On Bogomolny equations and Cauchy problems

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The goals

- deriving Bogomolny equations for:
  1. the restricted baby Skyrme model in (2+0) dimensions, by using strong necessary conditions method (SNCM), (more references can be find in the companion document)
  2. the (2+0)-dimensional Heisenberg model of ferromagnet
- solving exactly Cauchy problem for these equations (an answer to B. Szafirski's question on the Cauchy problem associated to Bogomolny equation derived using SNCM)
- nothing about the form of the potential V of the restricted baby Skyrme model will be assumed

The results presented here, were published in: Ł. T. S. "Strong Necessary Conditions and the Cauchy Problem", *Symmetry*, vol. 15, No. 9, 1622 (2023); arXiv:1912.02609 (an older version).

Bogomolny equations - an introduction I

- The Euler-Lagrange equations for model ϕ^4 with spontaneous symmetry breaking, have the following form, [?]
  $\frac{d^2\phi}{dx^2} = 2\lambda\phi(\phi^2 - \gamma^2)$
- Let Q be topological charge given by $Q = \phi(\infty) - \phi(-\infty)$
  Then, one writes energy functional E as an integral of total differential

$$E = \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{d\phi}{dx} + \sqrt{\lambda}(\phi^2 - \gamma^2)\right)^2 dx + \frac{2\sqrt{\lambda}}{3}\gamma^2 \mid Q \mid, \qquad (1)$$

- now one requires reaching the minimum by the functional (1), so the first term must vanish, and one gets Bogomolny equations, sometimes called also, as "Bogomol'nyi equations" or "BPS equations" (more references can be find in the companion document):

$$\frac{d\phi}{dx} = \sqrt{\lambda}(\gamma^2 - \phi^2) \qquad (2)$$

  The well-known solution of (2), so called "kink" $\phi(x) = \gamma \tanh(\gamma\sqrt{\lambda}(x - x_0))$.
  So, the functional (1) attends the minimum: $E_{min} = \frac{2\sqrt{\lambda}}{3}\gamma^2 \mid Q \mid$.
- the Bogomolny equations (2) were derived using classical method called as "completing to square"
- some other method of deriving Bogomolny equations:

Bogomolny equations - an introduction II

1. strong necessary conditions method SNCM, (introduced by K.Sokalski in 1979, and developed by him and his collaborators: P. Jochym, Z. Lisowski (died in 2021), T. Wietecha; a systematic approach to deriving Bogomolny equations using SNCM, was published by K. Sokalski, Ł. T. S. and D. Sokalska in 2002)
2. first-order formalism (Bazeia, Brito, Costa, Gomes, Losano, Marques, Menezes, Oliveira, Rodrigues, Rosenfeld, Zafalan 2006 - 2017)
3. On-Shell method (Atmaja and Ramadhan 2014)
4. BPS Lagrangian method (Atmaja 2015)
5. FOEL formalism (Adam and Santamaria 2016) - a development of SNCM

- in further part of this talk we present deriving of the Bogomolny equations using SNCM .

Baby Skyrme model - an introduction

- baby Skyrme model - an analogical model (on plane) to the Skyrme model in three-dimensional space, applied to describe quantum Hall effect (more references can be found in the companion document)
- the target space of Skyrme model is $SU(2)$, [**?**] and the target space of baby Skyrme model is S^2, magnetic skyrmions are considered to be applied in spintronic devices, [**?**], (more references can be found in the companion document)
- in these both models: Skyrme and baby Skyrme, static field configurations can be classified topologically by their winding numbers (more references can be found in the companion document)
- this is the lagrangian of baby Skyrme model (more references can be found in the companion document): $\mathcal{L} = \partial_\mu \vec{S} \cdot \partial^\mu \vec{S} - \beta(\partial^\mu \vec{S} \times \partial^\nu \vec{S})^2 - V(\vec{S})$, where $\mid \vec{S} \mid^2 = 1$.
- one considers the energy functional for restricted baby Skyrme model in (2+0) dimensions (the static σ term is absent)

$$H = \frac{1}{2}\int d^2x\mathcal{H} = \frac{1}{2}\int d^2x\left(\frac{\beta}{4}(\epsilon_{ij}\partial_i\vec{S} \times \partial_j\vec{S})^2 + \gamma^2 V(\vec{S})\right). \quad (3)$$

The concept of strong necessary conditions I

- After making stereographic projection $\vec{S} = \left[\frac{\omega+\omega^*}{1+\omega\omega^*}, \frac{-i(\omega-\omega^*)}{1+\omega\omega^*}, \frac{1-\omega\omega^*}{1+\omega\omega^*}\right]$, where $\omega = \omega(x, y) \in \mathbb{C}$ and $x, y \in \mathbb{R}$, the density of the energy functional (3) has the form

$$\mathcal{H} = -4\beta \frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})^2}{(1+\omega\omega^*)^4} + V(\omega, \omega^*) \tag{4}$$

- from the extremum principle, applied to the functional (4), the Euler-Lagrange equations follow
$\mathcal{H}_{,\omega} - \frac{d}{dx}\mathcal{H}_{,\omega_{,x}} - \frac{d}{dy}\mathcal{H}_{,\omega_{,y}} = 0, \mathcal{H}_{,\omega^*} - \frac{d}{dx}\mathcal{H}_{,\omega^*_{,x}} - \frac{d}{dy}\mathcal{H}_{,\omega^*_{,y}} = 0,$
- instead of Euler-Lagrange equations, one considers strong necessary conditions (obviously, all solutions of the following system, satisfy the Euler-Lagrange equation):

$$\mathcal{H}_{,\omega} = 0, \;\; \mathcal{H}_{,\omega_{,x}} = 0, \;\; \mathcal{H}_{,\omega_{,y}} = 0, \tag{5}$$

$$c.c. \tag{6}$$

The concept of strong necessary conditions II

- in order to get a chance for obtaining some non-trivial solutions, one makes gauge transformation of the functional $H = \int_{E^2} \mathcal{H} dxdy$: $H \to H + Inv$, where Inv is such functional that its local variation with respect to $u(x,t)$ vanishes: $\delta Inv \equiv 0 \Longrightarrow$ E.-L. equations are **invariant** with respect to the above gauge transformation, in contrary to the strong necessary conditions, (Sokalski, Ł. S., Sokalska 2002), (Ł.S. PhD Thesis 2003).
- the gauge transformation in the case of baby Skyrme model, has the form,

$$\mathcal{H} \longrightarrow \tilde{\mathcal{H}} = -4\beta \frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})^2}{(1+\omega\omega^*)^4} + V(\omega,\omega^*) + \sum_{k=1}^{3} I_k, \tag{7}$$

where I_k are the densities of the invariants: $I_1 = G_1(\omega,\omega^*)(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})$ is the density of topological invariant,
$I_2 = D_x G_2(\omega,\omega^*)$, $I_3 = D_y G_3(\omega,\omega^*)$, $D_x \equiv \frac{d}{dx}$, $D_y \equiv \frac{d}{dy}$
$\omega = \omega(x,y), \omega^* = \omega^*(x,y) \in \mathcal{C}^2$ and $G_k = G_k(\omega,\omega^*) \in \mathcal{C}^2$, $(k = 1,2,3)$, are some functions, which are to be determinated.

The concept of strong necessary conditions III

- If one applies the concept of strong necessary conditions to (7), the dual equations are, as follows, (Ł.T. S. 2012), (Ł. T. S. 2013)

$$\tilde{\mathcal{H}}_{,\omega} = 16\beta\frac{(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})^2\omega^*}{(1+\omega\omega^*)^5} + V_{,\omega}(\omega,\omega^*) + G_{1,\omega}(\omega,\omega^*)(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x}) + D_x G_{2,\omega}(\omega,\omega^*) + D_y G_{3,\omega}(\omega,\omega^*) = 0, \tag{8}$$

$$\tilde{\mathcal{H}}_{,\omega^*} = 16\beta\frac{(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})^2\omega}{(1+\omega\omega^*)^5} + V_{,\omega^*}(\omega,\omega^*) + G_{1,\omega^*}(\omega,\omega^*)(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x}) + D_x G_{2,\omega^*}(\omega,\omega^*) + D_y G_{3,\omega^*}(\omega,\omega^*) = 0, \tag{9}$$

$$\tilde{\mathcal{H}}_{,\omega_{,x}} = -8\beta\frac{(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})\omega^*_{,y}}{(1+\omega\omega^*)^4} + G_1(\omega,\omega^*)\omega^*_{,y} + G_{2,\omega} = 0, \tag{10}$$

$$\tilde{\mathcal{H}}_{,\omega_{,y}} = 8\beta\frac{(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})\omega^*_{,x}}{(1+\omega\omega^*)^4} - G_1(\omega,\omega^*)\omega^*_{,x} + G_{3,\omega} = 0, \tag{11}$$

$$\tilde{\mathcal{H}}_{,\omega^*_{,x}} = 8\beta\frac{(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})\omega_{,y}}{(1+\omega\omega^*)^4} - G_1(\omega,\omega^*)\omega_{,y} + G_{2,\omega^*} = 0, \tag{12}$$

$$\tilde{\mathcal{H}}_{,\omega^*_{,y}} = -8\beta\frac{(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})\omega_{,x}}{(1+\omega\omega^*)^4} + G_1(\omega,\omega^*)\omega_{,x} + G_{3,\omega^*} = 0. \tag{13}$$

The concept of strong necessary conditions IV

- Now, one needs to make the equations (8) - (13) self-consistent $\Rightarrow$ the necessity of the reduction of the number of independent equations by an appropriate choice of the functions $G_k, (k = 1, 2, 3)$.
- usually, such ansatzes exist only for some special $V(\omega, \omega^*)$ $\Rightarrow$ in most cases of $V(\omega, \omega^*)$ for many nonlinear field models, the reduction of the system of corresponding dual equations, to Bogomolny equations, is impossible.
- by using the equations (24) - (13), one needs to eliminate in the equations (8) - (9), all terms including $\omega_{,x}, \omega_{,y}, \omega^*_{,x}, \omega^*_{,y}$, (Ł. T. S. 2015)
- we integrate obtained relations, with respect to ω and ω^*, correspondingly. Next, the relation between the potential and function G_1, has been obtained, (Ł. T. S. 2015)

$$V(\omega, \omega^*) = -\frac{1}{16\beta} G_1^2(\omega, \omega^*)(1 + \omega\omega^*)^4 + c_1, c_1 = const, \tag{14}$$

$$G_2(\omega, \omega^*) = const, \quad G_3(\omega, \omega^*) = const. \tag{15}$$

- Thus

$$G_1 = \frac{4i\sqrt{\beta}}{(1 + \omega\omega^*)^2} \sqrt{V(\omega, \omega^*) - c_1} \tag{16}$$

The concept of strong necessary conditions V

- Hence, after inserting (15) into (24) - (13) and simplifying, one gets one equation, (Ł. T. S. 2012), (Ł. T. S. 2013), (Ł. T. S. 2015) $\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x} = \frac{1}{8\beta} G_1(\omega, \omega^*)(1+\omega\omega^*)^4$.
- by using the relation (16), one obtains from the above equation, the Bogomolny decomposition for the given potential $V(w, w^*)$, [**?**] (in [**?**], [**?**], some special form of the condition (16) was presented, in [**?**] some exact solutions of Bogomolny decomposition for zero value of the constant c_1, were presented)

$$\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x} = \frac{i}{2\sqrt{\beta}} \sqrt{V(\omega, \omega^*) - c_1}(1+\omega\omega^*)^2. \qquad (17)$$

Then, the equation (17) is Bogomolny decomposition (Bogomolny equation) for restricted baby Skyrme model in (2+0) dimensions, for *arbitrary* potential.
An analogical result can be obtained also in the case of (3+0)-dimensional BPS Skyrme model (the next page).

The (2+0)-dimensional Heisenberg model I

We consider the continuous Heisenberg model represented by the following Hamiltonian, (Belavin and Polyakov 1975):

$$H = \int_{E^2} \mathcal{H} dxdy = \int_{E^2} \left(\frac{\nabla w \cdot \nabla w^*}{(1 + w \cdot w^*)^2} \right) dxdy, \tag{18}$$

where the complex field variable w consists of classical spin components:

$$w = \frac{(S^x + iS^y)}{(1 + S^z)}, \tag{19}$$

where S^x, S^y, S^z are the components of the classical spin. In this case, we have the homotopy group $\pi_2(S^2)$, refs. (Morandi 1991, (Saxena and Kevrekidis and Cuevas-Maraver 2020)). The Bogomolny equations for this model were derived by applying classical completing to square in (Belavin and Polyakov 1975; one can also find this in (Saxena and Kevrekidis and Cuevas-Maraver 2020)).
Here, we apply the SCNM for the Hamiltonian:

$$\tilde{H} = \int_{E^2} \tilde{\mathcal{H}} dxdy = \int_{E^2} \left(\frac{\nabla w \cdot \nabla w^*}{(1 + w \cdot w^*)^2} + I_1 + I_2 + I_3 \right) dxdy, \tag{20}$$

where, as mentioned above, I_1 is density of the topological invariant, being the so-called winding number and Pontryagin index, (Morandi 1991) (c.f. for e.g., (Balakrishnan 1990, Balakrishnan 2023)):

$$I_1 = G_1(w, w^*)(w_{,x} w^*_{,y} - w_{,y} w^*_{,x}), \tag{21}$$

The (2+0)-dimensional Heisenberg model II

where $G_1(w, w^*) \in \mathcal{C}^1$ is the function to be determined later. I_2, I_3 are the so-called divergent invariants: $I_2 = \frac{dG_2}{dx}, I_3 = \frac{dG_3}{dy}$, and $G_k = G_k(w, w^*) \in \mathcal{C}^1, (k = 2, 3)$ are the functions to be determined later, during the further computations.
We apply strong necessary conditions to (20) and we obtain the system of dual equations, which can also be obtained as a two-dimensional version of the system of the dual equations derived in (Sokalski and Ł S. and Sokalska 2002):

$$-\frac{2w^*\nabla w\nabla w^*}{(1+ww^*)^3} + G_{1,w}(w_{,x}w^*_{,y} - w_{,y}w^*_{,x}) + D_x G_{1,w}(w, w^*) + D_y G_{2,w}(w, w^*) = 0, \tag{22}$$

$$c.c., \tag{23}$$

$$\frac{w^*_{,x}}{(1+ww^*)^2} + G_1 w^*_{,y} + G_{2,w} = 0, \tag{24}$$

$$\frac{w^*_{,y}}{(1+ww^*)^2} - G_1 w^*_{,x} + G_{3,w} = 0, \tag{25}$$

$$c.c. \tag{26}$$

We make this system self-consistent by choosing $G_k = const$ $(k = 2, 3)$ and (similarly to (Sokalski and Ł S. and Sokalska 2002)) by choosing $G_1 = \frac{i}{(1+ww^*)^2}$. Next, expressing the complex fields w and w^* by real fields:

The (2+0)-dimensional Heisenberg model III

$$w = U(x,y) + iV(x,y), w^* = U(x,y) - iV(x,y), \tag{27}$$

we derive from (24)–(26) the pair of equations governing real fields $V(x,y)$ and $U(x,y)$:

$$\frac{\partial U(x,y)}{\partial x} + \frac{\partial V(x,y)}{\partial y} = 0, \tag{28}$$

$$\frac{\partial U(x,y)}{\partial y} - \frac{\partial V(x,y)}{\partial x} = 0. \tag{29}$$

An exact solution (in terms ω, ω^*) for this model was published in (Belavin and Polyakov 1975) (however, the Bogomolny equations were derived there using a classical Bogomolny trick, i.e., completing to a square). As we have indicated, we see this issue from the point of view of the Cauchy problem. Then, solving (28) and (29), we get, (Ł. T. S. 2023):

$$U(x,y) = F_1(x - iy) + F_2(x + iy), \tag{30}$$

$$V(x,y) = -iF_1(x - iy) + iF_2(x + iy) + C_1, \tag{31}$$

where $F_1(\cdot)$ and $F_2(\cdot)$ are some functions. After taking into account the formula (27), we see that F_1, F_2 are connected with w, w^* by the formulas, (Ł. T. S. 2023):

The (2+0)-dimensional Heisenberg model IV

$$F_1 = \frac{1}{2}(w - iC_1), \tag{32}$$

$$F_2 = \frac{1}{2}(w^* + iC_1) \tag{33}$$

and C_1 is an arbitrary real constant. Based on the general solutions (30) and (31) of (28) and (29), we present the Cauchy problem for partial differential equations of the first order created by the strong necessary conditions. The considered example consists of two independent variables, x and y, and two functions. Therefore, it is possible to formulate the following constraints for the general solutions:

$$U(x,0) = f_1(x), \quad V(x,0) = f_2(x), \tag{34}$$

where $f_1(x)$ and $f_2(x)$ are given functions. It is possible for the considered Heisenberg model to derive analogous relations to $U(0,y)$ and $V(0,y)$, which relate integration constants to initial or boundary conditions. Constraining (30) and (31) to (34) and substituting $y = 0$, we obtain, (Ł. T. S. 2023):

$$f_1(x) = F_1(x) + F_2(x), \tag{35}$$

$$f_2(x) = -iF_1(x) + iF_2(x) + C_1. \tag{36}$$

Since $f_1(x)$ and $f_2(x)$ are given, F_1 and F_2 cannot be arbitrary, (Ł. T. S. 2023):

The (2+0)-dimensional Heisenberg model V

$$F_1(x) = \frac{1}{2}(f_1(x) + if_2(x) - iC_1), \tag{37}$$

$$F_2(x) = \frac{i}{2}(f_1(x) - if_2(x) + iC_1). \tag{38}$$

As such, $F_1 = F_2^*$. Then, the only freedom for F_1 and F_2 is gauge transformation regarding the C_1 constant.

The (3+0)-dimensional BPS Skyrme model

In (Stepien 2016), Bogomolny equations were derived for (3+0)-dimensional BPS Skyrme model

$$\mathcal{L} = f(\omega, \omega^*)(\varepsilon^{\mu\nu\rho\sigma}\chi_{,\nu}\omega_{,\rho}\omega^*_{,\sigma})^2 - \mu^2 V, \qquad (39)$$

where $\chi_{,\nu} = \frac{\partial\chi}{\partial x^\nu}$ etc. and the potential is a function of χ.
The derived BPS equations have the form, [**?**]

$$2\sqrt{f} \cdot (\varepsilon^{kmn} i\chi_{,k}\omega_{,m}\omega^*_{,n}) = \mp\sqrt{4V + c_2}, \qquad (40)$$

where $V(\chi, \omega, \omega^*) = -\frac{1}{4}\frac{G_1^2 - fc_1}{f}$, $c_1 = -c_2 = const.$.
For $c_2 = 0$ the Bogomolny equation for the BPS Skyrme model (where function f is chosen in the proper way) have been re-derived. For $c_2 > 0$ non-Bogomolny equation has been obtained - in fact, this coincides with non-zero pressure equation for the BPS Skyrme model (Adametal 2014), (Stepien 2016). Therefore, the constant found in our construction can be related with the pressure. This is an interesting, unexpected observation that our method not only leads to the Bogomolny equation, but also includes the non-zero pressure generalization.

An exact solution of BPS equation for the restricted baby-Skyrme model I

Now let me come back to derived Bogomolny equations for restricted baby BPS Skyrme model. Now, we will find an exact solution of Bogomolny decomposition (17), for $V = (\omega\omega^* - \gamma^2)^2$ - "Mexican hat" potential, i.e. it is the model with spontaneously broken symmetry.

We find now an exact localized static solution (with localized density of energy) of the Bogomolny decomposition (17) for the case of the so-called "Mexican hat" potential: $V = \lambda_3(\omega\omega^* - \gamma^2)^2$, when $c_1 = 0$. We use "hedgehog ansatz":

$$\omega = \frac{\sin(f(r))\cos(N\theta) + i\sin(f(r))\sin(N\theta)}{1 + \cos(f(r))}, \ c.c., \tag{41}$$

where (r, θ) are polar coordinates in the cartesian $x - y$ plane.

We insert this ansatz into (17), and we formulate the Cauchy problem, (Ł. T. S. 2023):

$$\frac{(\cos(f(r)) + 1)Nf'(r)\sin(f(r))}{r} = \sqrt{\frac{\lambda_3}{\beta}}\left[\cos(f(r))(\gamma^2 + 1) + \gamma^2 - 1\right], \tag{42}$$

$$f(0) = c_0 = const, \tag{43}$$

where, in this case, we put $c_0 = 2$.

We are interested in obtaining a localized solution, so we also impose the asymptotic conditions:

An exact solution of BPS equation for the restricted baby-Skyrme model II

$$\lim_{r\to\pm\infty} f(r) = const, \tag{44}$$

$$\lim_{r\to\pm\infty} \mathcal{H} = const. \tag{45}$$

We solve this problem and we have, (Ł. T. S. 2023), after some simplification

$$f(r) = \arccos(X_1), \tag{46}$$

where:

$$X_1 = \frac{1}{\gamma^2+1}\left((\cos(2)(\gamma^2+1)+\gamma^2-1)\exp\left(-\frac{1}{\sqrt{\beta N}}Y\right)-\gamma^2+1\right) \tag{47}$$

and:

$$Y = Lambert\left(\frac{1}{2}(\cos(2)(\gamma^2+1)+\gamma^2-1)\exp\left(\frac{Y_2}{\sqrt{\beta N}}\right)\right)\sqrt{\beta N}-\frac{1}{2}Y_2 \tag{48}$$

and:

An exact solution of BPS equation for the restricted baby-Skyrme model III

$$Y_2 = N(\cos(2)(\gamma^2+1)+\gamma^2-1)\sqrt{\beta} - \frac{\sqrt{\lambda_3}r^2(\gamma^2+1)^2}{2} \tag{49}$$

and $Lambert(Z)$ is the so-called Lambert function, which satisfies the equation $Lambert(Z)\exp(Lambert(Z)) = Z$. For $\gamma = 2, N = 1, \lambda_3 = 1, \beta = 1$, (Ł. T. S. 2023):

$$f(r) = \arccos\left\{\left[\frac{2Lambert\left(\frac{1}{2}\exp\left(-\frac{25}{4}r^2+\frac{5\cos(2)+3}{2}\right)(5\cos(2)+3)\right)-3}{5}\right]\right\}. \tag{50}$$

We present a figure of this above solution in Figure 1, (Ł. T. S. 2023).

An exact solution of BPS equation for the restricted baby-Skyrme model IV

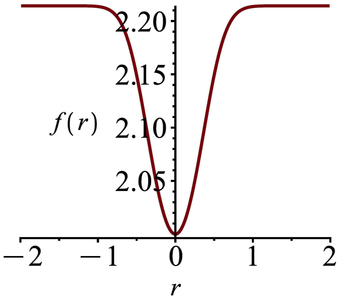


Figure: Figure of solution (50).

An exact solution of BPS equation for the restricted baby-Skyrme model V

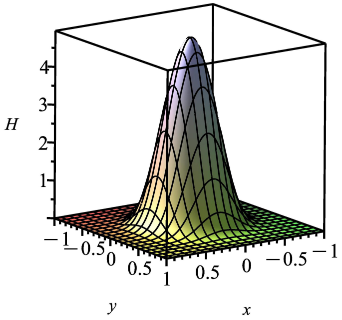
Let's notice that if we insert the found solution of the Cauchy problem into the ungauged and gauged Hamiltonian densities (4) and (7), correspondingly, then the ungauged Hamiltonian density is nonzero (Figure 2).
One can see that both the found solution and the density of the ungauged Hamiltonian, corresponding to this, are localized. Thus, one can tell this solution is a soliton solution (or, at least, a soliton-like solution). For this solution, the gauged Hamiltonian density is zero (of course, the conditions (15), (16) and Bogomolny Equation (17) hold), (Ł. T. S. 2023):

$$\tilde{\mathcal{H}} = 0. \tag{51}$$

Hence, we can see here the degeneracy of the Hamiltonian (the problem of a degenerate Hamiltonian, in the case of theory of gravity, was investigated in (Sanyal 2019); in (Del Castillo and Mirón and Rojas 2013, the existence of an infinite number of Lagrangians for a given second-order ODE was proven). This corresponds to the fact that if one considers two versions of a field-theoretical Lagrangian, ungauged and gauged on total derivatives of any function of field variables, then the energy-momentum tensors corresponding to each of these Lagrangians will be different, ref. (Arodz 1997). The Euler–Lagrange equations are invariant with respect to gauging of the Lagrangian on the term $\frac{\partial \omega}{\partial x}$, but there is an impact of this gauging on the energy of the ground state of the medium, (Kiselev and Batalov 2023). Such a term can appear only in crystals not possessing the inversion center, and this causes the

An exact solution of BPS equation for the restricted baby-Skyrme model VI

spiral ordering of magnetic moments, (Kiselev and Batalov 2023) (and the references [1] - [4] cited there)
Hence, we mention the breaking of some symmetry: the Euler–Lagrange equations are the same for both Hamiltonians, gauged and ungauged; however, on the other hand, the density of the ungauged Hamiltonian is nonzero and the density of the gauged Hamiltonian is zero.



An exact solution of BPS equation for the restricted baby-Skyrme model VII

Figure: The figure of the ungauged Hamiltonian density for solution (50).

The effect of vanishing of energy-momentum tensor (when topological invariants occur in an action functional), was established in (Hosoya 1978), for some two Yang-Mills family models in $SU(2)$ case (and for certain other field-theoretical models in refs. (Dimopoulos and Eguchi 1977, de Vega and Schaposnik 1976, Belavin and Burlankov 1976). However, this result had been obtained there for a version of BPS equations derived by using the method of classical completing action functional to square, so the forms of the invariants used there, had been special (in contrary to this paper, where we have used generalized forms of the invariants). The effect of degeneracy of hamiltonian for the restricted baby Skyrme model (and the exact localized solution for this), had been presented for the first time in (Ł. T. S. 2023), to our best knowledge.

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