Noncommutative exterior product in lattice integrable systems

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Noncommutative Integrable Systems, 11.03.2024

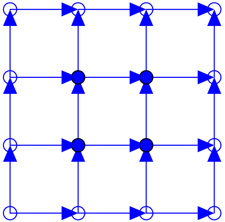
Overview

1. $-\mathrm{div} \to \partial$
2. $\mathrm{d} \to \delta$
3. $\wedge \to \smile$
4. $\lrcorner \to \frown$
5. $\int \to \epsilon$

Step 1

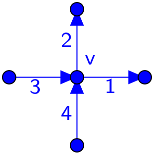
$$-\mathrm{div} \to \partial$$

Domain $\to$ Grid (or a triangulation):



- a grid of tubes with sources at the boundary, pumping fluid in and out.
- a grid of unit resistors with current sources at the boundary.

A *conserved current* j is a real-valued function defined on the set of edges satisfying



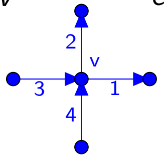
$$-j(1) - j(2) + j(3) + j(4) = 0$$

A *conserved current* j is a real-valued function defined on the set of edges satisfying

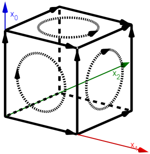
$$\partial j = 0,$$

where

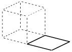
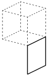
$$\partial j(v) = \sum_{e \text{ ending at } v} j(e) - \sum_{e \text{ starting at } v} j(e).$$



$$-j(1) - j(2) + j(3) + j(4) = 0$$



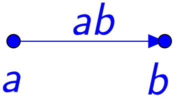
ϕ - function on faces

$[\partial\phi]($  $) = \phi($  $) - \phi($  $) + \phi($  $) - \phi($  $)$

Step 2

$$\mathrm{d} \to \delta$$

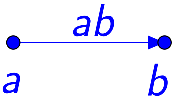
Potential flow



$$j(ab) = \phi(a) - \phi(b),$$

where ϕ is a function on vertices

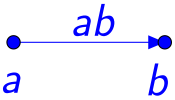
Potential flow



$$j = -\delta\phi,$$

where ϕ is a function on vertices and

$$[\delta\phi](ab) = \phi(b) - \phi(a)$$



$$j = -\delta\phi,$$

where ϕ is a function on vertices and

$$[\delta\phi](ab) = \phi(b) - \phi(a)$$

— discrete (exterior) derivative (de Rham)

Variational formulation

$$\sum_{\text{edges } e} [\delta\phi](e)^2 \to \min,$$

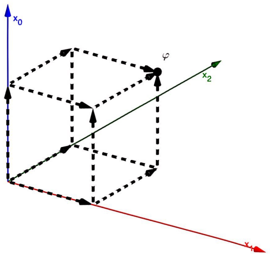
values on the boundary fixed

0-form = scalar function

$$\phi(x_0, x_1, x_2)$$

$$\downarrow$$

function on vertices

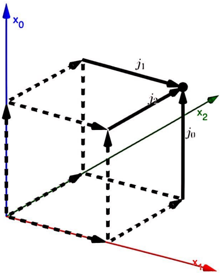


1-form $=^*$ vector function

$$j_0\,dx_0 + j_1\,dx_1 + j_2\,dx_2$$

$$\downarrow$$

function on edges

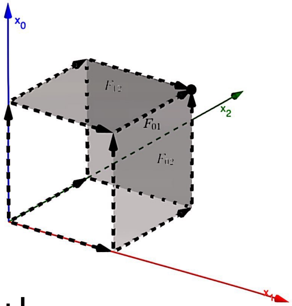


Take edges with a common endpoint!

k-form $\to$ function on k-dimensional faces

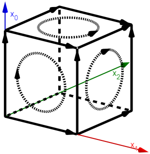
$$F_{12}\, dx_1 \wedge dx_2 + F_{02}\, dx_0 \wedge dx_2 + F_{01}\, dx_0 \wedge dx_1$$

$$|$$

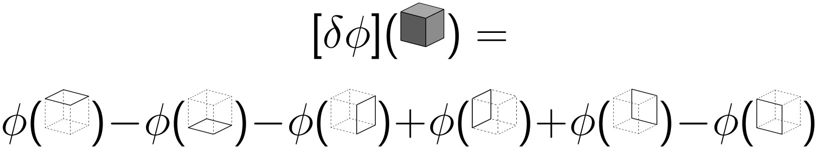


Take faces with a common maximal point!

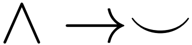
Coboundary operator



ϕ - function on faces

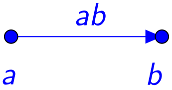


Step 3



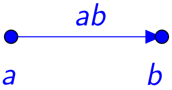
Cup-product: anticommutative vs associative

Cup product	
anticommutative	associative

Cup product	
anticommutative	associative
Kolmogorov–Alexander  $[\phi \smile j](ab) = \frac{\phi(a)+\phi(b)}{2} j(ab)$	

Cup-product: anticommutative vs associative

Cup product	
anticommutative	associative
Kolmogorov–Alexander	Whitney



$[\phi \smile j](ab) = \frac{\phi(a)+\phi(b)}{2} j(ab)$	$[\phi \smile j](ab) = \phi(a) j(ab)$

Cup-product: anticommutative vs associative

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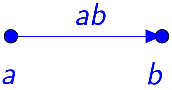

Kolmogorov–Alexander	Whitney
$[\phi \smile j](ab) = \frac{\phi(a)+\phi(b)}{2} j(ab)$	$[\phi \smile j](ab) = \phi(a) j(ab)$
$[j \smile \phi](ab) = j(ab) \frac{\phi(a)+\phi(b)}{2}$	$[j \smile \phi](ab) = j(ab) \phi(b)$

Cup-product: anticommutative vs associative

Cup product	
anticommutative	associative
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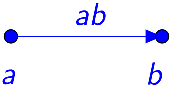

$[\phi \smile j](ab) = \frac{\phi(a)+\phi(b)}{2} j(ab)$	$[\phi \smile j](ab) = \phi(a) j(ab)$
$[j \smile \phi](ab) = j(ab) \frac{\phi(a)+\phi(b)}{2}$	$[j \smile \phi](ab) = j(ab) \phi(b)$
requires orientation	requires vertices ordering

Cup-product: anticommutative vs associative

Cup product	
anticommutative	associative
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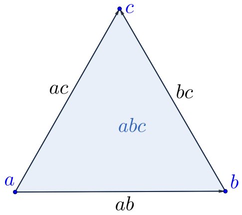
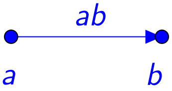


$[i \smile j](abc) = \frac{1}{6} i(ab)j(bc) - \dots$	$[i \smile j](abc) = i(ab)j(bc)$
requires orientation	requires vertices ordering

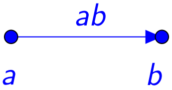
Cup-product: anticommutative vs associative

Cup product	
anticommutative	associative
Kolmogorov–Alexander	Whitney
$[\phi \smile j](ab) = \frac{\phi(a)+\phi(b)}{2} j(ab)$	$[\phi \smile j](ab) = \phi(a) j(ab)$
$[j \smile \phi](ab) = j(ab) \frac{\phi(a)+\phi(b)}{2}$	$[j \smile \phi](ab) = j(ab) \phi(b)$
requires orientation	requires vertices ordering
common in DEC	common in topology



Cup-product: anticommutative vs associative

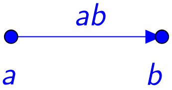
Cup product	
anticommutative	associative
Kolmogorov–Alexander	Whitney



$[\phi \smile j](ab) = \frac{\phi(a)+\phi(b)}{2} j(ab)$	$[\phi \smile j](ab) = \phi(a) j(ab)$
$[j \smile \phi](ab) = j(ab) \frac{\phi(a)+\phi(b)}{2}$	$[j \smile \phi](ab) = j(ab) \phi(b)$
requires orientation	requires vertices ordering
common in DEC	common in topology
—	curvature $F = \delta A + A \smile A$

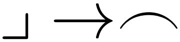
Cup-product: anticommutative vs associative

Cup product	
anticommutative (for a commutative ring)	associative (for an associative ring)
Kolmogorov–Alexander	Whitney

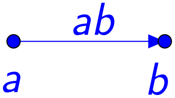


$[\phi \smile j](ab) = \frac{\phi(a)+\phi(b)}{2} j(ab)$	$[\phi \smile j](ab) = \phi(a) j(ab)$
$[j \smile \phi](ab) = j(ab) \frac{\phi(a)+\phi(b)}{2}$	$[j \smile \phi](ab) = j(ab) \phi(b)$
requires orientation	requires vertices ordering
common in DEC	common in topology
—	curvature $F = \delta A + A \smile A$

Step 4

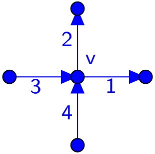


Dual: cap product



$$[j \smile \phi](ab) = j(ab)\phi(b)$$

$$[j \frown \phi](ab) = j(ab)\phi(a)$$



$$[j \frown j](v) = j(3)j(3) + j(4)j(4)$$

Leibnitz rule

$$\partial(\psi \frown \phi) = (-1)^{\dim \phi}(\partial\psi \frown \phi - \psi \frown \delta\phi)$$

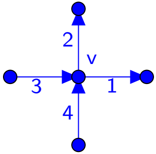
Step 5

$$\int \to \epsilon$$

Potential flow revisited

$$\mathcal{S} := \sum_{\text{edges } e} [\delta\phi](e)^2 \to \min$$

$$\mathcal{S} = \underbrace{\sum_{\text{vertices v}}}_{\varepsilon} \underbrace{[\delta\phi \frown \delta\phi](v)}_{\mathcal{L}[\phi]}$$



$$[\delta\phi \frown \delta\phi](v) = \delta\phi(3)\delta\phi(3) + \delta\phi(4)\delta\phi(4)$$

Spacetime M is an arbitrary finite simplicial or cubical complex with fixed vertices ordering. For a cubical complex, we require that the minimal and the maximal vertex of each 2-face are opposite.

A k-dimensional *field* or *k-cochain* ϕ is a real-valued function defined on the set of k-dimensional faces. Notation: $C^k(M;\mathbb{R}) = C_k(M;\mathbb{R})$.

A *Lagrangian* is a function

$$\mathcal{L}\colon C^k(M;\mathbb{R}) \to C_0(M;\mathbb{R}).$$

ϕ is *stationary* if $\frac{\partial}{\partial t}\epsilon\mathcal{L}[\phi + t\Delta] = 0\ \forall \Delta \in C^k(M;\mathbb{R})$.

Discretization algorithm

In a Lagrangian, replace				
continuum operation		$\to$	discrete one	
exterior derivative	d		coboundary	δ
exterior product	$\wedge$		cup product	$\smile$
interior product	$\lrcorner$		cap product	$\frown$
connection 1-form	A		connection	A
curvature 2-form	F		curvature	F

Examples

Table 2: Discretization of basic field theories

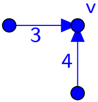
	Continuum		Discrete				
Field theory	Field	Lagrangian	Lagrangian	Equation of motion	Conserved current	Energy-momentum tensor	**Reference**
Electric network	$\mathbb{R}$-valued 0-form ϕ	$\frac{1}{2}d\phi \lrcorner d\phi - s \lrcorner \phi$	$\frac{1}{2}\delta\phi \frown \delta\phi - s \frown \phi$	$\partial\delta\phi = s$	$\delta\phi$	$\delta\phi \times \delta\phi$	§2.1
Electrodynamics	$\mathbb{R}$-valued 1-form A	$-\frac{1}{2}\sharp dA \lrcorner dA - j \lrcorner A$	$-\frac{1}{2}\#\delta A \frown \delta A - j \frown A$	$-\partial\#\delta A = j$	j	$-\#\delta A \times \delta A$	§2.2
Gauge theory	connection 1-form A	$-\mathrm{ReTr}[\frac{1}{2}\sharp F^* \lrcorner F + j \lrcorner A]$, $F = dA + A \wedge A$	$-\mathrm{ReTr}[\frac{1}{2}\#F^* \frown F + j \frown A]$, $F = \delta A + A \smile A$	$\mathrm{Pr}_{T_U G}\, D_A^* \#F = -\mathrm{Pr}_{T_U G}\, j$	j	$-\mathrm{ReTr}[\#F \times F]$	§2.3
Klein–Gordon field	$\mathbb{C}$-valued 0-form ϕ	$\sharp d\phi \lrcorner d\phi^* - m^2\phi \lrcorner \phi^*$	$\#\delta\phi \frown \delta\phi^* - m^2\phi \frown \phi^*$	$\partial\#\delta\phi = m^2\phi$	$-2\mathrm{Im}[\#\delta\phi^* \frown \phi]$	$2\mathrm{Re}[\#\delta\phi^* \times \delta\phi - m^2\phi^* \times \phi]$	§A.2
Boson in a gauge field	$\mathbb{C}^{1\times n}$-valued k-form ϕ	$\sharp D_A\phi \lrcorner (D_A\phi)^* - m^2\phi \lrcorner \phi^*$	$\#D_A\phi \frown (D_A\phi)^* - m^2\phi \frown \phi^*$	$D_A^*\#D_A\phi = m^2\phi$	$-2\phi^* \smile \#D_A\phi$	unknown	§A.2

Informal definitions

A Lagrangian $\mathcal{L}$ is *local*, if its value at a vertex v depends only on the values of ϕ and $\delta\phi$ at the faces for which v is maximal. $\mathcal{L}[\phi](v) = L(\phi(v), \delta\phi(3), \delta\phi(4))$.

Partial derivatives $\frac{\partial \mathcal{L}}{\partial \phi}$ and $\frac{\partial \mathcal{L}}{\partial (\delta\phi)}$ are fields of dimension k and $k+1$ respectively, obtained by differentiating $\mathcal{L}$ as if ϕ and $\delta\phi$ were independent variables. $\frac{\partial \mathcal{L}}{\partial \phi}(v) = \frac{\partial L(x,y,z)}{\partial x}\Big|_{x=\phi(v), y=\delta\phi(3), z=\delta\phi(4)}$.

Lagrangian $\mathcal{L}[\phi]$	$\frac{\partial \mathcal{L}}{\partial \phi}$	$\frac{\partial \mathcal{L}}{\partial (\delta\phi)}$
$\frac{1}{2}\delta\phi \frown \delta\phi$	0	$\delta\phi$
$\frac{1}{2}\delta\phi \frown \delta\phi + \frac{1}{2}m^2\phi \frown \phi$	$m^2\phi$	$\delta\phi$



Conservation laws

$j \in C_1(M;\mathbb{R})$ is a *conserved current*, if $\partial j = 0$.

The *Noether theorem* gives a conserved current for each continuous symmetry of the Lagrangian.

Discrete Nöther's theorem

Theorem (Discrete Nöther's theorem, S)

Let $\mathcal{L}\colon C^k(M;\mathbb{R}) \to C_0(M;\mathbb{R})$ be a local Lagrangian and $\phi \in C^k(M;\mathbb{R})$ be a stationary field. The Lagrangian is invariant under an infinitesimal transformation $\Delta \in C^k(M;\mathbb{R})$, i.e.,

$$\frac{\partial}{\partial t}\mathcal{L}[\phi + t\Delta]\Big|_{t=0} = 0,$$

iff the following current is conserved:

$$j[\phi] = \frac{\partial \mathcal{L}[\phi]}{\partial(\delta\phi)} \frown \Delta.$$

E.g., for *electrical networks* $\mathcal{L}[\phi]$ is invariant under $\phi \mapsto \phi - t$, $t \in \mathbb{R}$. The conserved current is $j = -\delta\phi$.

Reference

S., Discrete field theory: symmetries and conservation laws, Math. Phys. Anal. Geom. 26:19 (2023).

THANKS!