ADHM Construction of Skyrmions

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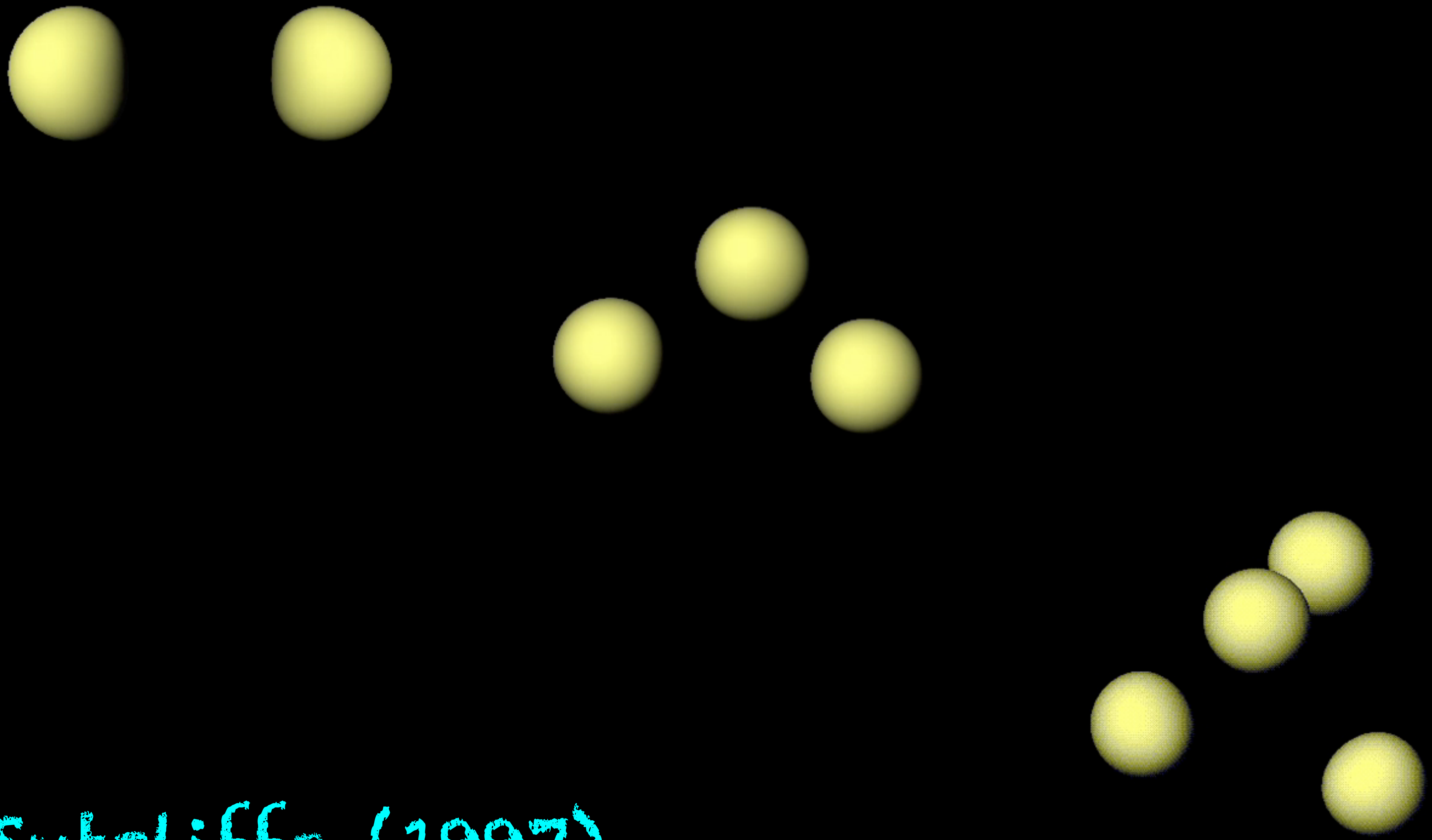
Harland and Sutcliffe, arXiv:2307.09355

Outline

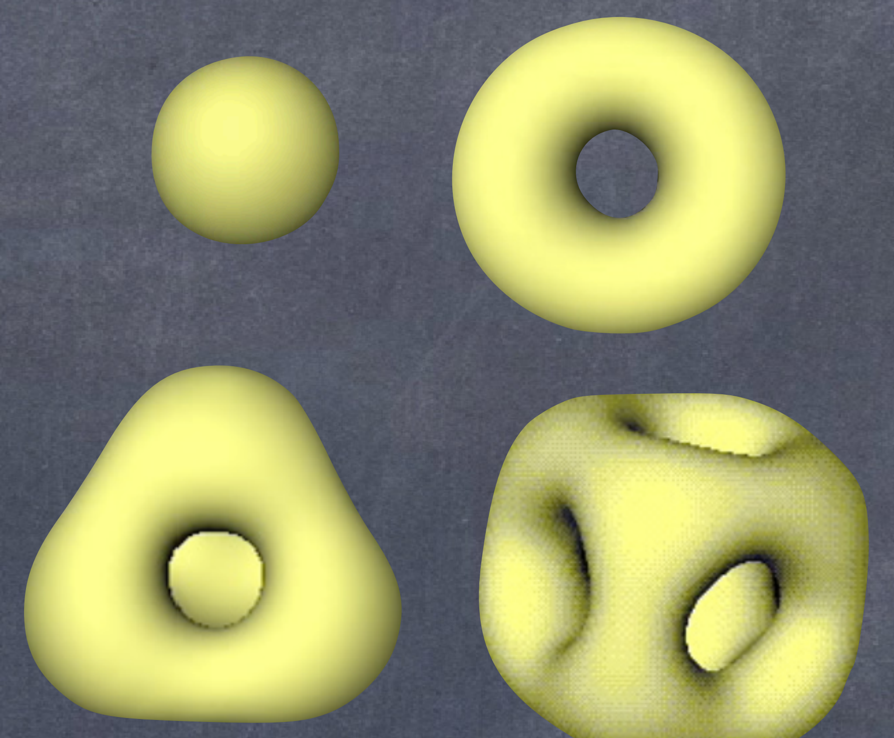
1. Introduction
2. Skyrmions
3. Rational maps
4. Instantons and ADHM
5. Atiyah-Manton approximation
6. ADHM construction of Skyrmions
7. Conclusion

Skyrmion dynamics



Battye+Sutcliffe (1997)

Motivation



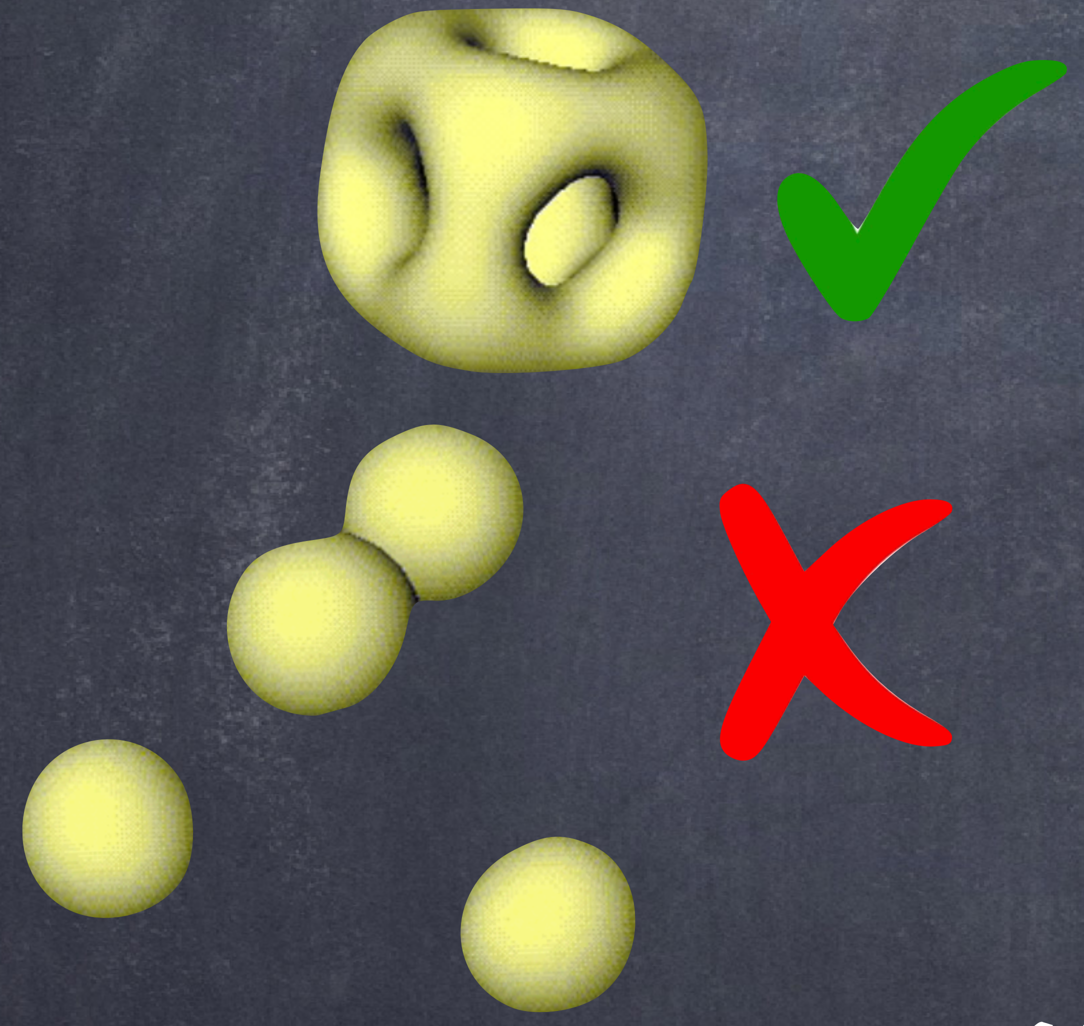
Skyrmions

Motivation

product ansatz

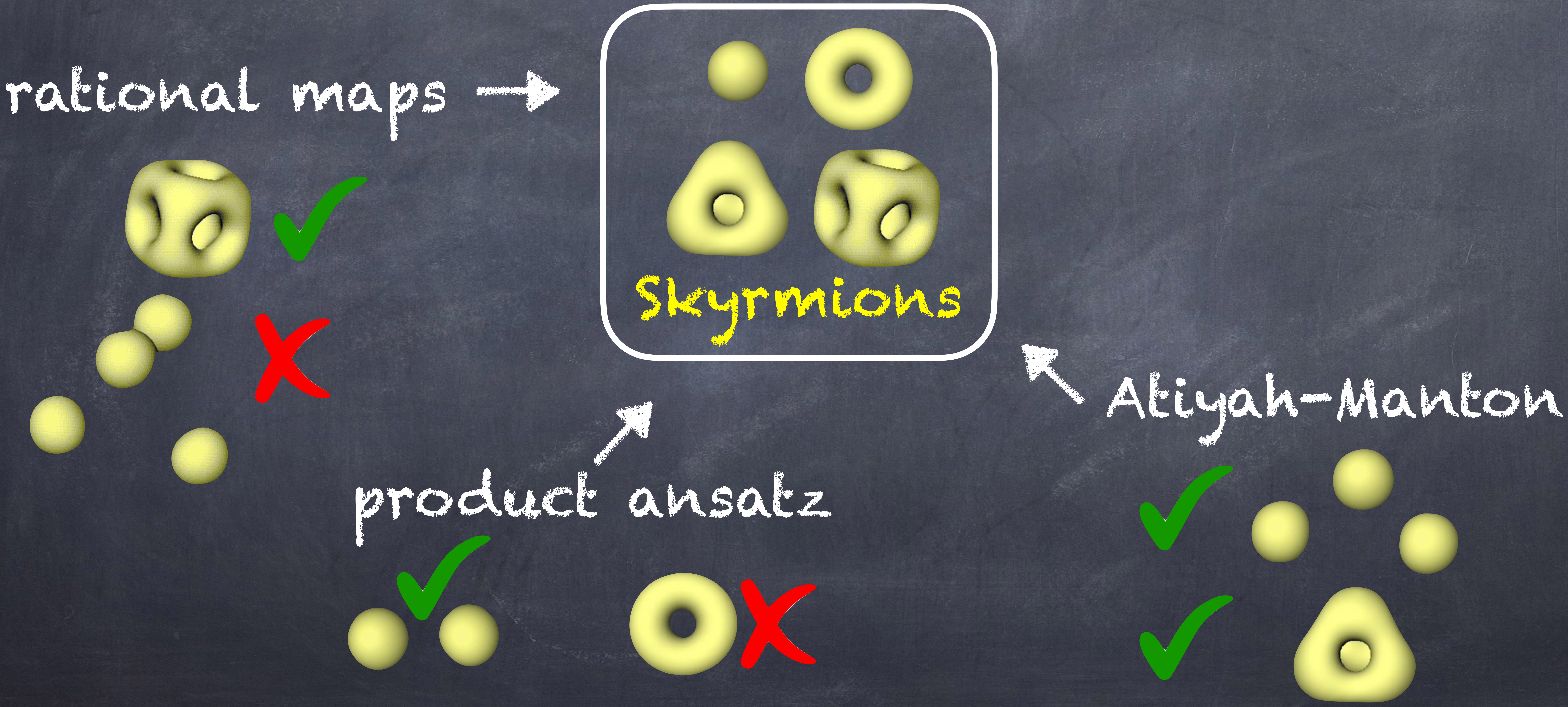
Motivation

rational maps →

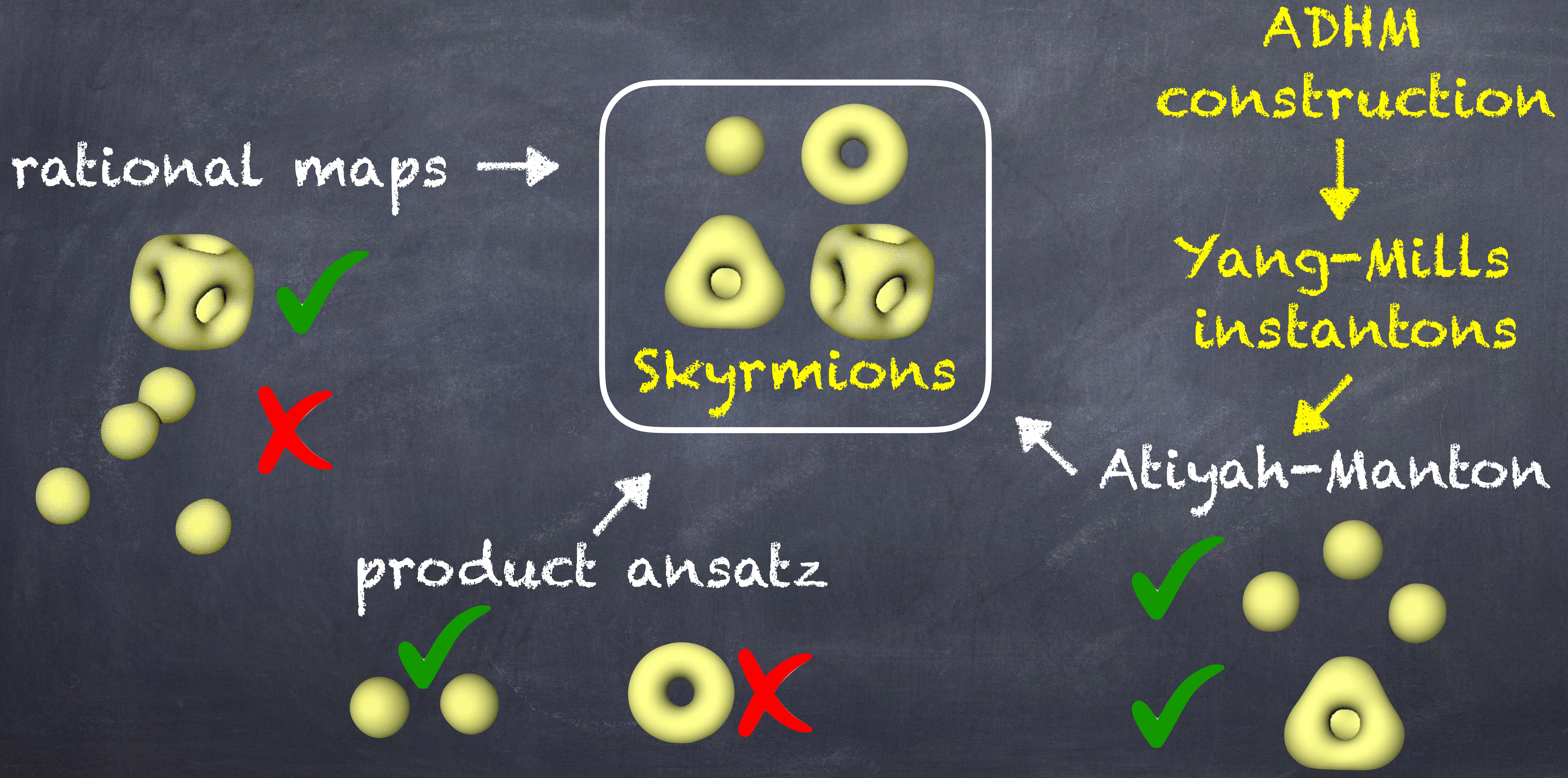


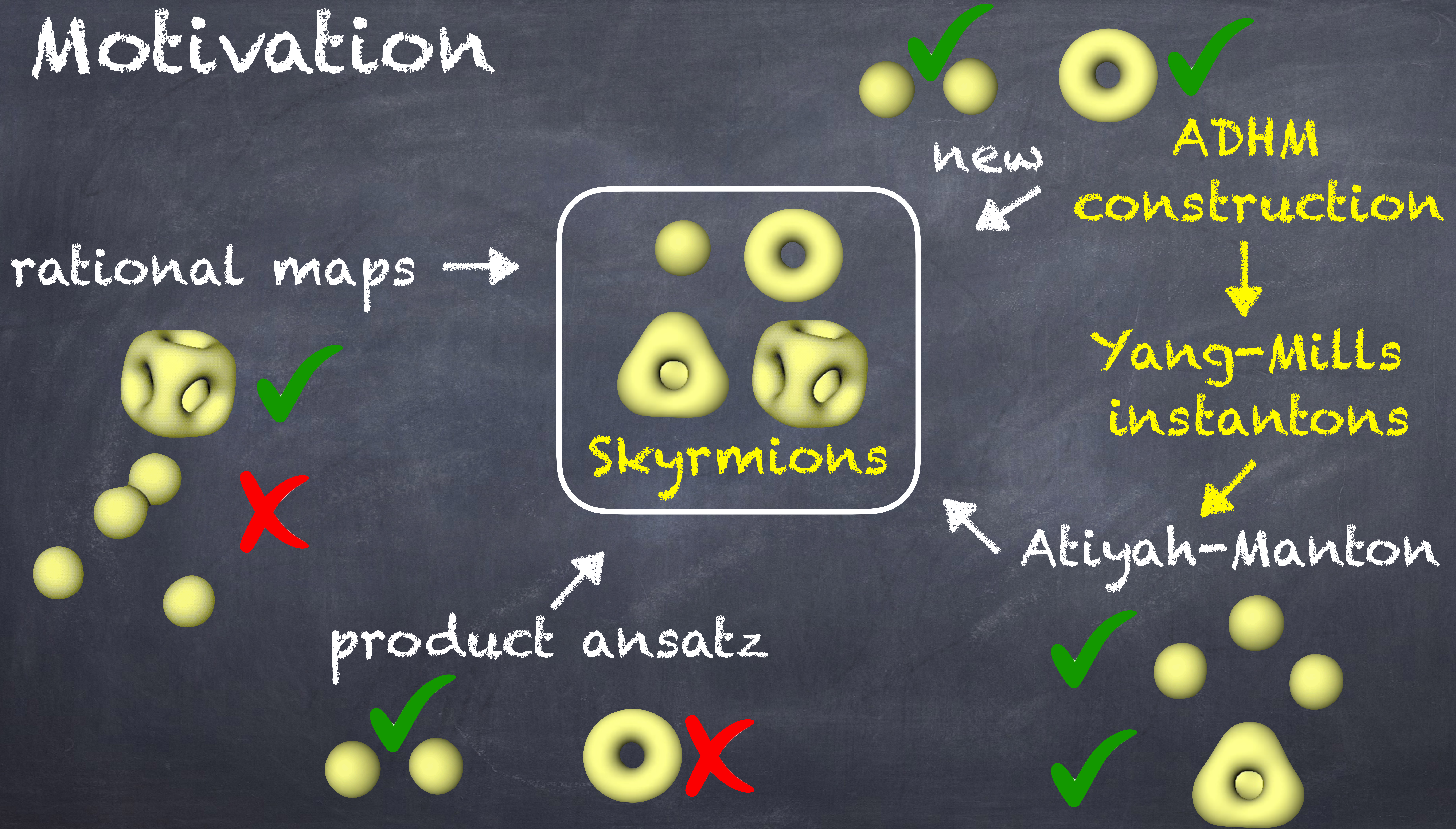
product ansatz

Motivation



Motivation


ADHM construction
Yang-Mills instantons
Atiyah-Manton
rational maps
Skyrmions
product ansatz


Motivation
new
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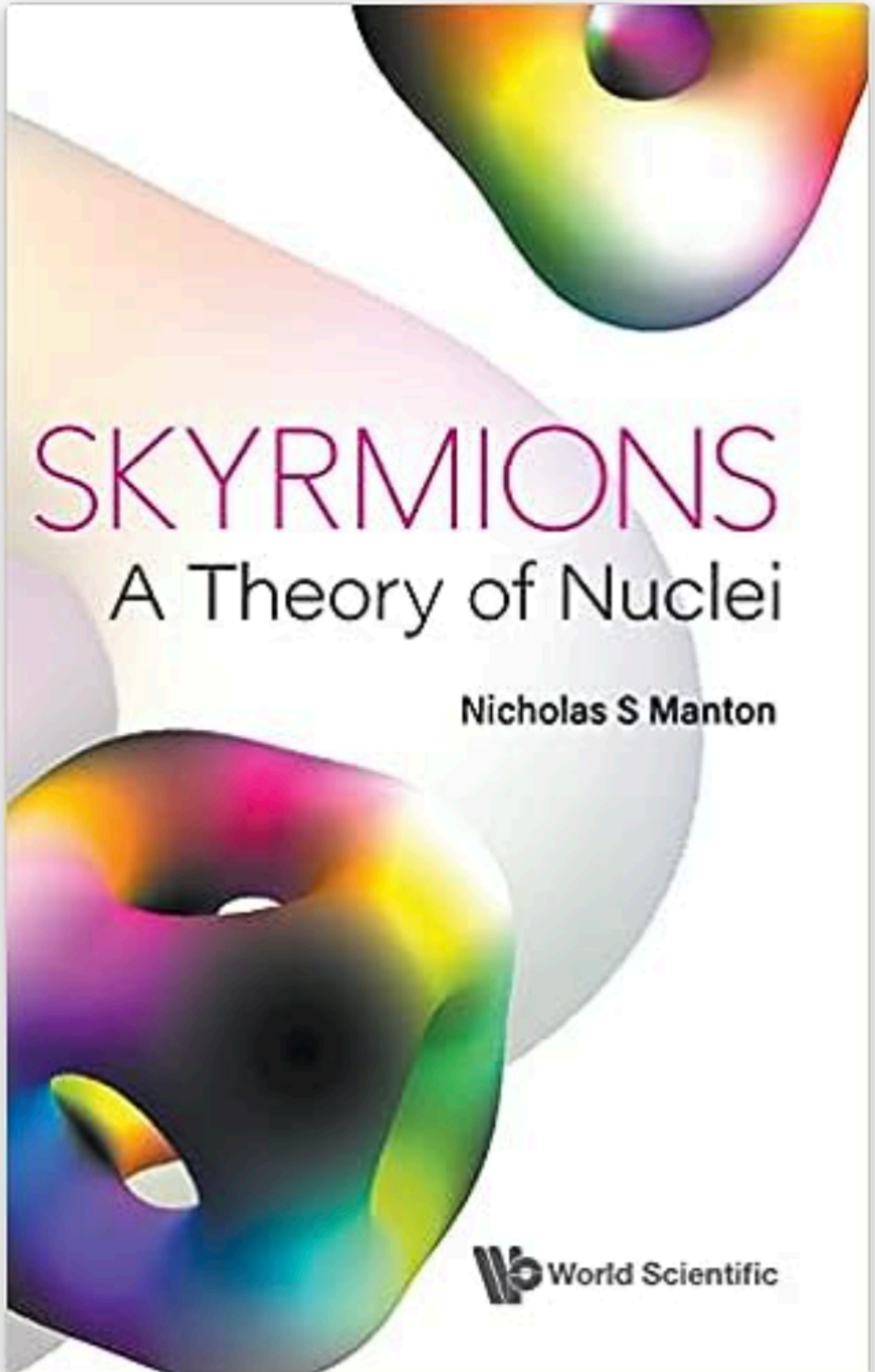

SKYRMIONS
A Theory of Nuclei
Nicholas S Manton
World Scientific

Skyrmions

Skyrme (1962)

$U(x) : \mathbb{R}^3 \mapsto SU(2)$, with $U \to 1$ as $|x| \to \infty$.

Topological charge $B \in \mathbb{Z} = \pi_3(SU(2))$

Baryon number $$B = \int \frac{1}{24\pi^2} \varepsilon_{ijk} \mathrm{Tr}(R_i R_k R_j)\, d^3x,$$

baryon density

$R_i = (\partial_i U)U^{-1} \in \mathfrak{su}(2)$, $i = 1,2,3$.

$$E = \frac{1}{12\pi^2} \int -\mathrm{Tr}\left\{ \frac{1}{2} R_i^2 + \frac{1}{16} [R_i, R_j]^2 \right\} d^3x \geq B\,.$$

B=1 hedgehog Skyrmion

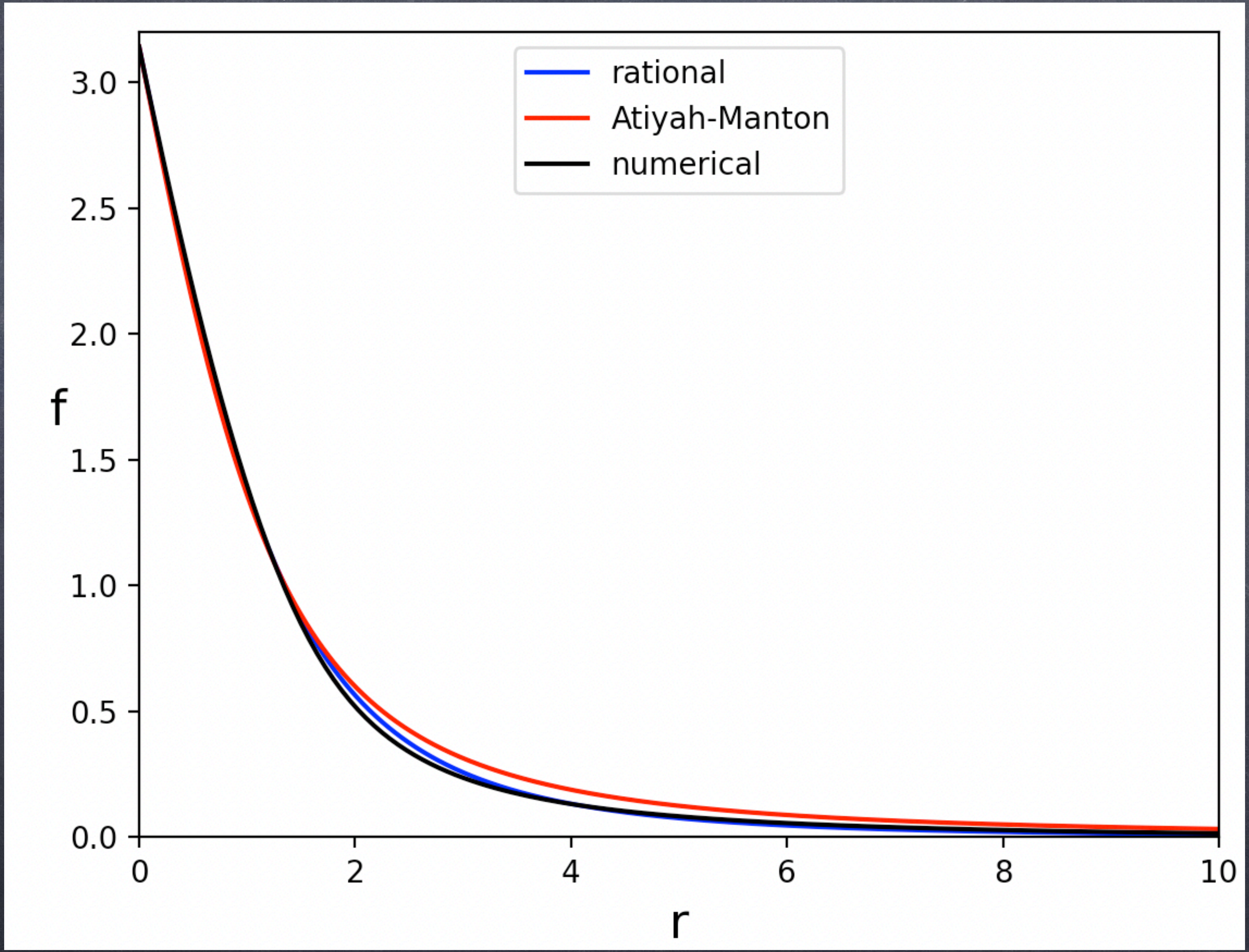
$$U = \cos f + i\frac{\sin f}{r}\,\boldsymbol{x}\cdot\boldsymbol{\tau},$$ isospin symmetry $U \mapsto \mathcal{O}U\mathcal{O}^{-1}$

$f(r)$ is a profile function with $f(0) = \pi,\, f(\infty) = 0.$

$$E = \frac{1}{3\pi}\int_0^\infty \left(r^2\left(\frac{df}{dr}\right)^2 + 2\left(1 + \left(\frac{df}{dr}\right)^2\right)\sin^2 f + \frac{\sin^4 f}{r^2}\right) dr\,.$$

Numerical minimization gives $E = 1.232$

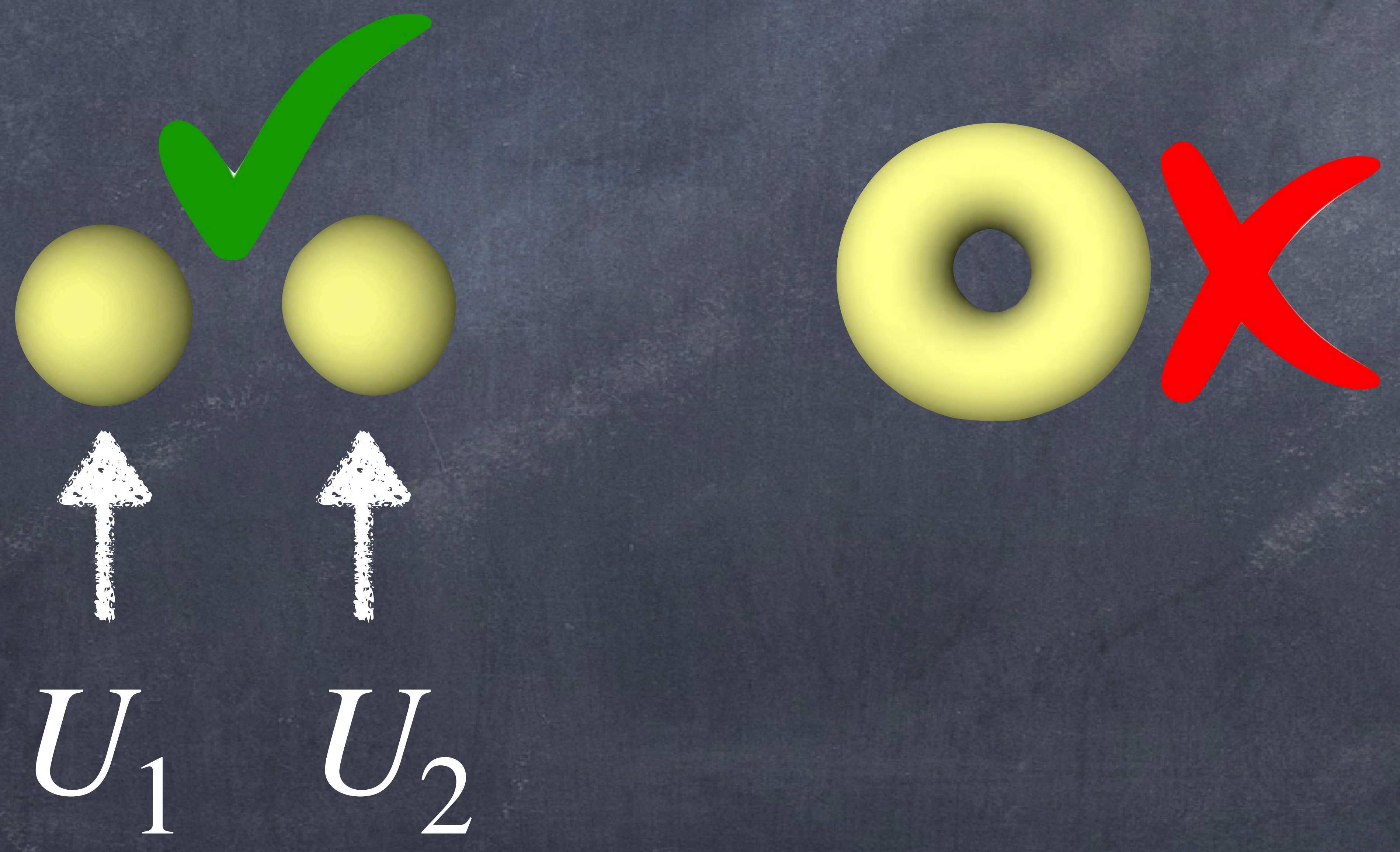
B=1 profile function



Product ansatz

$$U = U_1 U_2$$

only works for well-separated Skyrmions



Rational map approximation

Houghton+Manton+Sutcliffe (1997)

Rewrite the hedgehog ansatz using spherical polar and Riemann sphere coordinates.

$$\boldsymbol{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta), \quad z = e^{i\phi}\tan(\theta/2)\,.$$

$$U = \exp\left(\frac{i f(r)}{1+|W|^2}\begin{pmatrix} 1-|W|^2 & 2\overline{W} \\ 2W & |W|^2-1 \end{pmatrix}\right)$$

$W = z,$ is a degree 1 rational map $W(z) : \mathbb{CP}^1 \mapsto \mathbb{CP}^1$.

Rational map approximation

Take a degree B rational map $W(z) : \mathbb{CP}^1 \mapsto \mathbb{CP}^1$,

$$W(z) = \frac{\alpha_0 + \alpha_1 z + \ldots + \alpha_B z^B}{\beta_0 + \beta_1 z + \ldots + \beta_B z^B}$$

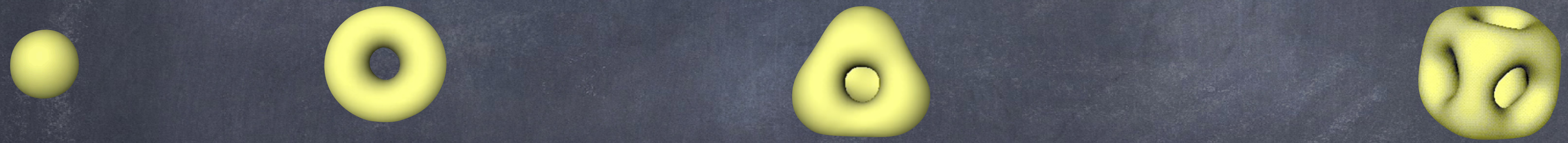
$$E = \frac{1}{3\pi}\int_0^\infty \left(r^2\left(\frac{df}{dr}\right)^2 + 2B\left(1 + \left(\frac{df}{dr}\right)^2\right)\sin^2 f + I\frac{\sin^4 f}{r^2}\right) dr,$$

$$I = \frac{1}{4\pi}\int \left(\frac{1+|z|^2}{1+|W|^2}\left|\frac{dW}{dz}\right|\right)^4 \frac{2i\, dz d\bar{z}}{(1+|z|^2)^2}.$$

Rational map approximation

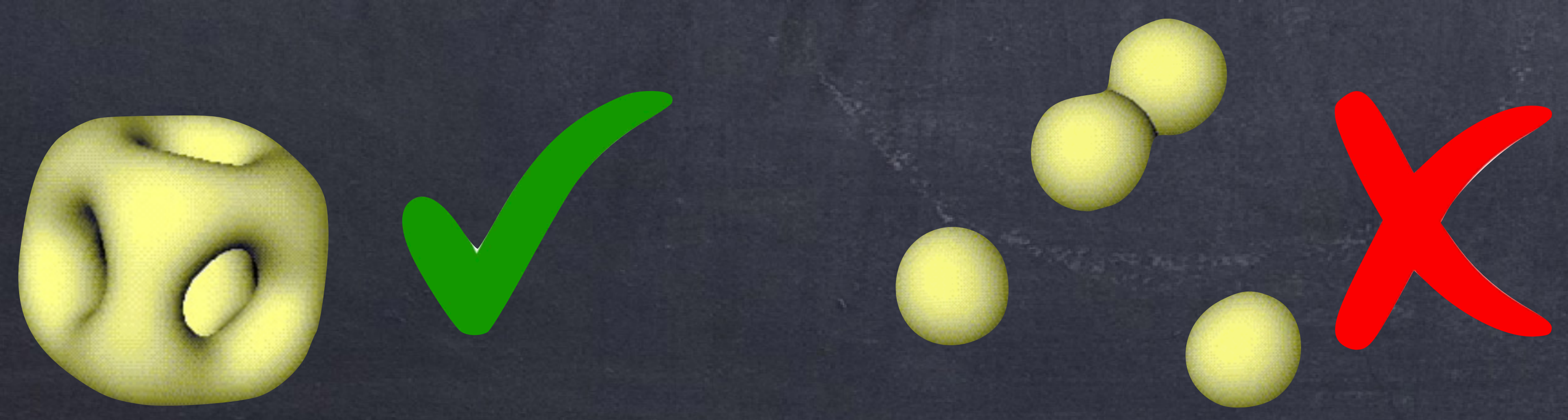
Maps that minimize I for B=1,2,3,4 are

$$W = z, \qquad W = z^2, \qquad W = \frac{z^3 - \sqrt{3}iz}{\sqrt{3}iz^2 - 1}, \qquad W = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}.$$



Eg. $B = 2$, numerical $E = 2.358$, approximation $E = 2.416$,

error $\approx 2.5\,\%$



Yang-Mills instantons

$SU(2)$ Yang-Mills in $\mathbb{R}^4$,

$$A_\mu \in \mathfrak{su}(2), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \qquad \mu = 1,2,3,4.$$

$$E_{\mathrm{YM}} = -\frac{1}{16\pi^2}\int \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})\, d^4x \geq N = -\frac{1}{32\pi^2}\int \mathrm{Tr}(\varepsilon_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta})\, d^4x\,.$$

instanton number

$E_{\mathrm{YM}} = N$, for self-dual fields, $F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$

$8N$–dimensional moduli space of charge N instantons.

ADHM construction of instantons

ADHM = Atiyah+Drinfeld+Hitchin+Manin (1978)

ADHM data: $\widehat{M} = \begin{pmatrix} L \\ M \end{pmatrix},$

row of N quaternions

symmetric NxN matrix of quaternions

with $\widehat{M}^\dagger \widehat{M}$ a real non-singular NxN matrix,
where † is quaternionic conjugate transpose.

ADHM construction of instantons

1 Given the ADHM data, form the operator

$$\Delta(x, x_4) = \begin{pmatrix} L \\ M - (ix_1 + jx_2 + kx_3 + x_4)1_N \end{pmatrix},$$

i,j,k are the quaternions

NxN identity matrix

2 Solve $\Psi(x, x_4)^\dagger \Delta(x, x_4) = 0, \qquad \Psi(x, x_4)^\dagger \Psi(x, x_4) = 1,$

where $\Psi(x, x_4)$ is an N+1 component column vector.

3 $A_\mu = \Psi(x, x_4)^\dagger \partial_\mu \Psi(x, x_4)$, is a pure quaternion.

Atiyah-Manton approximation

Atiyah+Manton (1989)

Skyrmions ≈ holonomy instantons

$U(x) = \Omega(x, \infty)$, where $\Omega(x, x_4)$ is the solution of

$$\partial_4 \Omega(x, x_4) = \Omega(x, x_4) A_4(x, x_4), \quad \text{with } \Omega(x, -\infty) = 1.$$

↗ gauge potential of a charge N instanton

This gives a Skyrme field with B=N.

B=1, Atiyah-Manton approximation

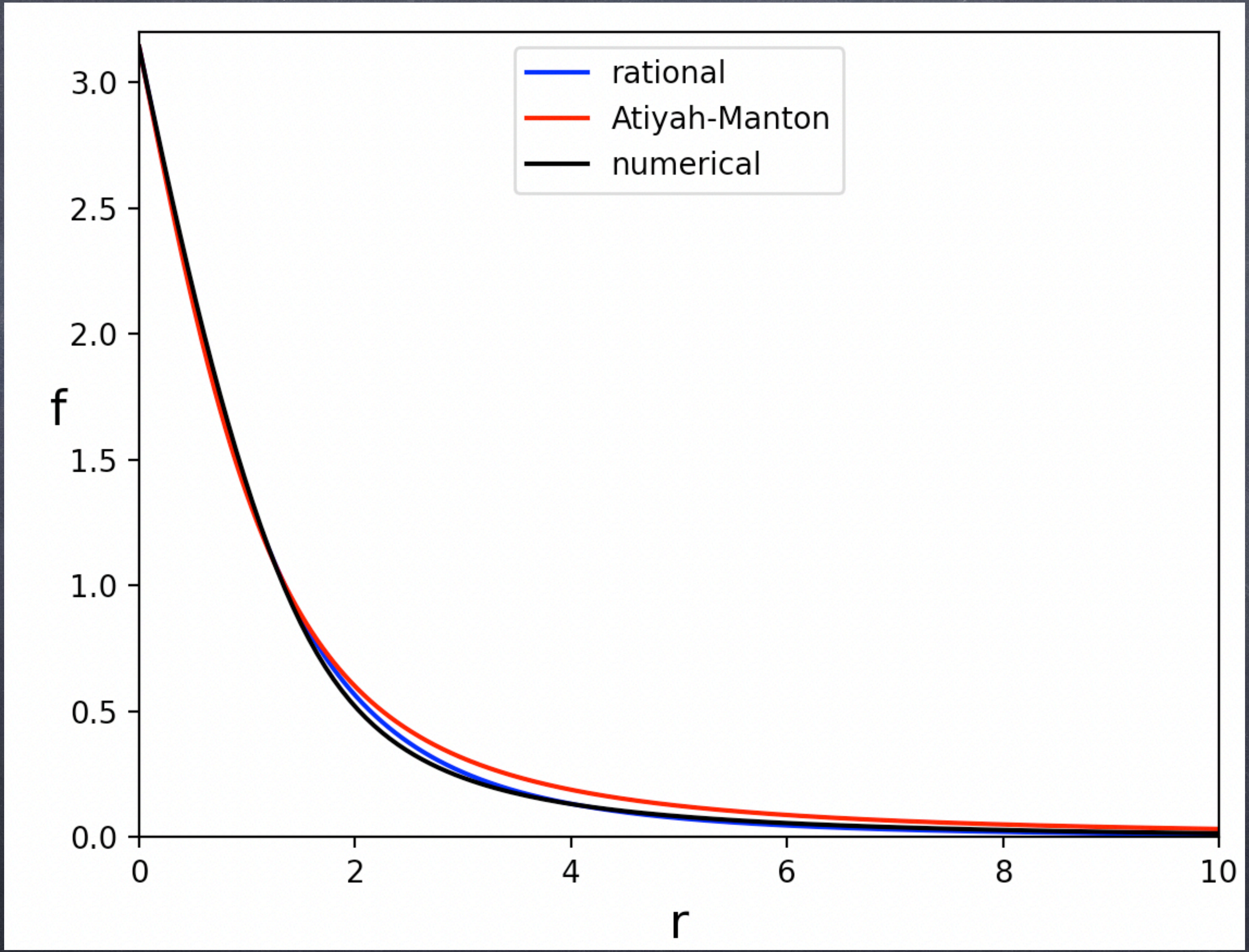
Taking an N=1 instanton with size λ
gives a hedgehog Skyrmion with a profile function

$$f(r) = \pi\left(1 - \frac{r}{\sqrt{\lambda^2 + r^2}}\right).$$

For $\lambda = 1.45$, the energy is $E = 1.243$

Comparing with the true value $E = 1.232$, error $< 1\,\%$.

B=1 profile function



Atiyah-Manton approximation

Why does it work?

Consider approximating an instanton with a Skyrmion.

In the gauge $A_4 = 0$, set $A_i = -\frac{1}{2}(\partial_i U)U^{-1}(1 + \tanh(x_4))$.

This gives $N = B$ and $E_{\mathrm{YM}} = E$ and $E_{YM} \geq N \implies E \geq B$.

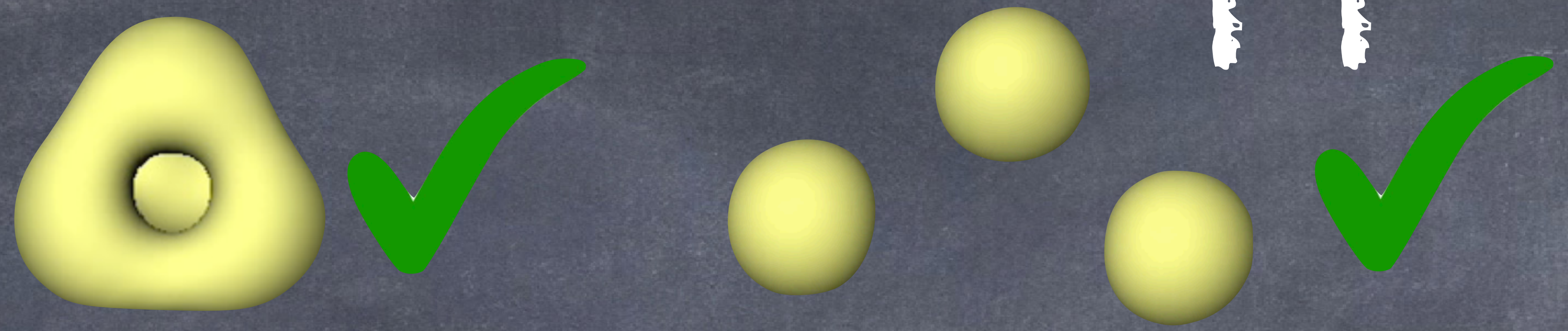
This is the leading term in an expansion in x_4.

More terms gives a Skyrme model with vector mesons.

Yang-Mills = Skyrme + ∞ tower of vector mesons.

Sutcliffe (2010)

Atiyah-Manton approximation



The instanton moduli spaces induce spaces of Skyrmions that include good approximations to all known minimal energy Skyrmions, and allow these to be separated into individual Skyrmions.

The ODE $\partial_4 \Omega(\boldsymbol{x}, x_4) = \Omega(\boldsymbol{x}, x_4) A_4(\boldsymbol{x}, x_4)$ can only be solved explicitly for a spherically symmetric Skyrmion and must be computed numerically in all other cases.

ADHM construction of Skyrmions

Harland+Sutcliffe (2023)

ADHM data → rational approximations to Skyrmions

$\partial_4\Omega(\boldsymbol{x}, x_4) = \Omega(\boldsymbol{x}, x_4)A_4(\boldsymbol{x}, x_4),$ set $A_4 = \Psi(\boldsymbol{x}, x_4)^\dagger\partial_4\Psi(\boldsymbol{x}, x_4),$

$$\partial_4\Omega(\boldsymbol{x}, x_4) = \Omega(\boldsymbol{x}, x_4)\Psi(\boldsymbol{x}, x_4)^\dagger\partial_4\Psi(\boldsymbol{x}, x_4)\,.$$

Discretize this equation by replacing $x_4 \in \mathbb{R}$ by a lattice $-\infty = t_1 < t_2 < \ldots < t_p = \infty$, with forward difference approximations

$$\Omega(\boldsymbol{x}, t_{i+1}) - \Omega(\boldsymbol{x}, t_i) = \Omega(\boldsymbol{x}, t_i)\Psi(\boldsymbol{x}, t_i)^\dagger(\Psi(\boldsymbol{x}, t_{i+1}) - \Psi(\boldsymbol{x}, t_i))$$

and as Ψ has unit length $\Omega(\boldsymbol{x}, t_{i+1}) = \Omega(\boldsymbol{x}, t_i)\Psi(\boldsymbol{x}, t_i)^\dagger\Psi(\boldsymbol{x}, t_{i+1})$

ADHM construction of Skyrmions

As $\Omega(\boldsymbol{x}, t_1) = 1$ the solution is

$$\Omega(\boldsymbol{x}, \infty) = \Omega(\boldsymbol{x}, t_p) = \Psi(\boldsymbol{x}, t_1)^\dagger \Psi(\boldsymbol{x}, t_2) \Psi(\boldsymbol{x}, t_2)^\dagger \Psi(\boldsymbol{x}, t_3) \ldots \Psi(\boldsymbol{x}, t_{p-1})^\dagger \Psi(\boldsymbol{x}, t_p) \,.$$

Also, $\Psi(\boldsymbol{x}, t_1) = \Psi(\boldsymbol{x}, t_p) = e_1 = (1,0,...,0)^t$, hence

$$\Omega(\boldsymbol{x}, \infty) = e_1^\dagger \Psi(\boldsymbol{x}, t_2) \Psi(\boldsymbol{x}, t_2)^\dagger \Psi(\boldsymbol{x}, t_3) \ldots \Psi(\boldsymbol{x}, t_{p-1})^\dagger e_1 \,.$$

Now, $\Psi(\boldsymbol{x}, x_4) \Psi(\boldsymbol{x}, x_4)^\dagger = Q(\boldsymbol{x}, x_4)$ is the projector onto the kernel of $\Delta(\boldsymbol{x}, x_4)$,

$$Q(\boldsymbol{x}, x_4) = 1_{N+1} - \Delta(\boldsymbol{x}, x_4) \left(\Delta(\boldsymbol{x}, x_4)^\dagger \Delta(\boldsymbol{x}, x_4) \right)^{-1} \Delta(\boldsymbol{x}, x_4)^\dagger ,$$

$$\Omega(\boldsymbol{x}, \infty) = e_1^\dagger Q(\boldsymbol{x}, t_2) Q(\boldsymbol{x}, t_3) \ldots Q(\boldsymbol{x}, t_{p-1}) e_1 \,.$$

ADHM construction of Skyrmions

Crazy idea: use only 3 interior lattice points, $t_2, t_3, t_4 = -\mu, 0, \mu$

$$U(x) = \frac{e_1^\dagger Q(\boldsymbol{x}, -\mu) Q(\boldsymbol{x}, 0) Q(\boldsymbol{x}, \mu) e_1}{|\, e_1^\dagger Q(\boldsymbol{x}, -\mu) Q(\boldsymbol{x}, 0) Q(\boldsymbol{x}, \mu) e_1 \,|}.$$

Looks algebraic but in fact all examples will be rational.

ADHM data has an axial symmetry in the (x_3, x_4)-plane

$$\Longrightarrow \quad e_1^\dagger \Big(Q(\boldsymbol{x}, 0) Q(\boldsymbol{x}, \mu) - Q(\boldsymbol{x}, -\mu) Q(\boldsymbol{x}, 0) \Big) e_1 = 0$$

$\Longrightarrow$ $U(x)$ is rational in the Cartesian coordinates.

B=1 rational Skyrmion

$\widehat{M} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix},$ λ is the size of the instanton

$$U = \frac{r^2(r^2+\lambda^2+\mu^2)^2 - \lambda^4\mu^2 + 2i\lambda^2\mu(r^2+\lambda^2+\mu^2)\boldsymbol{x}\cdot\boldsymbol{\tau}}{\left((r^2+\lambda^2)^2+\mu^2 r^2\right)\left(r^2+\mu^2\right)},$$

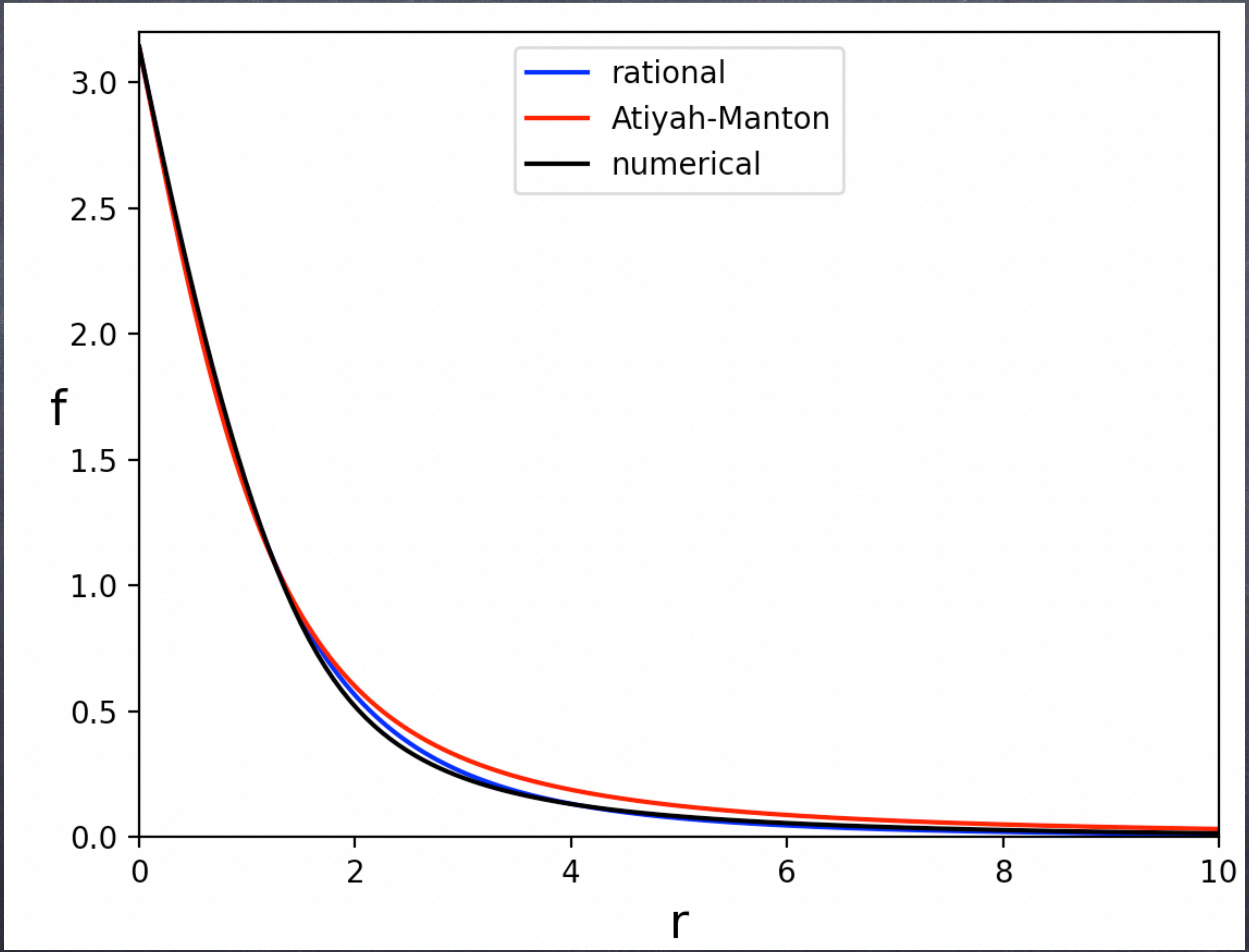
hedgehog with profile function $f(r) = \tan^{-1}\left(\frac{2\lambda^2\mu r\left(r^2+\lambda^2+\mu^2\right)}{r^2\left(r^2+\lambda^2+\mu^2\right)^2 - \lambda^4\mu^2}\right).$

True: $E = 1.232$

Atiyah-Manton: $E = 1.243$, error $0.9\,\%$, $\lambda = 1.45$.

Rational Skyrmion: $E = 1.236$, error $0.3\,\%$, $\lambda = 2.03$, $\mu = 1.43$.

B=1 profile function



B=1 rational Skyrmion

$\lambda = 2.03 \mapsto 2, \quad \mu = 1.43 \mapsto \sqrt{2}$

$$U = \frac{r^6 + 12r^4 + 36r^2 - 32 + 8\sqrt{2}i(r^2+6)\boldsymbol{x}\cdot\boldsymbol{\tau}}{\left(r^2+8\right)\left(r^2+2\right)^2}.$$

B=2 rational axial Skyrmion

$$\widehat{M} = \frac{\lambda}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2}k \\ i & j \\ j & -i \end{pmatrix},$$

$$\lambda = \mu = 2$$

$$U = \frac{\widetilde{\sigma} + i\widetilde{\pi}\cdot\boldsymbol{\tau}}{\sqrt{\widetilde{\sigma}^2 + \widetilde{\pi}_1^2 + \widetilde{\pi}_2^2 + +\widetilde{\pi}_3^2}}$$

$$\widetilde{\pi}_1 = 64\left(x_1^2 - x_2^2\right)\left(r^2+4\right)\left(\left(r^2+8\right)^2 - 4\rho^2\right)$$

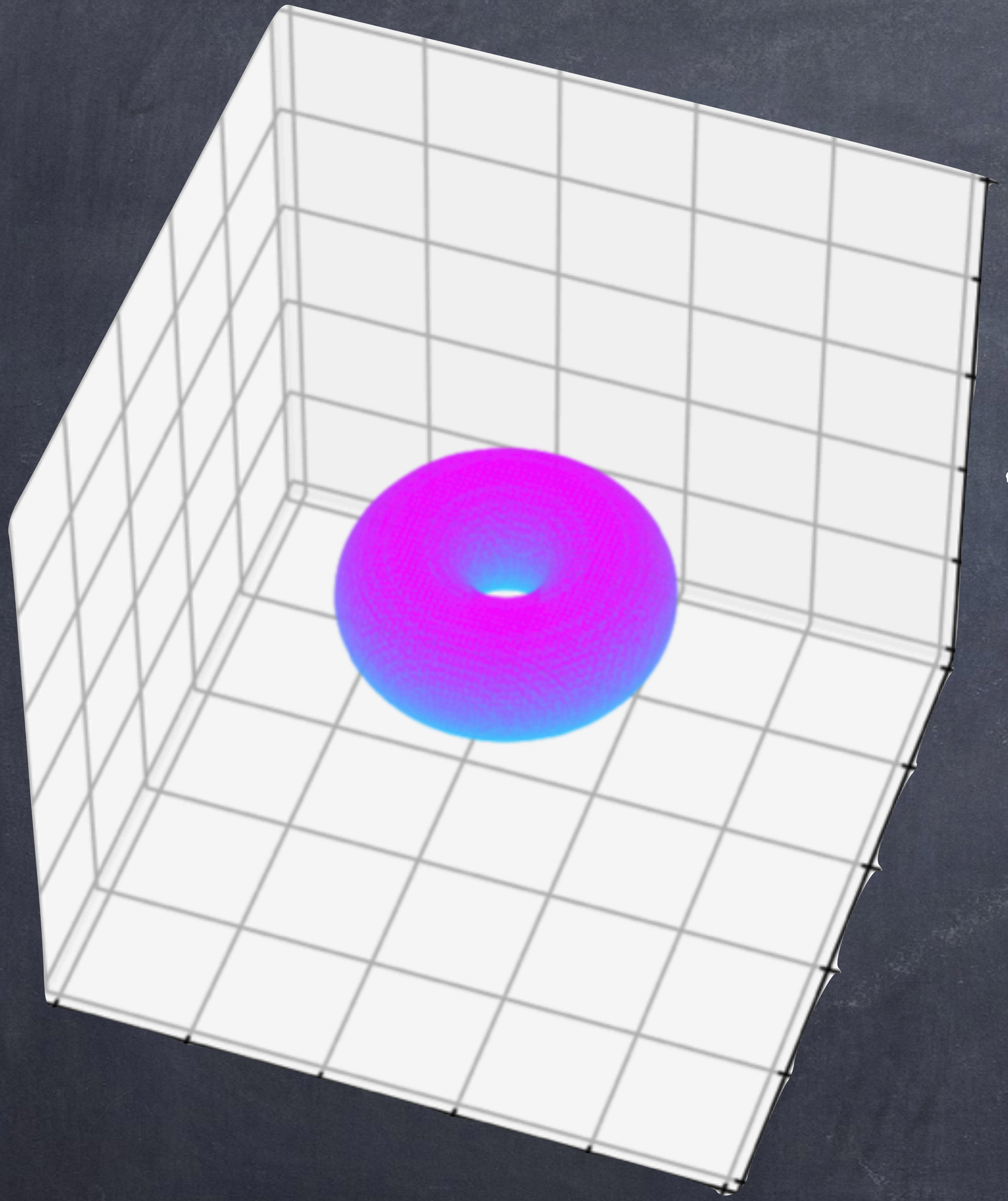
$$\widetilde{\pi}_2 = 128 x_1 x_2\left(r^2+4\right)\left(\left(r^2+8\right)^2 - 4\rho^2\right)$$

$$\widetilde{\pi}_3 = 16 x_3\left(r^4 + 12r^2 + 4\rho^2 + 32\right)\left(\left(r^2+8\right)^2 - 4\rho^2\right)$$

$$\widetilde{\sigma} = r^{12} + 36r^{10} + \left(-12\rho^2 + 512\right)r^8 + \left(-288\rho^2 + 3520\right)r^6 + \left(48\rho^4 - 2496\rho^2 + 11008\right)r^4 + \left(576\rho^4 - 9728\rho^2 + 8192\right)r^2 - 64\rho^6 + 1792\rho^4 - 16384(\rho^2+1)\,.$$

$$\rho^2 = x_1^2 + x_2^2$$

B=2 rational axial Skyrmion



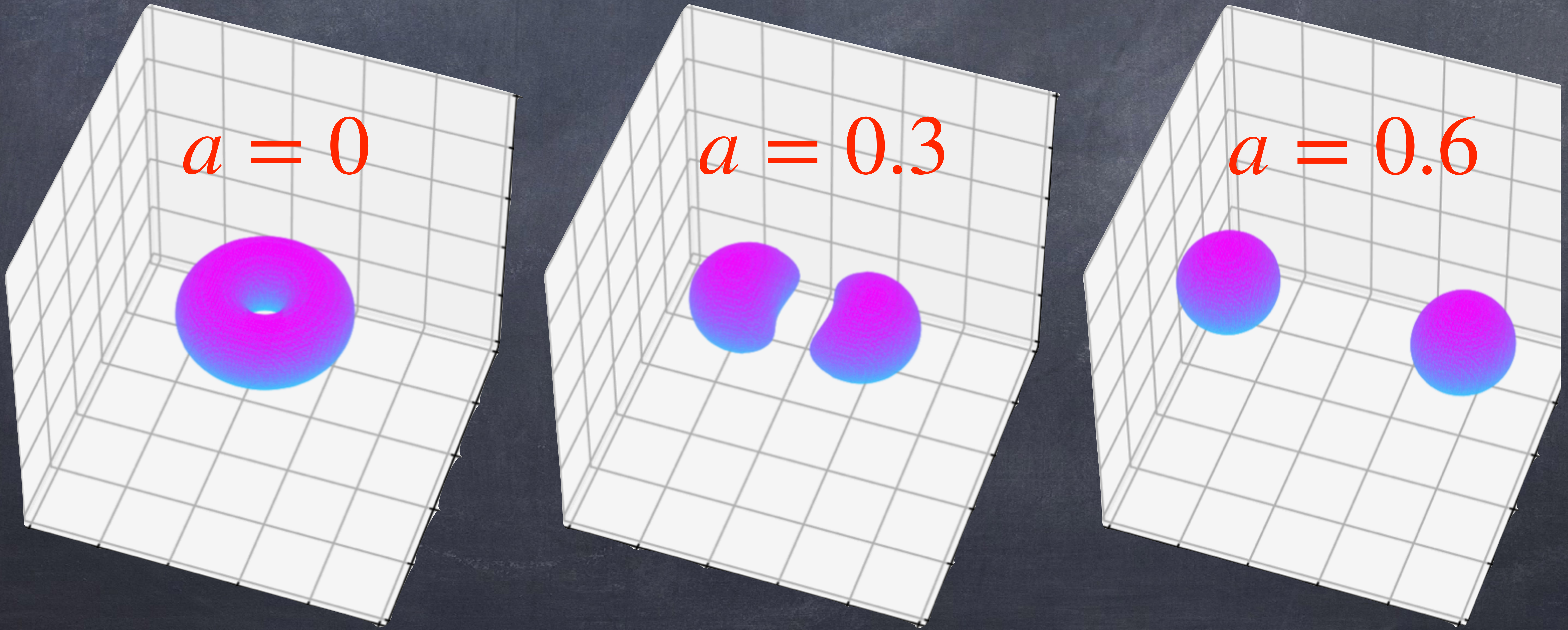
	energy	error
true	2.358	0.0%
Atiyah-Manton	2.384	1.1%
rational map	2.416	2.5%
rational	2.418	2.5%

B=2 attractive channel

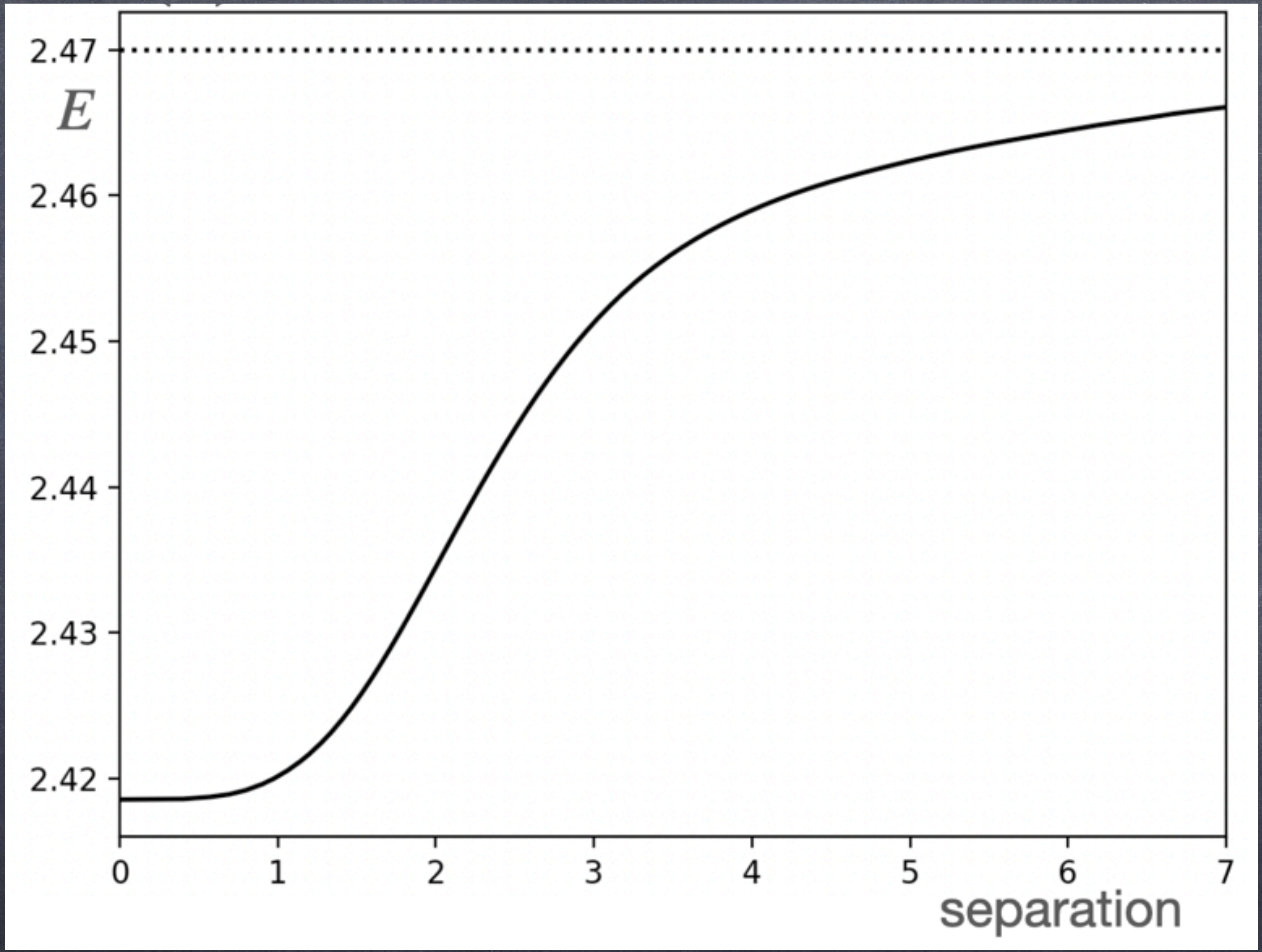
$$\widehat{M} = \frac{\lambda}{2}\begin{pmatrix} \sqrt{2(1-a^2)} & \sqrt{2(1-a^2)}k \\ (1+a)i & (1-a)j \\ (1-a)j & -(1+a)i \end{pmatrix}$$

$-1 < a < 1$

controls the separation



B=2 attractive channel



Conclusion

- Skyrmions deform when they overlap to produce symmetric shapes .
- Various approximations are valid in different regimes.
- Introduced a new approximation based on an ADHM type construction.
- Algebraic Skyrmions are obtained by using only 2 internal points.
- Applications: quantization, nuclear force.

The End