Novel transition dynamics of topological solitons

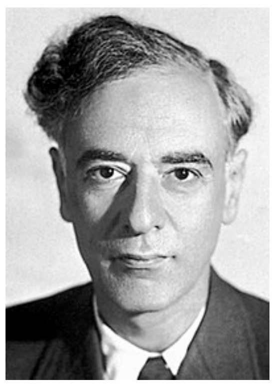
Noriyuki Sogabe
University of Illinois Chicago (UIC)

Topological Solitons, Sep 27, 2023

In collaboration with Kentaro Nishimura (KEK)

Based on arXiv: 2304.01264

Classification of transitions



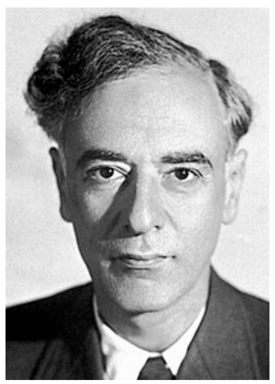
Lev Landau

Phase transition

Discontinuous → First order

Continuous → Second order

Classification of transitions



Lev Landau

P. G. de Gennes

Phase transition

Discontinuous → First order

Continuous → Second order

Topologically

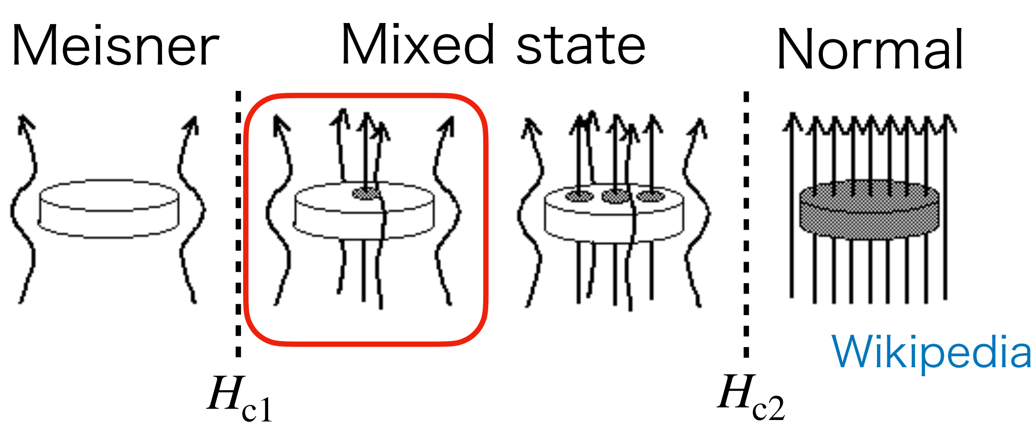
Trivial — Local order parameter

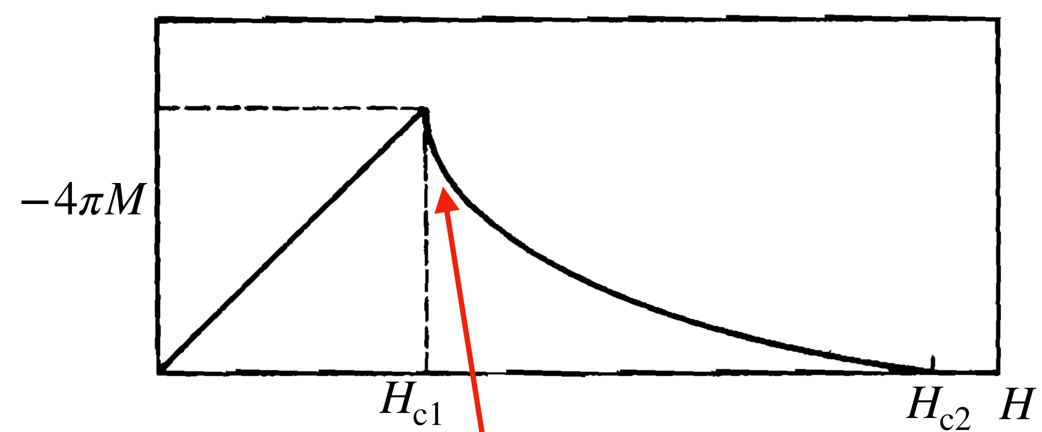
Non-trivial — with solitons

P. G. de Gennes, “Phase transition and turbulence: An introduction,” in Fluctuations, Instabilities, and Phase Transitions edited by T. Riste (Springer, 1975)

Topologically nontrivial transition

- Order parameter with a topological constraint
- E.g., Type-II superconductors, chiral magnets, cholesteric liquid crystals, etc.





$$-4\pi M \propto \log\left(\frac{1}{H - H_{c1}}\right)$$

Universal logarithmic behavior

From “Superconductivity,” Volume 2, Edited by R. D. Parks (1969)

See also I. E. Dzyaloshinsky (1964) and P. G. de Gennes and J. Prost (1993)

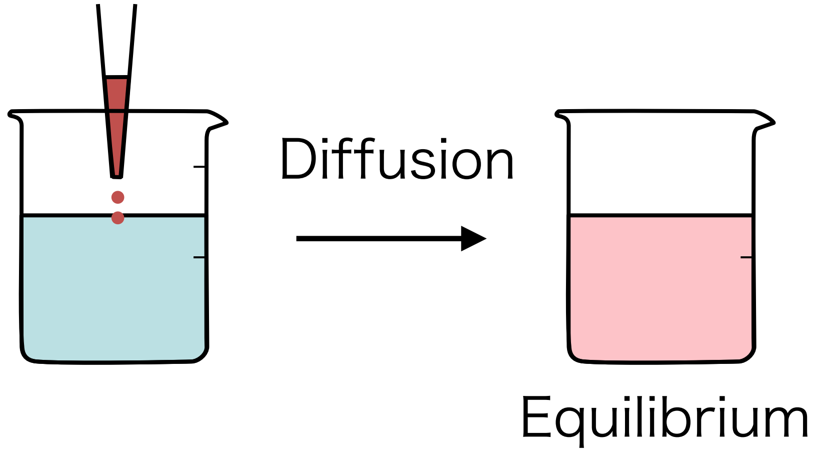
second-order phase transition dynamics

- Topologically trivial cases:

  Critical slowing down of hydrodynamic modes,
  E.g., relaxation, diffusion, and Nambu-Goldstone modes

  E.g., Liquid-gas critical point:

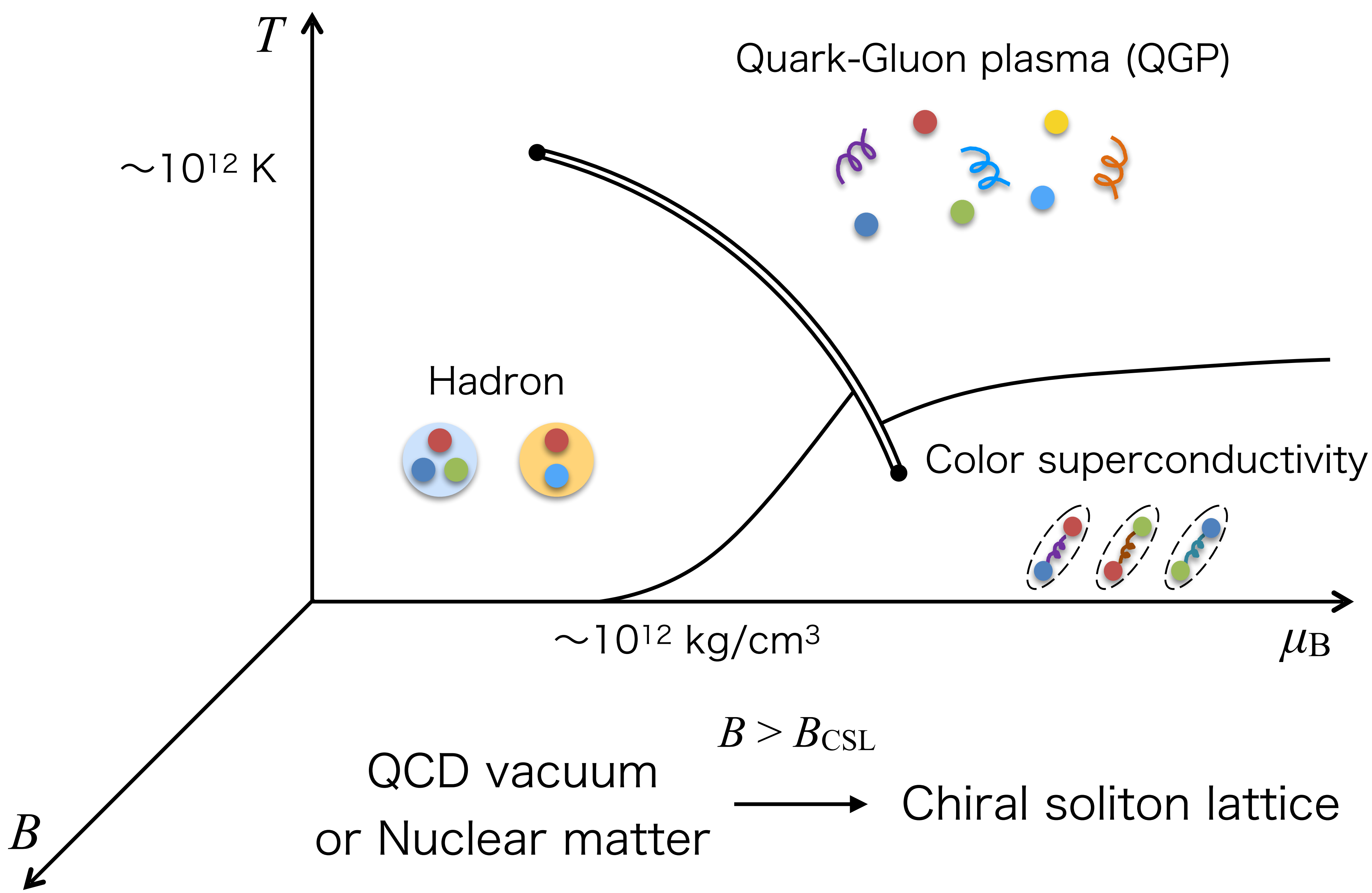
  Diffusion rate $\to 0$



- What about the nontrivial case?

  - The motion of topological objects, e.g., domain walls, vortices, etc.

QCD phase diagram



D. T. Son and M. A. Stephanov (2008), T. Brauner and N. Yamamoto (2017)

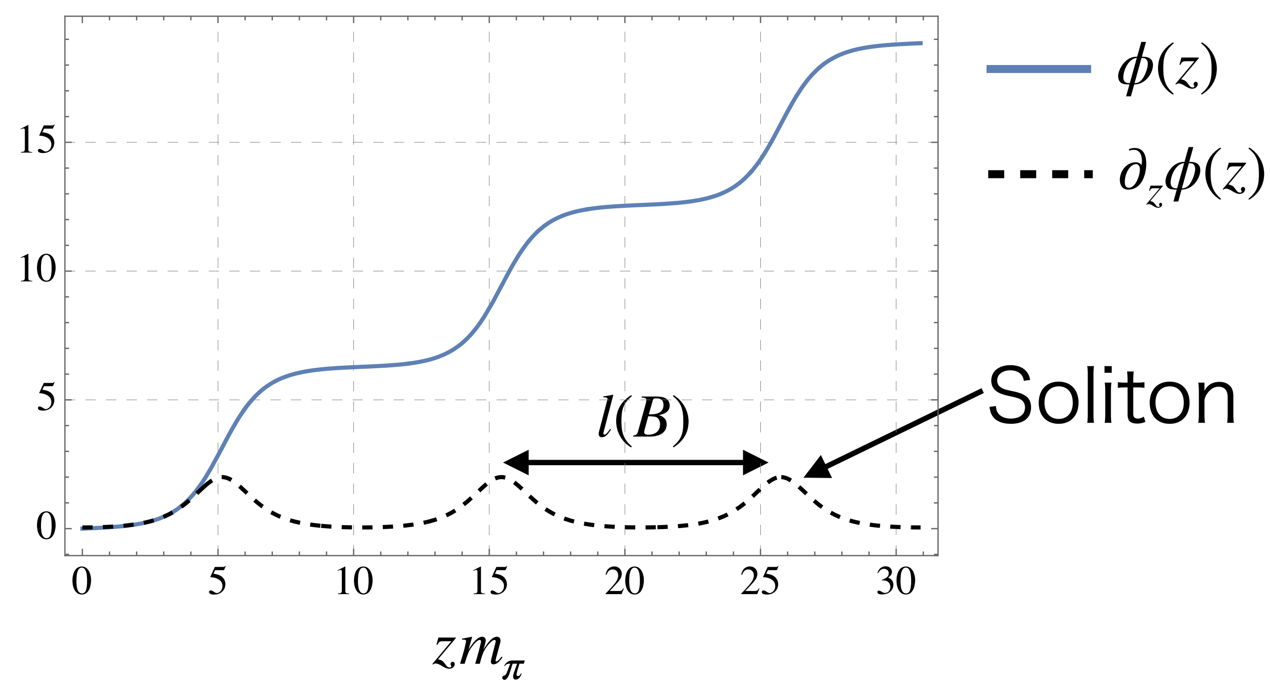
Chiral soliton lattice (CSL)

T. Brauner and N. Yamamoto (2017)

The anomaly-related ground state of QCD at finite μ_B & $B > B_{CSL}$

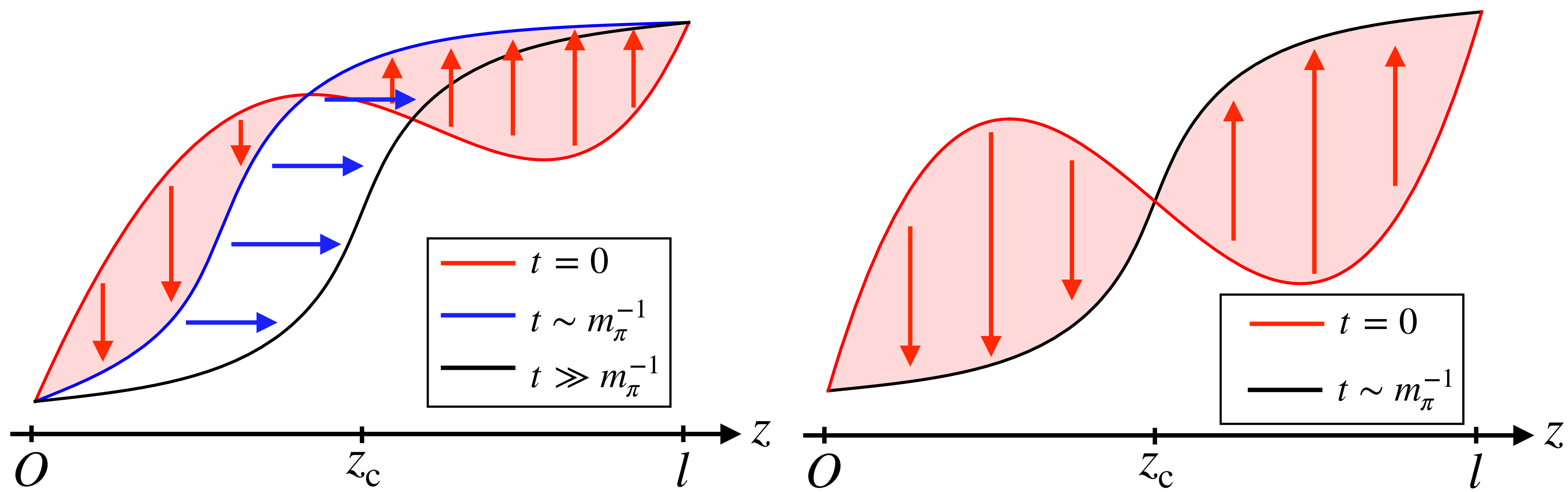
Neutral pion configuration: $\phi(z)$

- Violating parity
- Carrying topological charges
- Breaks translational symmetry



Relaxational dynamics

K. Nishimura and NS (2023)



Near the CSL transition

- Relaxation to the domain wall $t \sim m_\pi^{-1}$ Fast
- Translational motion (moduli space) $t \gg m_\pi^{-1}$ Slow

Outline

1. Setup
2. Formulation
3. Chiral soliton lattice (CSL)
4. Dynamics near the CSL transition

Setup

- Two-flavor QCD at finite T, μ_{B}, and B
- Finite quark mass
- Hydrodynamic variables:

$$\bar{q}q \sim \mathrm{e}^{2\mathrm{i}\phi t^3} \qquad \rho = \bar{q}\gamma^0\gamma^5 t^3 q$$

Neutral pion — Axial isospin charge

Outline

1. Setup
2. **Formulation**
3. Chiral soliton lattice (CSL)
4. Dynamics near the CSL transition

Hamiltonian

- Most general Hamiltonian density at finite μ_{B} and B

$$\mathscr{H}_0 = \frac{f_\pi^2}{2}(\nabla\phi)^2 - f_\pi^2 m_\pi^2 \cos\phi$$

- Kinetic + mass

C.f., chiral Lagrangian

$$\mathscr{H}_\rho = \frac{1}{2\chi}\rho^2$$

- χ : Axial isospin susceptibility

$$\mathscr{H}_{\mathrm{anom}} = -\frac{\mu_{\mathrm{B}}}{4\pi^2}\boldsymbol{B}\cdot\nabla\phi$$

favors inhomogeneity

- See D. Son and M. Stephanov (2008)

Low-energy dynamics

- A low-energy effective description with dissipation

Chaikin and Lubensky (1995)

See also D. T. Son (2000)

$$\partial_t \phi(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{y} \left[\phi(\boldsymbol{x}), \rho(\boldsymbol{y})\right] \frac{\delta H}{\delta \rho(\boldsymbol{y})} - \gamma_\phi(\nabla) \frac{\delta H}{\delta \phi(\boldsymbol{x})}$$

Reversible Dissipative

- Poisson bracket: $\left[\phi(\boldsymbol{x}), \rho(\boldsymbol{y})\right] = \delta(\boldsymbol{x} - \boldsymbol{y})$

- Derivative expansion: $\gamma_\phi(\nabla) = \kappa + \mathcal{O}(\nabla^2)$

κ ← Relaxation rate

Low-energy dynamics

K. Nishimura and NS (2023)

$$\partial_t \phi = \frac{\rho}{\chi} + f_\pi^2 \kappa \left(-m_\pi^2 \sin\phi + \nabla^2 \phi\right)$$

$$\partial_t \rho = f_\pi^2 \nabla^2 \phi - m_\pi^2 f_\pi^2 \sin\phi + \frac{\lambda}{\chi} \nabla^2 \rho$$

λ: Axial isospin conductivity

∵ Conservation law

- Nonlinear and dissipative generalization of previous works

D. Son (2000), D. Son and M. Stephanov (2002)

Both are essential for the relaxation process to the CSL

- 1+1 d $(\boldsymbol{B} = B\boldsymbol{e}_z)$

Outline

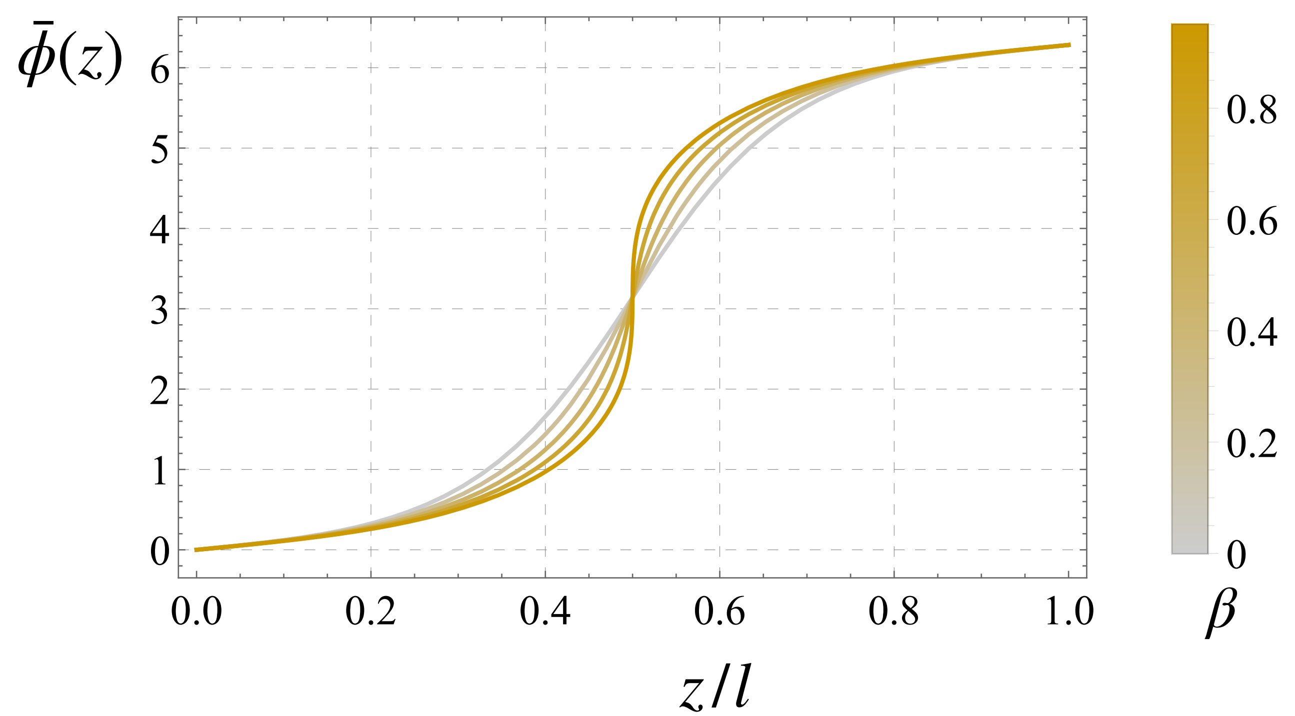
1. Setup
2. Formulation
3. Chiral soliton lattice (CSL)
4. Dynamics near the CSL transition

Stationary solution

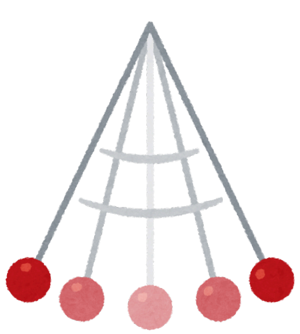
K. Nishimura and NS (2023)

$$\partial_z^2 \bar{\phi} = m_\pi^2 \sin \bar{\phi} - \beta \partial_z^2 \sin \bar{\phi}$$

$$\beta \equiv \kappa \lambda m_\pi^2$$



- Analytic solution at $\beta = 0$:



$$\cos \frac{\phi(zm_\pi/k)}{2} = \mathrm{sn}(zm_\pi/k, k)$$

Jacobi elliptic function   Elliptic modulus

Periodic length:

$$l(k) = \frac{2kK(k)}{m_\pi}$$

$K(k)$: The complete elliptic function of the first kind

B dependence

See T. Brauner and N. Yamamoto (2017) for details

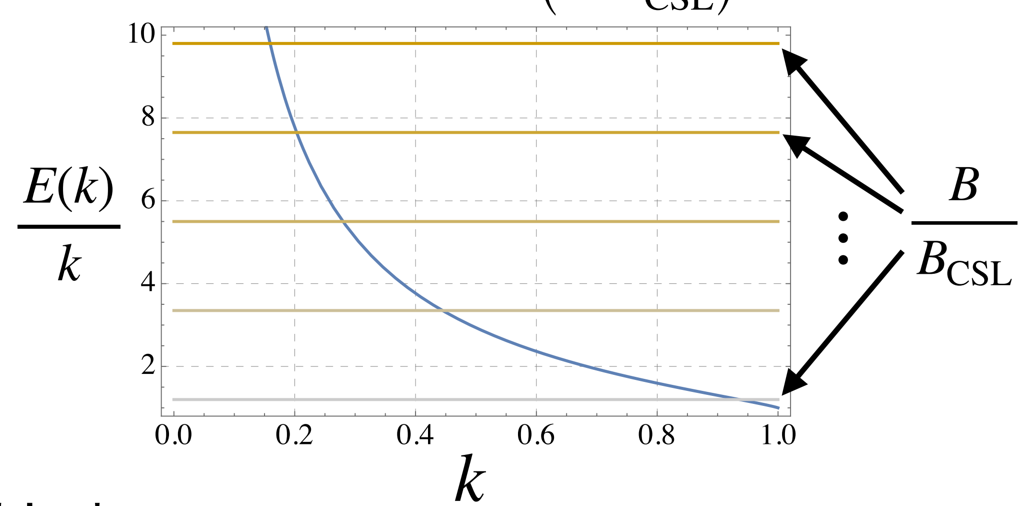
- Energy minimization condition (β=0):

$$\frac{\delta E_{\mathrm{tot}}\left[\phi=\bar{\phi}\right]}{\delta k}=0 \quad \Leftrightarrow \quad \frac{E(k)}{k}=\frac{B}{B_{\mathrm{CSL}}}$$

Critical magnetic field: $B_{\mathrm{CSL}}=\dfrac{16\pi m_\pi f_\pi^2}{\mu}$

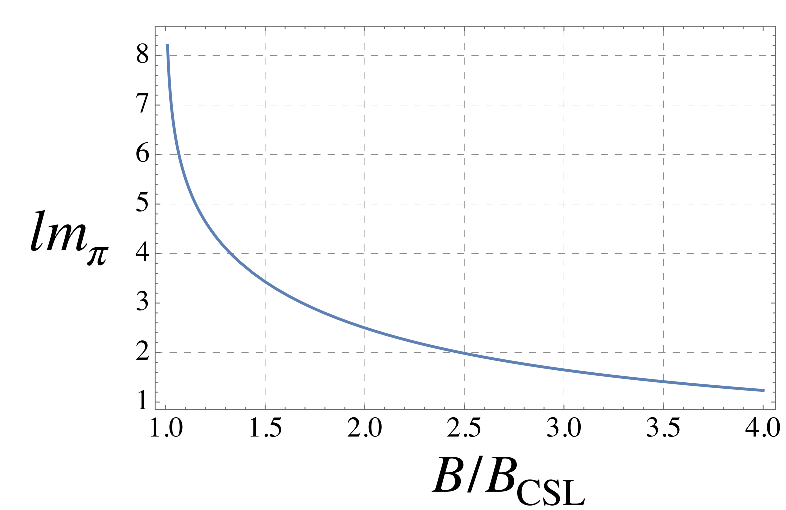
$E(k)$: The complete elliptic function of the second kind

Determine $k\left(B/B_{\mathrm{CSL}}\right)$



- Periodic length:

$$l(B)=l\left(k\left(B/B_{\mathrm{CSL}}\right)\right)\to\infty \qquad (B\to B_{\mathrm{CSL}})$$



Topological charges

D. T. Son and M. A. Stephanov (2008)
T. Brauner and N. Yamamoto (2017)

$$\mathscr{H} = \frac{f_\pi^2}{2}(\partial_z \phi)^2 - f_\pi^2 m_\pi^2 \cos\phi + \frac{1}{2\chi}\rho^2 - \frac{\mu_\mathrm{B} B}{4\pi^2}\partial_z \phi$$

Baryon number: $N_\mathrm{B} = -\int_0^l \mathrm{d}z \frac{\partial \mathscr{H}}{\partial \mu_\mathrm{B}} = \frac{B}{4\pi^2}\int_0^l \mathrm{d}z \partial_z \phi = \frac{B}{2\pi}$

Magnetization: $M = -\int_0^l \mathrm{d}z \frac{\partial \mathscr{H}}{\partial B} = \frac{1}{4\pi^2}\mu \int_0^l \mathrm{d}z \partial_z \phi = \frac{\mu_\mathrm{B}}{2\pi}$

Quantized per unit lattice

Logarithmic “critical” behavior of CSL in QCD

K. Nishimura and NS (2023)

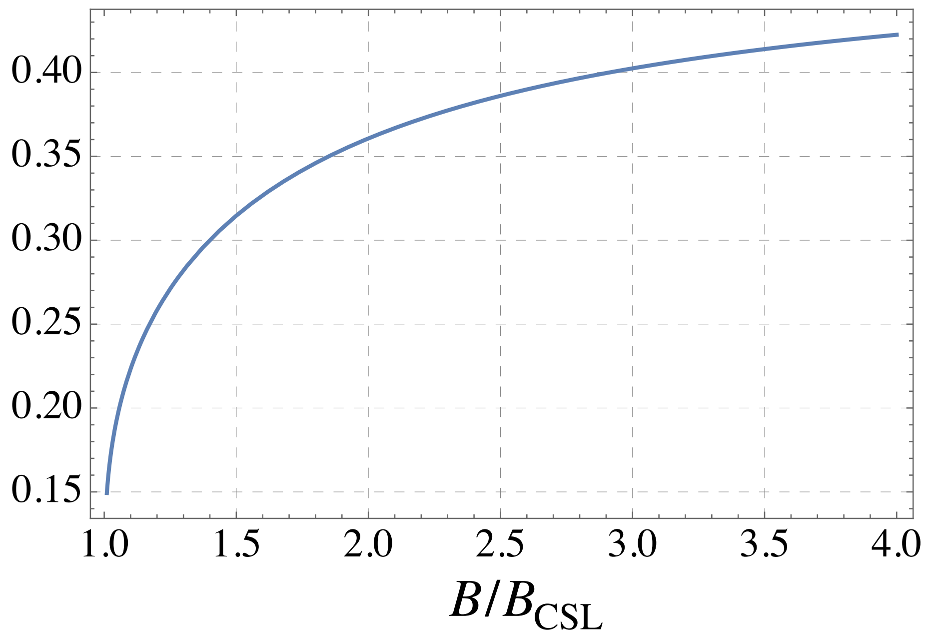
- An asymptotic form of $K(k)$ and $E(k)$ for $B \simeq B_{\mathrm{CSL}}$ $(k \simeq 1)$

Periodic length: $$l(B) \simeq \log \frac{8}{1 - B_{\mathrm{CSL}}/B}$$

- Total baryon number and magnetization:

$$(N_{\mathrm{B}})_{\mathrm{tot}} = \frac{L}{l(B)} N_{\mathrm{B}} \sim \frac{1}{\log \frac{8}{1 - B_{\mathrm{CSL}}/B}}$$

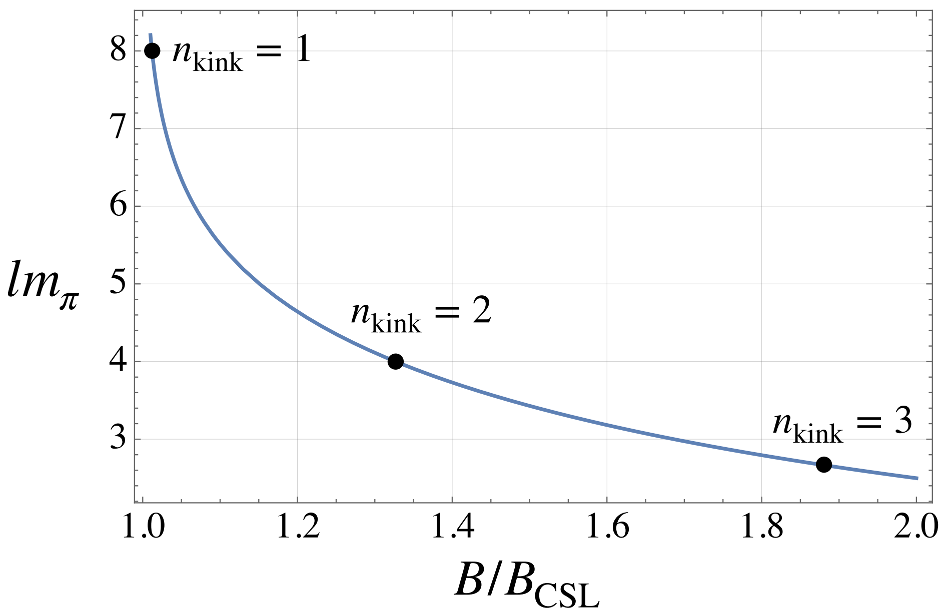
$\sim (N_{\mathrm{B}})_{\mathrm{tot}}$



Outline

Dynamics near $B_{\rm CSL}$

- System size $L = 8, 9, 10\ m_\pi^{-1}$ and vary $B(L/n_{\rm kink})$
- Relaxation process to the $n_{\rm kink}$ state
- Initial and boundary conditions:



(I) $\phi(0, z) = 0\,, \quad \phi(t, L) = 2\pi n_{\rm kink}\theta(t - t_0)$

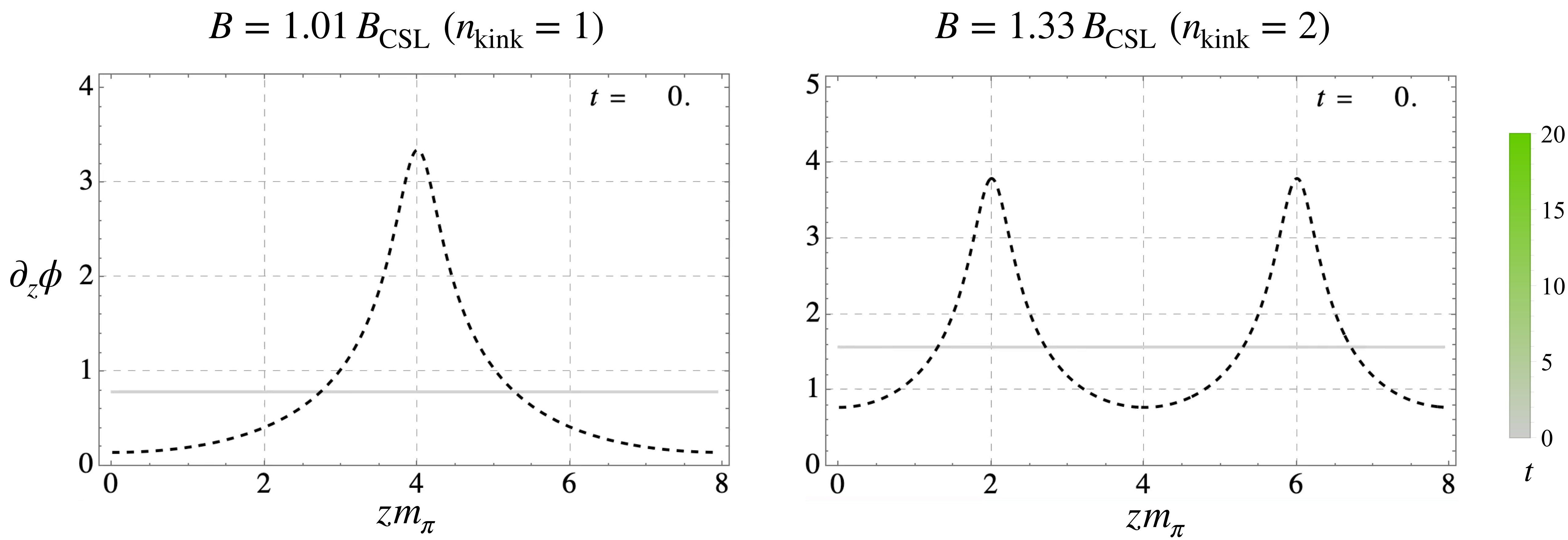
- $B(t) = B\left(L/n_{\rm kink}\right)\theta(t - t_0)$
- Transition from the vacuum $(B = 0)$ to the $n_{\rm kink}$ state at t_0

(II) $\phi(0, z) = 2n_{\rm kink}\pi z/L\,, \quad \phi(t, L) = 2n_{\rm kink}\pi$

- No translational motion
- Any odd functions: $\phi_{\rm ini}(z - z_{\rm c}) - \phi_{\rm ini}(z_{\rm c}) = \phi_{\rm ini}(z_{\rm c} - z) + \phi_{\rm ini}(z_{\rm c})$

$z_{\rm c}$: Center of the i-th kink state

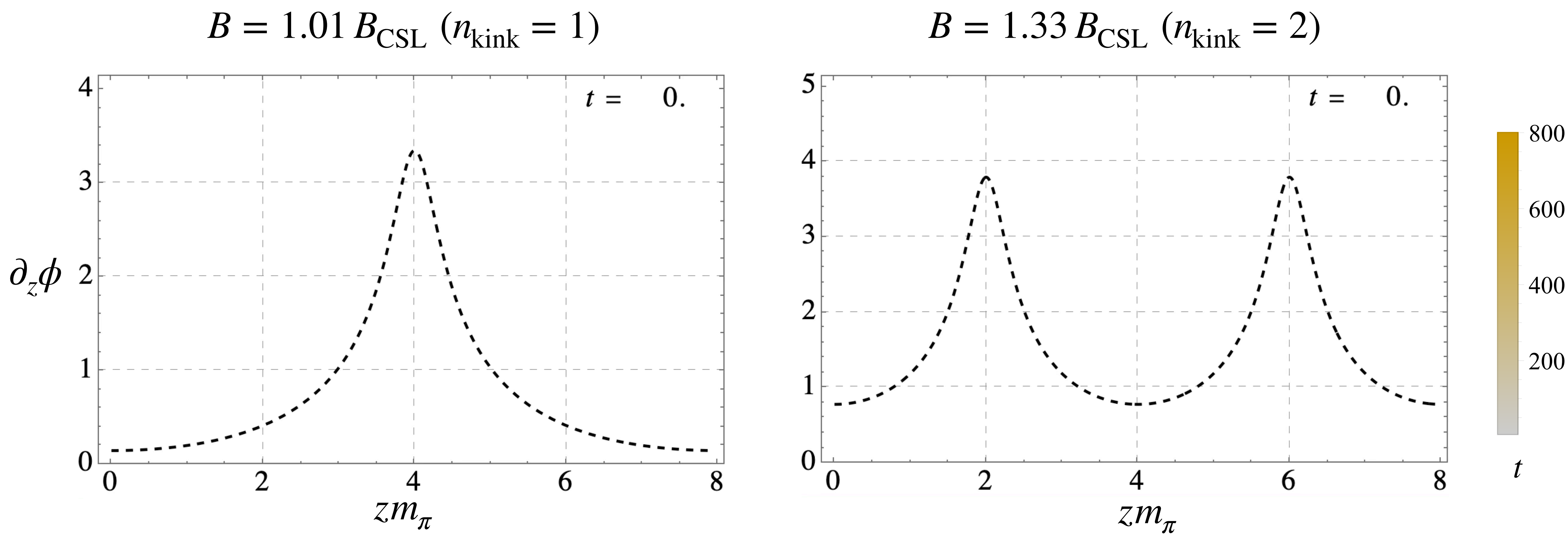
Case (II)



Movies

(time unit: $(m_\pi c)^{-1}$; c: pion velocity in the chiral limit, $L = 8$)

Case (I)



Movies

(time unit: $(m_\pi c)^{-1}$; c: pion velocity in the chiral limit, $L = 8$)

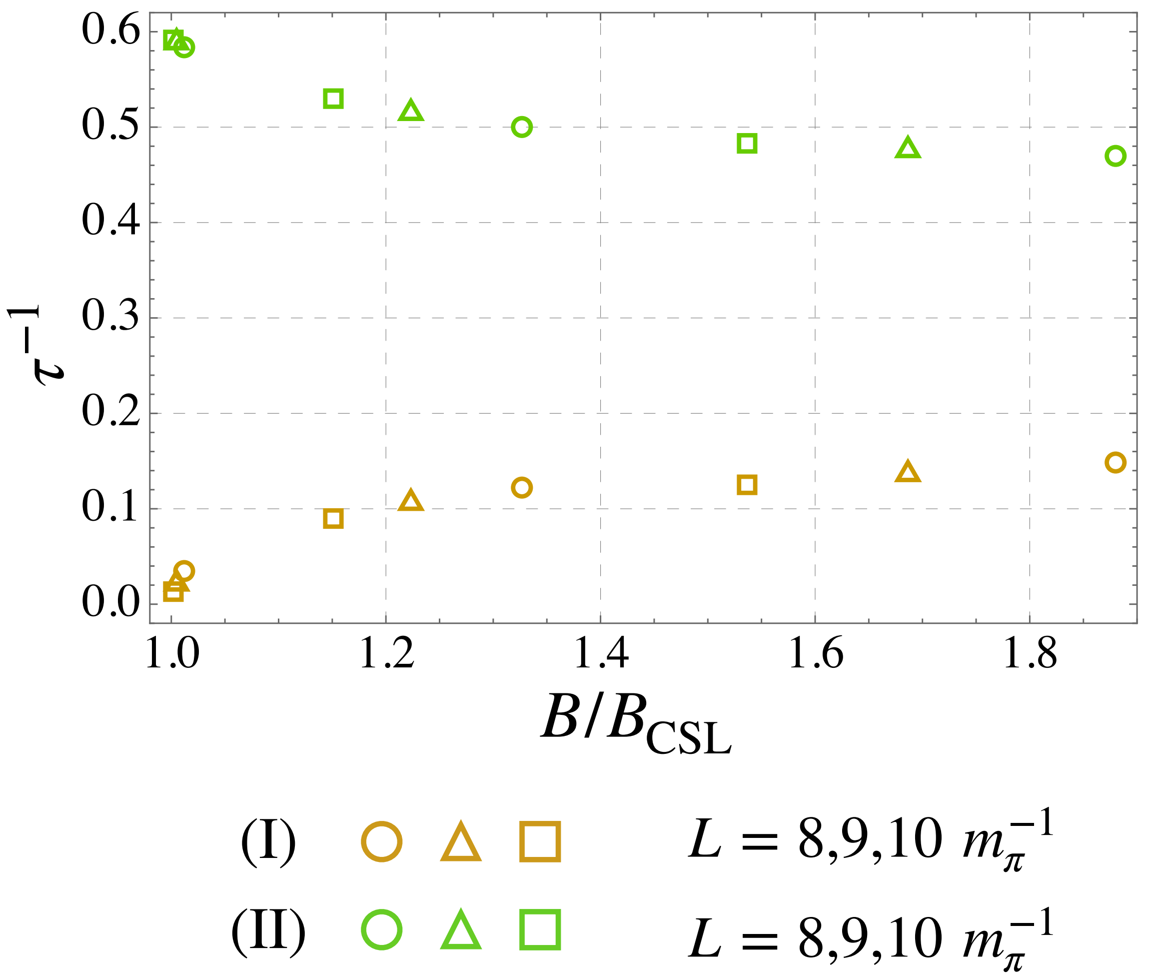
Characteristic rate

- How fast/slow does the system approach the stationary state near $B \sim B_{\rm CSL}$?

- A rate motivated by " $\propto \mathrm{e}^{-t/\tau}$ "

$$\Delta U(t) \equiv \int \mathrm{d}z \left| \phi(t,z) - \bar{\phi}(z) \right|$$

$$\longrightarrow \quad \tau \quad \text{s.t.} \quad \Delta U(\tau) = \frac{\Delta U(t_{\rm ini})}{\mathrm{e}}$$



Discussion

- As the CSL transition approaches, $B \to B_{CSL}$

  (I) Slowing down of the translational motion

  $\because$ Less repulsion from the other solitons

  (II) Finite relaxation to the domain wall

  $\sim$ Decay of gapped excited states $(\omega \sim m_\pi)$

A novel class of second-order transitions where the motion of a topological object (moduli) **only** slows down (not the relaxation to the solitonic state)

Summary

- Relaxation and soliton dynamics near the CSL transition
- Slow solitonic motion, whereas finite local relaxation rate

$\longrightarrow$ Characteristic time to the stationary state depends on the initial configuration whether it will include translational motion in the dynamical process

- Noel transition dynamics of topological solitons

Kentaro Nishimura and NS (2023), arXiv: 2304.01264