it becomes the context of Giveon-Kutasov duality. ($k+N_f$ is preserved under this treatment.)

5-3 Giveon-Kutasov and Seiberg-duality

- Aharony duality can be derived from the 4d Seiberg-duality as shown by [Aharony-Razamat-Seiberg-Willet 2013]

- (1) dimensional reduction
- (2) With suitable modification of the superpotential by monopole operators.

- Relates to the 4d Seiberg-duality.

6.Summary & discussion

6-1 Summary(Motivation)

- CS matter theory →Information of the Quantum gravity ?

⇓

- Study of the Phase structure is meaningful.
- We have investigated the phase structure of the CS matter theory

6-2 Summary(New salient phase)

- CS matter theory on $S^1 \times S^2$

$$\Downarrow$$

(1) Holonomy along the S^1 Linearly couple to the Magnetic flux on S^2

(2) Non-Propagating D.O.F of gauge fields

- New salient phases show up

→ New interesting information of the QG ?

6-3. Summary (Triality)

- We confirmed the CFT-CFT duality.

~~Parity~~ Vasiliev's gravity theory

CS theory coupled to regular fermions

Gravity side

Dual

CS theory coupled to critical bosons

Chern-Simons side

Level-rank duality in the pure CS theory

6-3. Summary (Triality)

- We confirmed the CFT-CFT duality.

~~Parity~~ Vasiliev's gravity theory

Gravity side

CS theory coupled to regular fermions

Dual

CS theory coupled to critical bosons

Chern-Simons side

Upper limit of the eigenvalue density plyas crucial role

6-3. Summary (Triality)

- We confirmed the CFT-CFT duality.

~~Parity~~ Vasiliev's gravity theory

Gravity side

CS theory coupled to regular fermions

Dual

CS theory coupled to critical bosons

Chern-Simons side

Anyon ?? (Connecting the fermion and boson)

6-4 SUSY self-duality

- This would be 3d version of the Seiberg-duality -> Non -SUSY extension ?? (Private communication to Adi Armoni, thanks)