International Seminar-Type Online Workshop on Topological Solitons @ Online (Sep. 27, 2023)

Born Sigma Models and Worldsheet Instantons in T-fold

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NPB **981** (2022) 115873 [arXiv:2202.03016],
JHEP **06** (2022) 119 [arXiv:2203.03272],
and arXiv:23xx.xxxxx

with Tetsuji Kimura (Osaka Electro-Communication U.) and Shin Sasaki (Kitasato U.)

Introduction

- We are interested in the **stringy nature of spacetime**
- The **worldsheet instantons** contribute to the string scattering amplitude as non-perturbative effects of α'
- **T-fold** is a space in which local charts are patched together with T-duality

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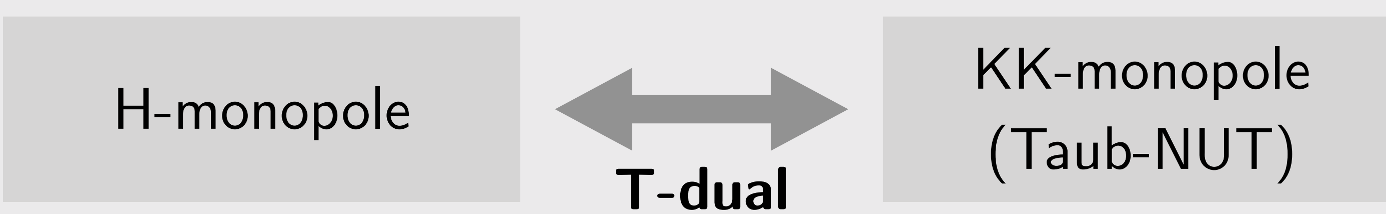
Our message in this talk is as follows:

1. The worldsheet instantons on T-fold are **multi-valued** and ill-defined **due to the T-duality monodromies**
2. The Born sigma model is a two-dimensional sigma model with T-duality of the target space as a manifest symmetry
3. Since doubled instantons in Born sigma models are described by T-duality covariant, the worldsheet instantons on T-fold are **well-defined by using doubled formalism**

Introduction — T-duality

- The transverse geometry of the KK-monopole in type II string theories is known as the four-dimensional Euclidean Taub-NUT space

- The T-duality transformation along the S^1 isometry direction in the Taub-NUT space yields the H-monopole



Introduction — worldsheet instantons

- The H-monopole is also obtained by periodically aligning and smearing NS5-branes to make the S^1 isometry
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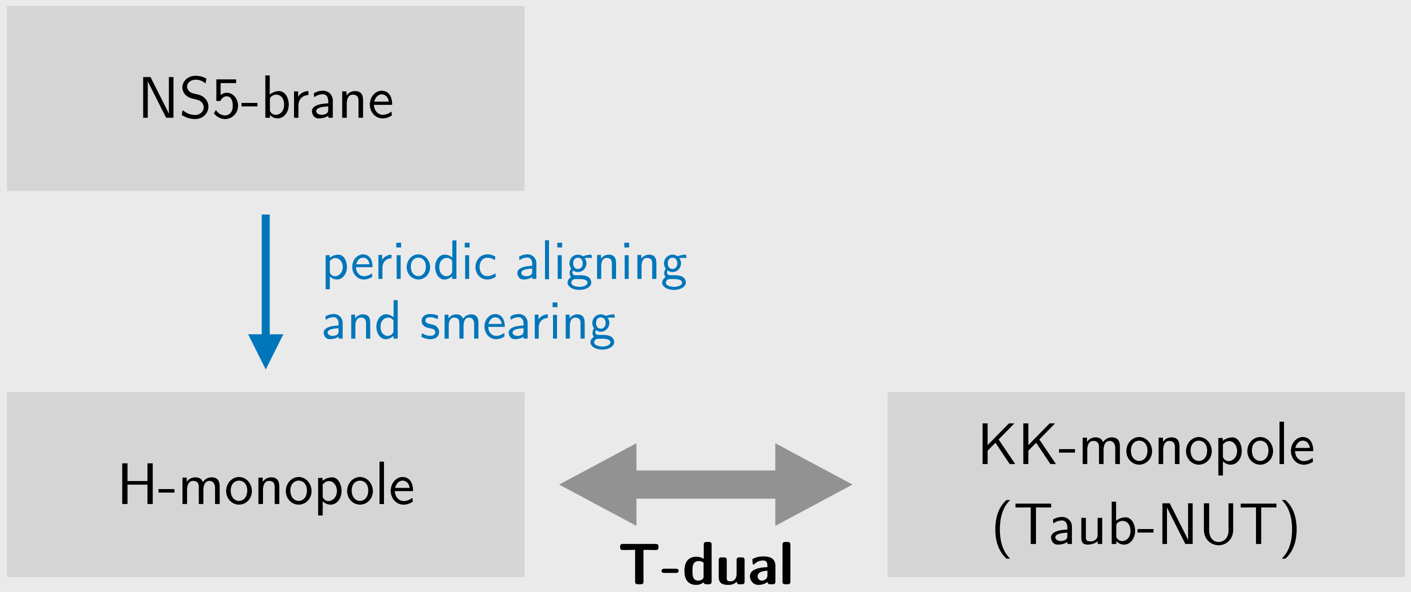
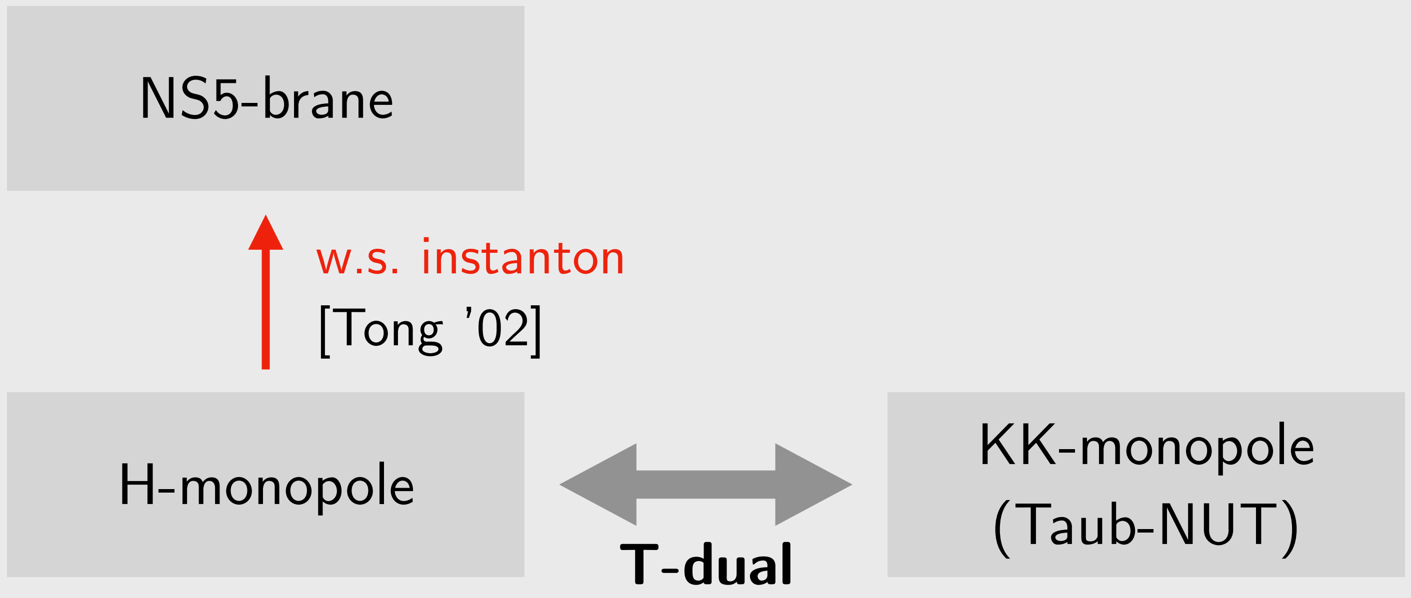


Introduction — worldsheet instantons

- The H-monopole is also obtained by periodically aligning and smearing NS5-branes to make the S^1 isometry
- In contrast, the NS5-brane is obtained by localizing the S^1 direction of the H-monopole
- The localization of the H-monopole in S^1 is identified by the worldsheet instanton effects

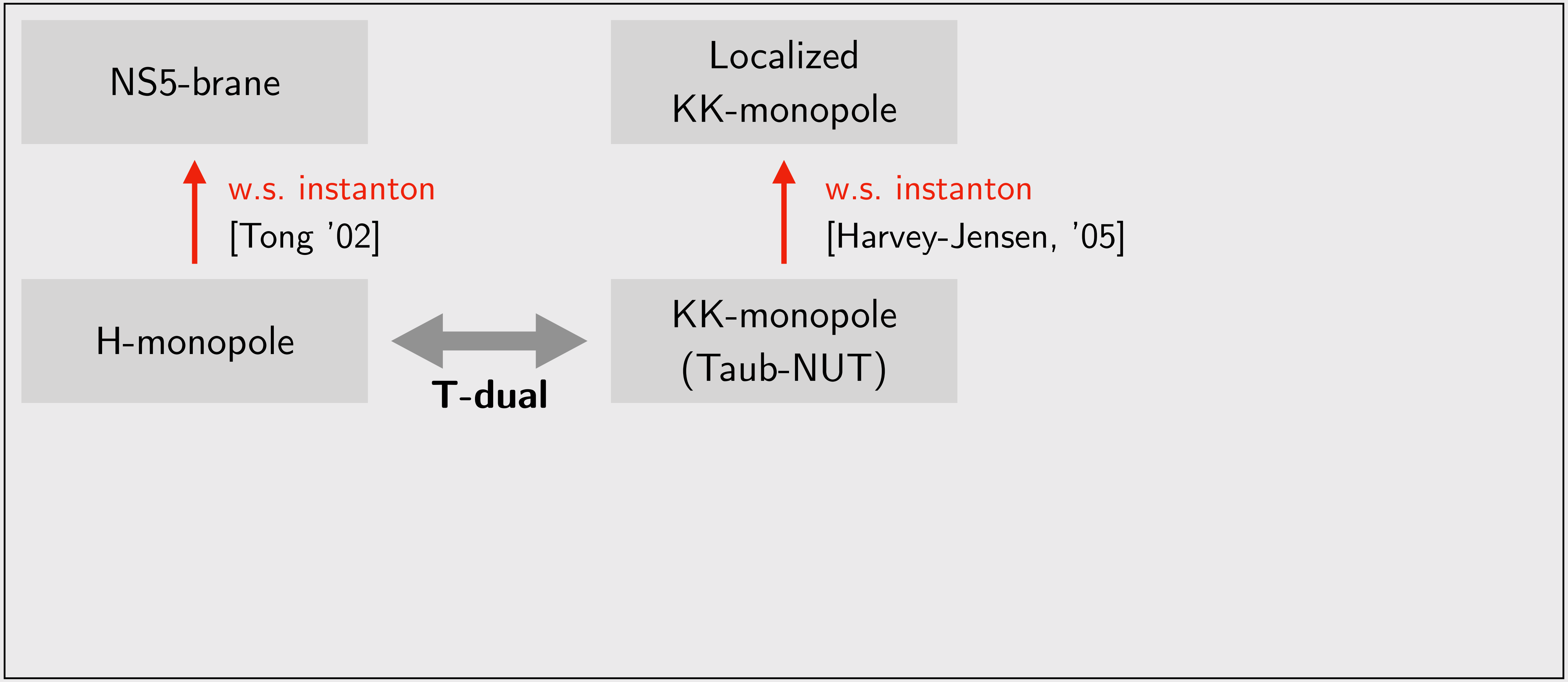
[Tong, '02]



Introduction — worldsheet instantons

- When we consider the worldsheet instanton correction to the KK-monopole, the S^1 isometry in the Taub-NUT is not broken and localized in the "winding space"
  [Harvey-Jensen, '05]

- The "winding coordinate" is a Fourier conjugate to the string winding number, and is a T-dual to the coordinate conjugate to the momentum



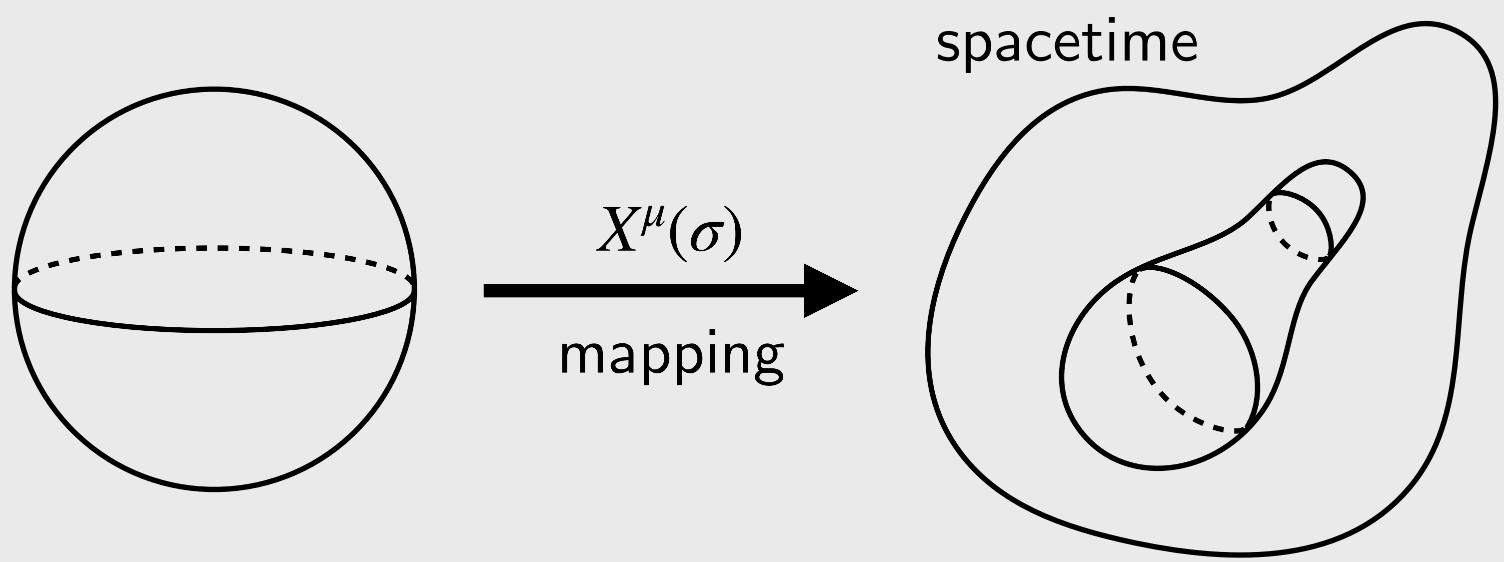
String worldsheet instantons

[Wen-Witten '86]

- Worldsheet instantons are configurations that minimize the Euclidean action of the fundamental string in a given topological sector

- We now focus on the tree-level worldsheet $\Sigma = S^2$

- The worldsheet instanton is a mapping from the worldsheet with S^2 topology to a 2-cycle in the target space



$\Sigma = S^2$
string worldsheet

$C_2 \simeq S^2$
2-cycle in spacetime

- This map is classified by the homotopy group $\pi_2(S^2) = \mathbb{Z}$

Worldsheet instanton equations

- The string worldsheet sigma model is given by

$$S = \frac{1}{2}\int \left(g_{\mu\nu}\,\mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu + B_{\mu\nu}\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu\right)$$

- We focus on the metric term in the action
- In the following, spacetime and the worldsheet have the Euclidean signature
- Since the metric is positive-definite, we have the Bogomol'nyi bound of the action

$$\begin{aligned} S_{\mathrm{E}} &= \frac{1}{4}\int \left[g_{\mu\nu}\left(\underline{\mathrm{d}X^\mu \pm J^\mu{}_\rho * \mathrm{d}X^\rho}\right) \wedge *\left(\mathrm{d}X^\nu \pm J^\nu{}_\sigma * \mathrm{d}X^\sigma\right) \pm 2\omega_{\mu\nu}\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu\right] \\ &\geq \frac{1}{2}\int \omega_{\mu\nu}\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu \end{aligned}$$

complex structure

$\omega = -\,gJ$

- The bound is saturated when the map X satisfies

$$\mathrm{d}X^\mu \pm J^\mu{}_\nu * \mathrm{d}X^\nu = 0 \longleftarrow$$

w.s. instanton equation

worldsheet instantons require a **complex structure** on the target space

Worldsheet instanton effects

- The worldsheet instantons contribute to the string scattering amplitude

$$e^{-S_{\mathrm{inst.}}} \sim \exp\left[-\frac{\mathcal{A}}{\alpha'}\right]$$

where α' is the square of the string length $\alpha' = \ell_{\mathrm{s}}^2$

- The contribution of the worldsheet instantons shows the non-perturbative effects of α'

$\longrightarrow$ the stringy nature of spacetime

Introduction — T-fold

- Smearing the KK-monopole and introducing another isometry, the solution is called the KK-vortex
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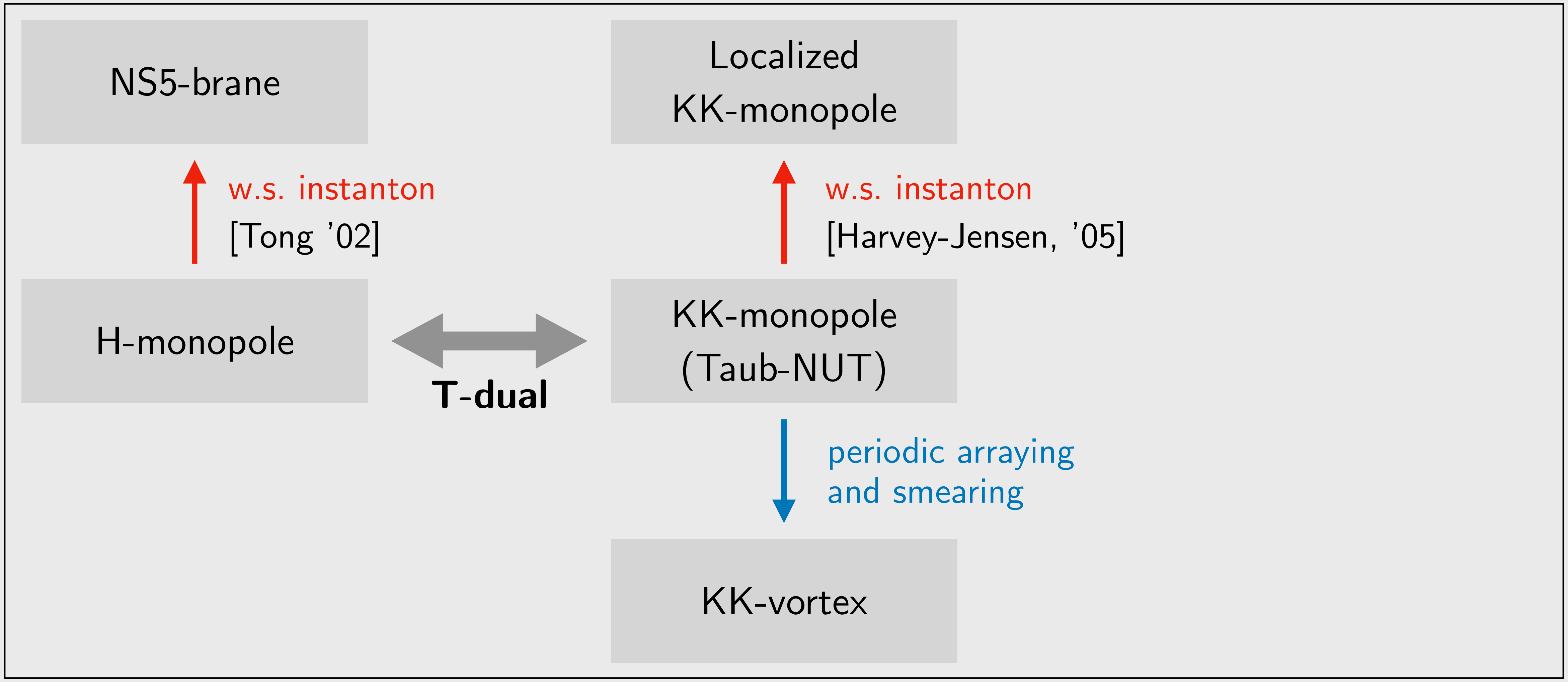

Introduction — T-fold

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- For the KK-vortex, the T-duality transformation along the isometry originated from the KK-monopole yields the defect NS5-brane
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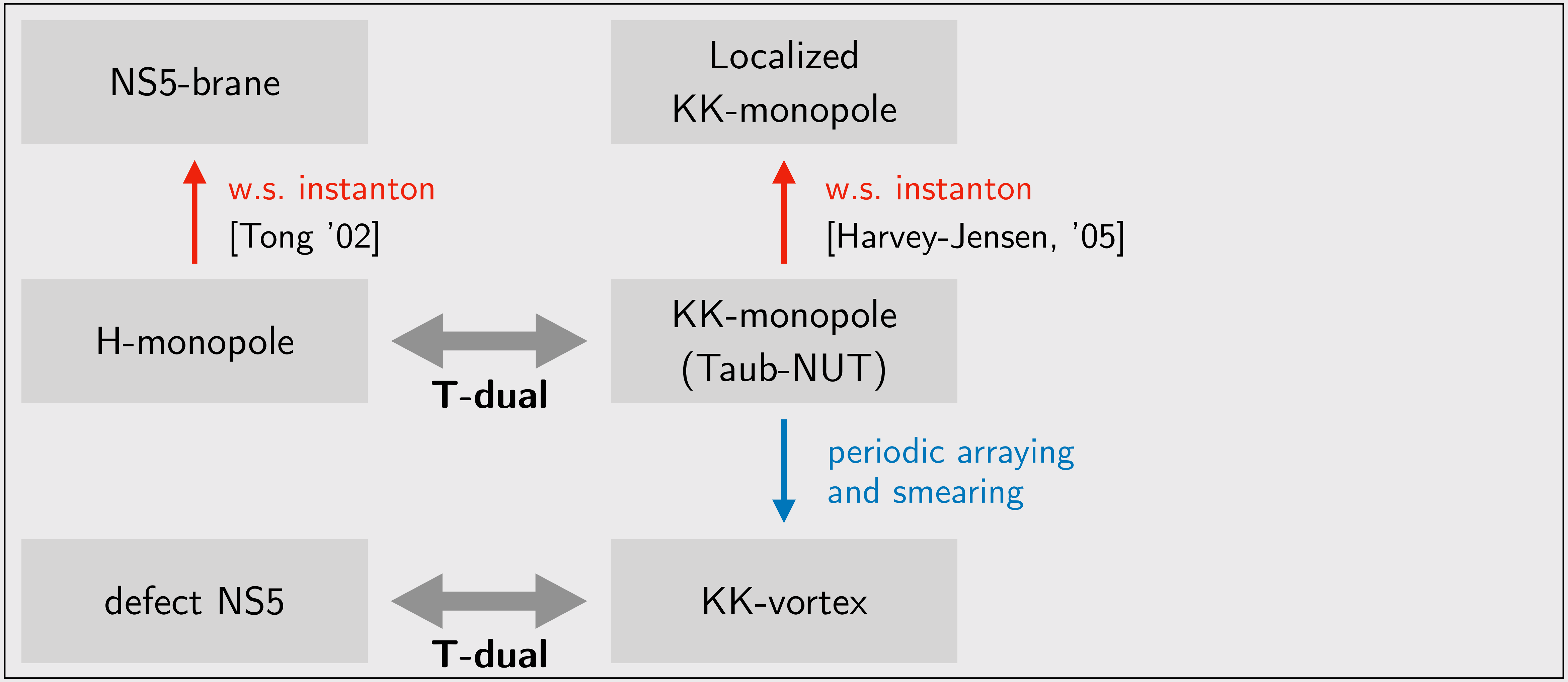
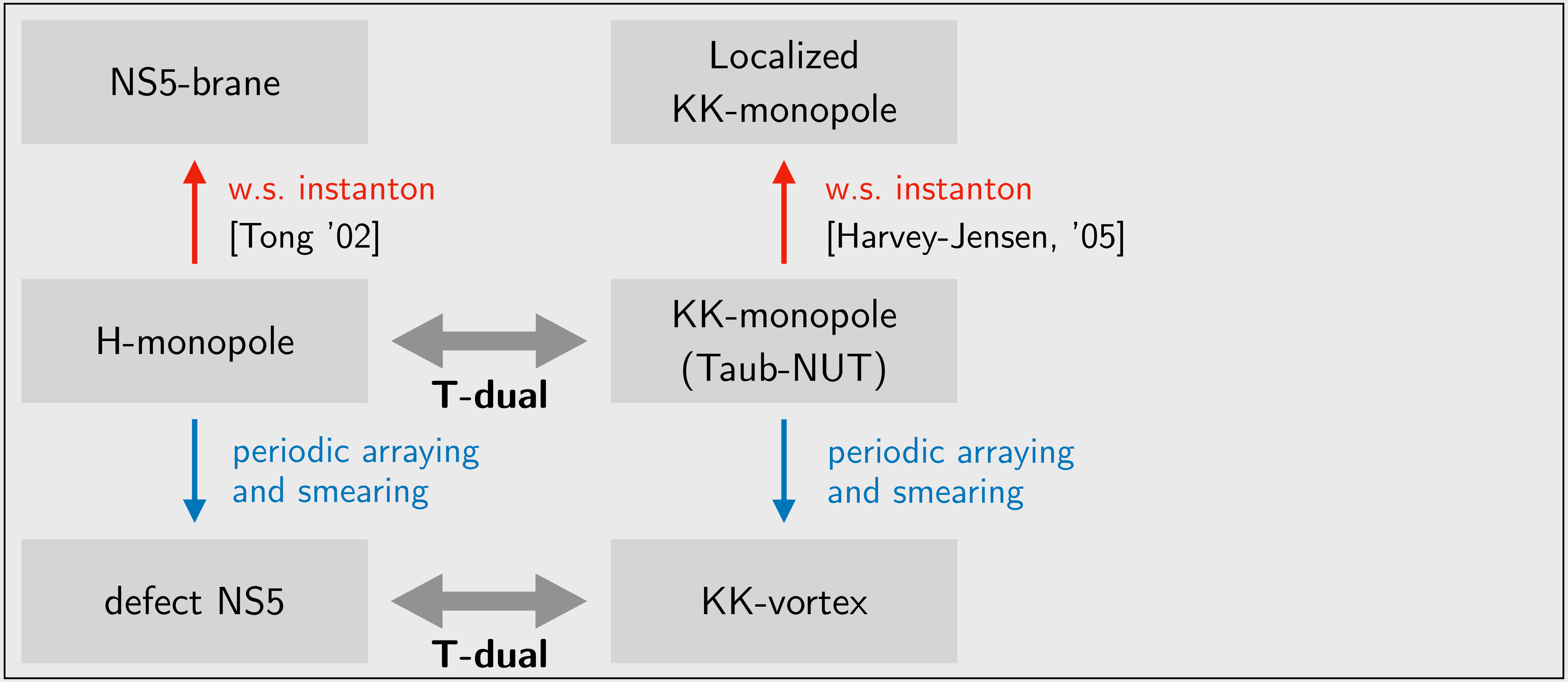


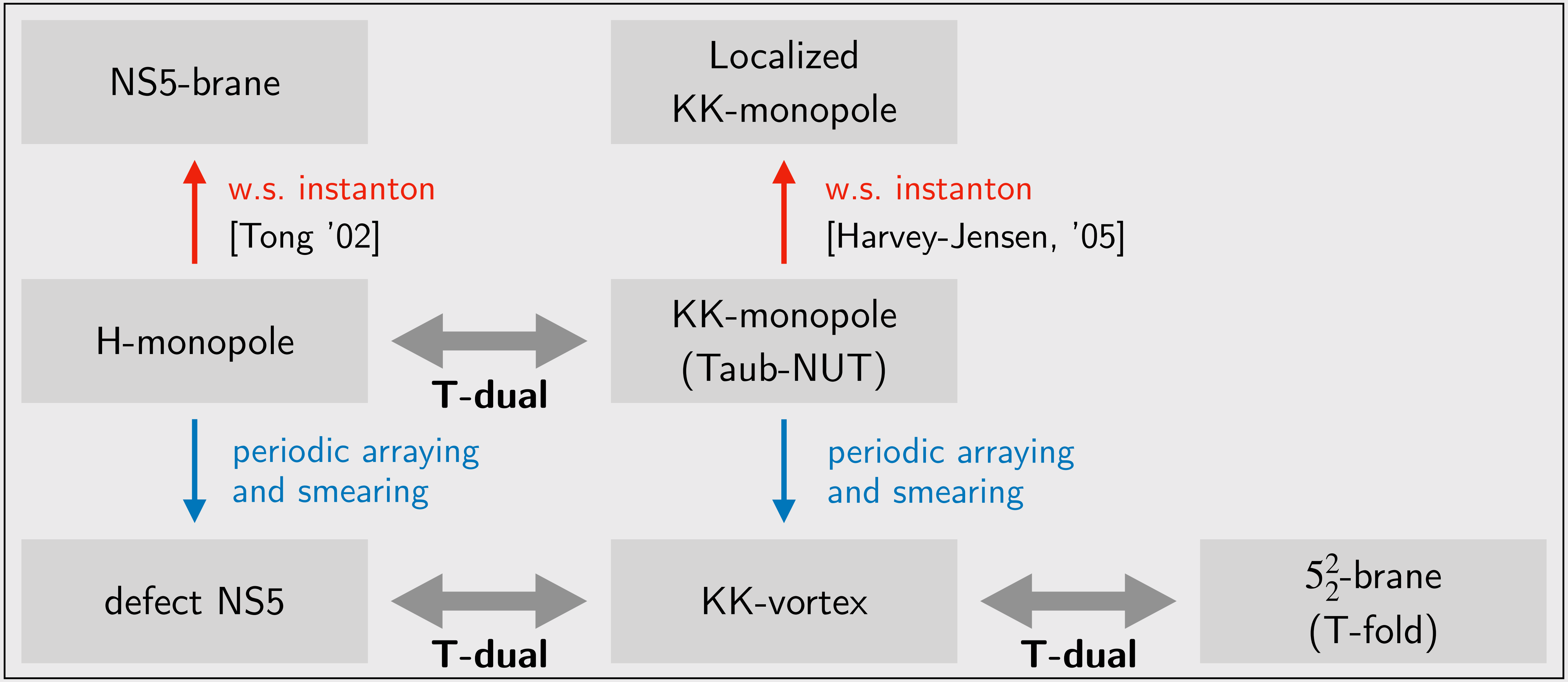
Introduction — T-fold

- Smearing the KK-monopole and introducing another isometry, the solution is called the KK-vortex
- For the KK-vortex, the T-duality transformation along the isometry originated from the KK-monopole yields the defect NS5-brane
- The defect NS5-brane is a smeared solution of the H-monopole



Introduction — T-fold

- Smearing the KK-monopole and introducing another isometry, the solution is called the KK-vortex
- The T-duality transformation along the newly introduced isometry of the KK-vortex yields the 5^2_2-brane [de Boer-Shigemori, '10, '12]
- A T-fold is a space where local charts are patched by T-duality, and the 5^2_2-brane is one of the examples



T-fold: 5^2_2-brane

[de Boer-Shigemori, 1004.2521, 1209.6056]

KK-vortex geometry

$$ds^2 = H\,dx^2_{123} + H^{-1}(dx^2_4 + A_i\,dx^i)^2$$

$$H = h_0 + \sigma \log\left(\frac{\mu}{\rho}\right)$$

(h_0, σ and μ are constants)

$$dA = \hat{*}_3\,dH$$

$(\rho^2 = (x^1)^2 + (x^2)^2,\ \theta = \arctan(x^2/x^1))$

T-duality transformation (Buscher rule) along $y = x^3$

$$g'_{ij} = g_{ij} - \frac{g_{iy}g_{jy} - B_{iy}B_{jy}}{g_{yy}}, \qquad g'_{iy} = \frac{B_{iy}}{g_{yy}}, \qquad g'_{yy} = \frac{1}{g_{yy}},$$

$$B'_{ij} = B_{ij} - \frac{B_{iy}g_{jy} - g_{iy}B_{jy}}{g_{yy}}, \qquad B'_{iy} = \frac{g_{iy}}{g_{yy}}, \qquad \phi' = \phi - \frac{1}{2}\log g_{yy}\,.$$

5^2_2-brane geometry

$$ds^2 = H\,dx^2_{12} + \frac{H}{H^2 + A_3^2}\,dx^2_{34}, \quad B = -\frac{A_3}{H^2 + A_3^2}\,dx^3 \wedge dx^4, \quad e^{2\phi} = e^{2\phi_0}\frac{H}{H^2 + A_3^2}\,.$$

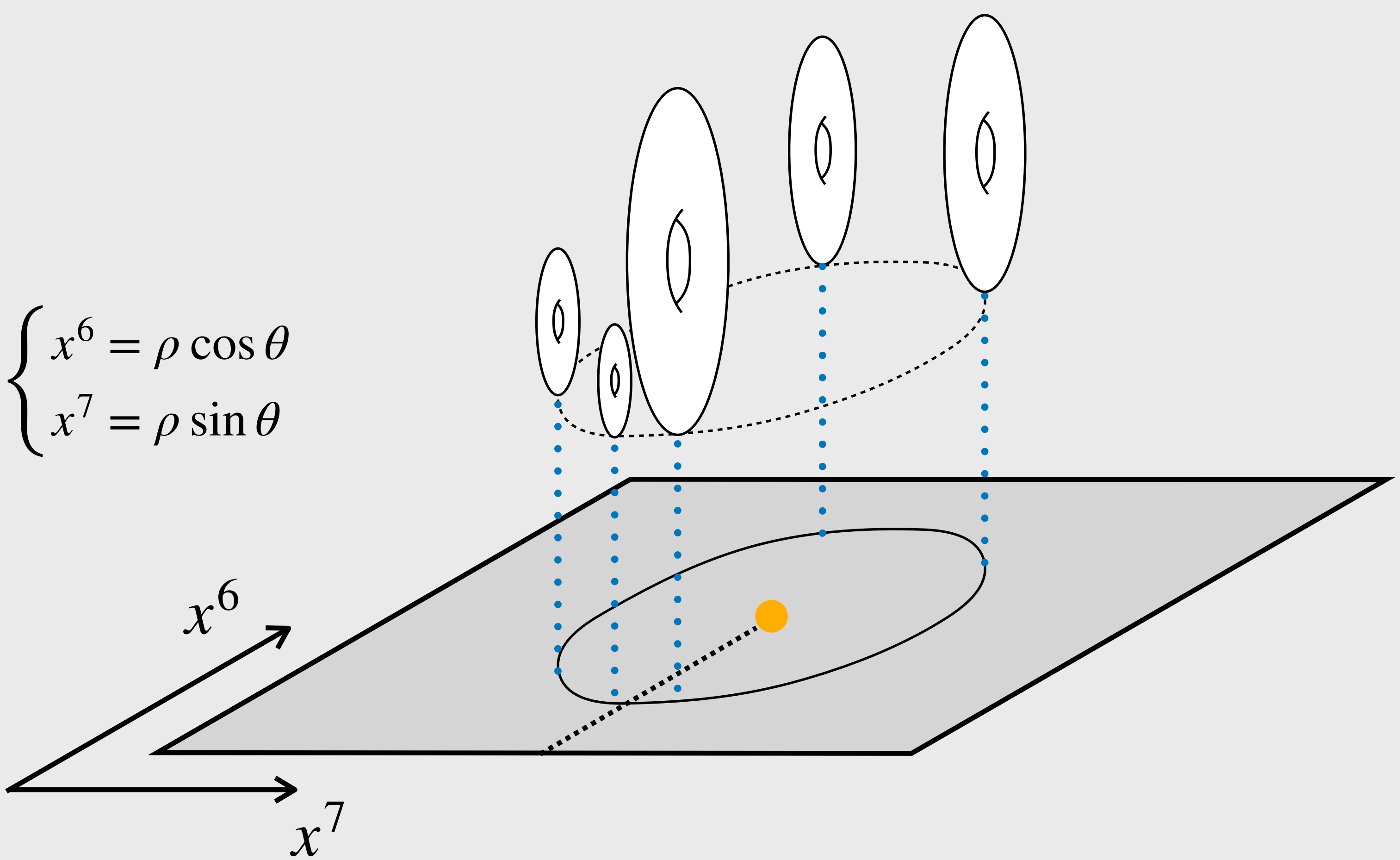
T-fold: 5^2_2-brane

[de Boer-Shigemori, 1004.2521, 1209.6056]

$$\boxed{\begin{aligned} & ds^2 = H\,dx_{12}^2 + \frac{H}{H^2 + A_3^2}\,dx_{34}^2, \\ & H = H(\rho), \quad A_3 = -\,\sigma\theta, \quad \sigma = \text{const}\,. \end{aligned}}$$

$$\begin{cases} \theta = 0: & \dfrac{H}{H^2 + A_3^2} = \dfrac{1}{H} \\ \theta = 2\pi: & \dfrac{H}{H^2 + A_3^2} = \dfrac{H}{H^2 + (2\pi\sigma)^2} \end{cases}$$

$$\begin{cases} x^6 = \rho\cos\theta \\ x^7 = \rho\sin\theta \end{cases}$$



- The geometry of 5^2_2-brane is torus fibered
- The torus radii do **not** match at $\theta = 0$ and 2π
- This geometry has a *monodromy*
- This monodromy is neither a diffeo. nor a B-field gauge transformation

→ T-duality

T-fold: 5^2_2-brane

[de Boer-Shigemori, 1004.2521, 1209.6056]

- The 5^2_2 monodromy is clearly evaluated in the **doubled formalism**
- The doubled formalism is the T-duality $O(D,D)$ covariant formulation
- The metric and B-fields are combined into an $O(D,D)$ covariant quantity called the *generalized metric*:

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

  - generalized metric for 5^2_2 at $\theta = 0$:

$$\mathcal{H}(\theta = 0) = \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H^{-1}\delta & 0 & 0 \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & 0 & 0 & H\delta \end{pmatrix}$$

  - generalized metric for 5^2_2 at $\theta = 2\pi$:

$$\mathcal{H}(\theta = 2\pi) = \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H^{-1}\delta & 0 & 2\pi\sigma H^{-1}\epsilon \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & H^{-1}\delta & 0 & (H + (2\pi\sigma)^2 H^{-1})\delta \end{pmatrix}$$

T-fold: 5^2_2-brane

[de Boer-Shigemori, 1004.2521, 1209.6056]

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  - generalized metric for 5^2_2 at $\theta = 2\pi$:

$$\mathcal{H}(\theta = 2\pi) = \underline{\begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & -2\pi\sigma\epsilon & 0 & \delta \end{pmatrix}} \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H^{-1}\delta & 0 & 0 \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & 0 & 0 & H\delta \end{pmatrix} \underline{\begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 2\pi\sigma\epsilon \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}}$$

$O(D,D)$ matrix known as the β-shift

T-duality

Introduction — T-fold

- The worldsheet instantons on the T-fold were analyzed using the *GLSM technique* [Kimura-Sasaki, '13][Kimura-Sasaki-KS, '18]
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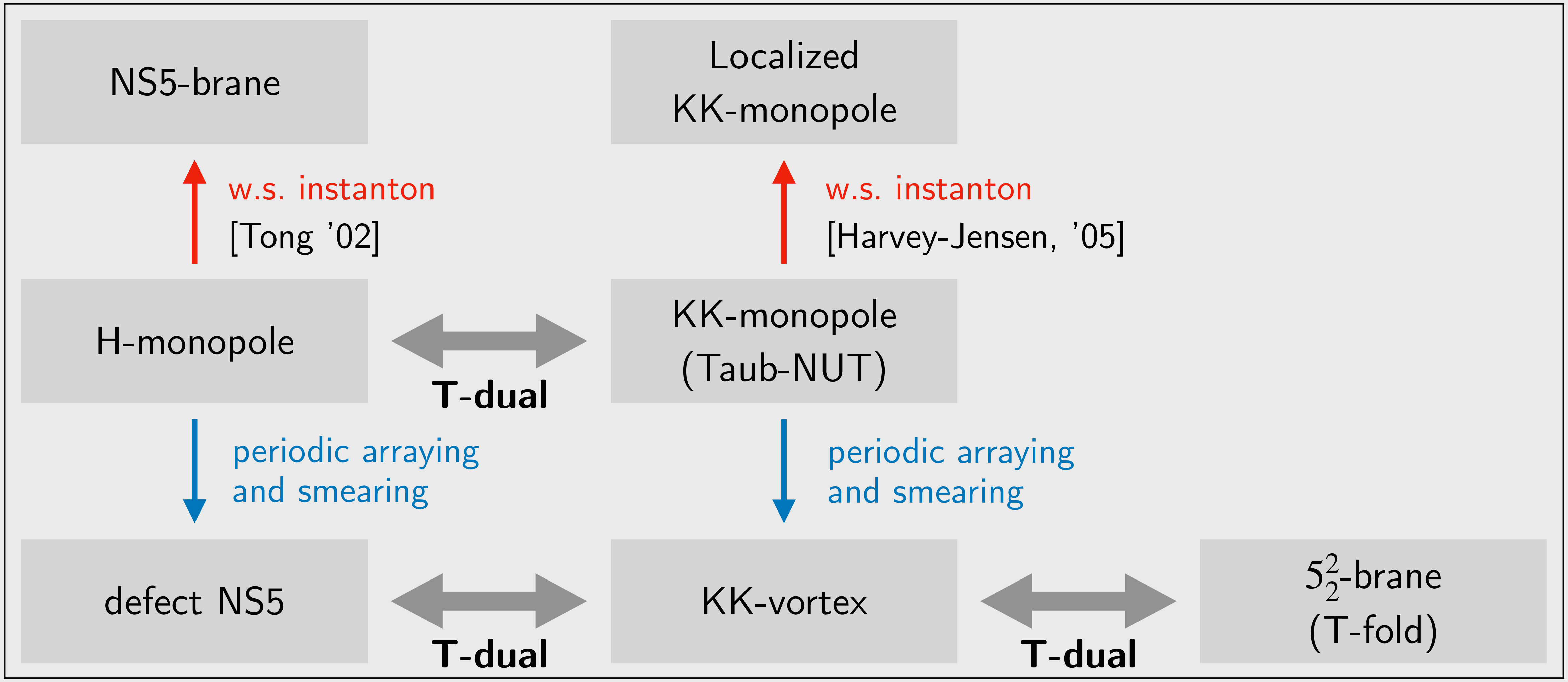

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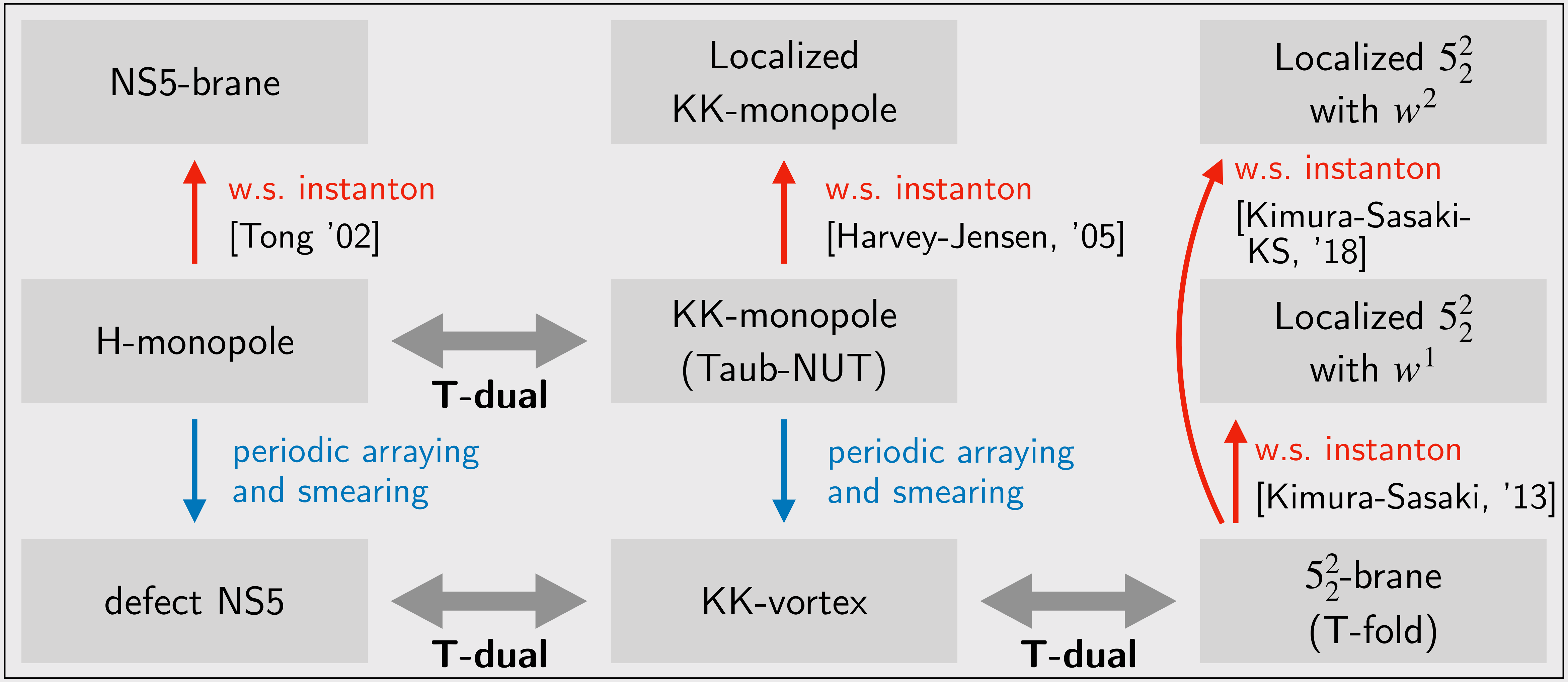
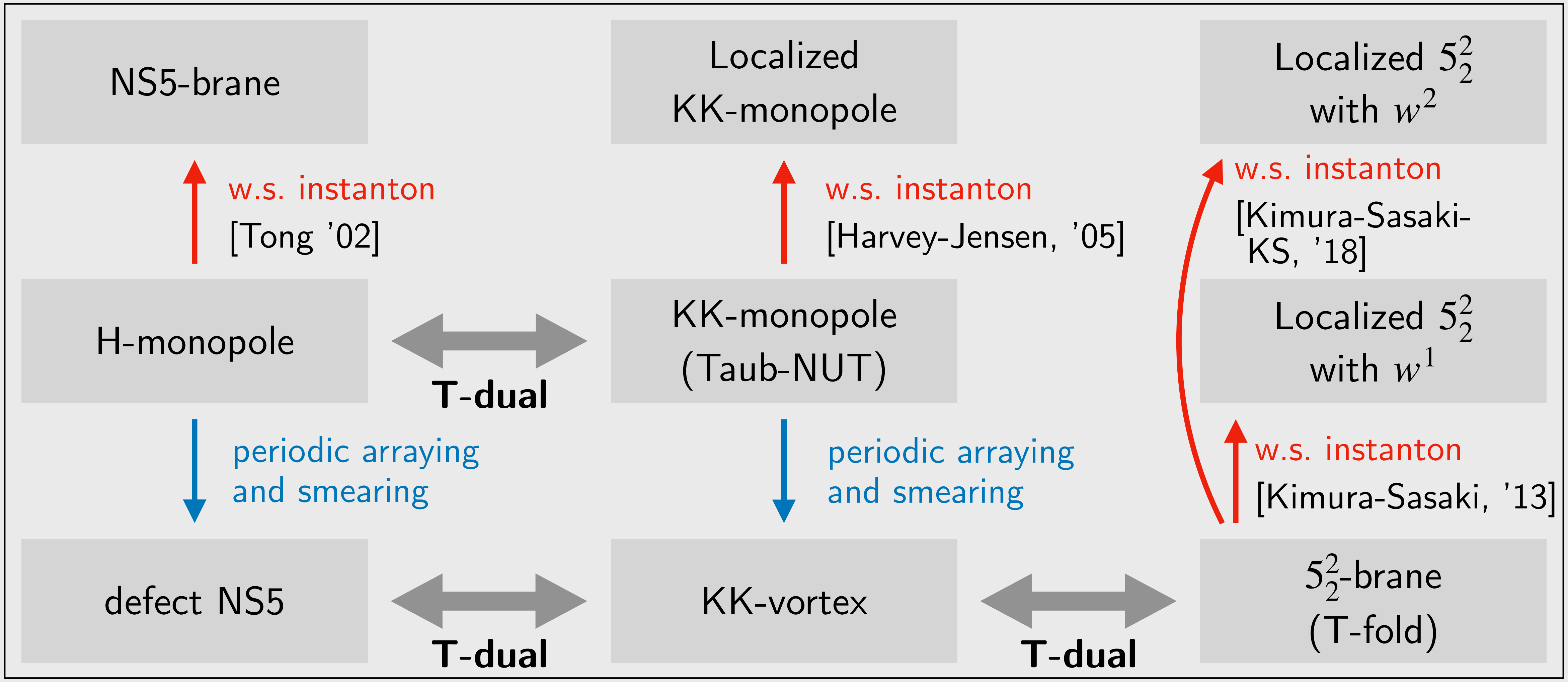


Introduction — T-fold

- The worldsheet instantons on the T-fold were analyzed using the *GLSM technique* [Kimura-Sasaki, '13][Kimura-Sasaki-KS, '18]
- The picture of the worldsheet instantons in NLSM is still not well understood
- In this talk, we will discuss the worldsheet instantons on T-fold using the doubled formalism



Complex structures on T-fold

T-duality b/w (J_a, ω_a) and $(J_{a,\pm}, \omega_{a,\pm})$

- The KK-vortex has the hyperkähler structure that originates from the Taub-NUT space

$$(J_1)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & -A_3H^{-1} & -H^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ H & -A_3 & 0 & 0 \end{pmatrix}, \qquad (\omega_1)_{\mu\nu} = \begin{pmatrix} 0 & 0 & A_3 & 1 \\ 0 & 0 & H & 0 \\ -A_3 & -H & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$(J_2)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -A_3H^{-1} & -H^{-1} \\ -1 & 0 & 0 & 0 \\ A_3 & H & 0 & 0 \end{pmatrix}, \qquad (\omega_2)_{\mu\nu} = \begin{pmatrix} 0 & 0 & -H & 0 \\ 0 & 0 & A_3 & 1 \\ H & -A_3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(J_3)^\mu{}_\nu = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -A_3H^{-1} & -H^{-1} \\ 0 & 0 & H + A_3^2H^{-1} & A_3H^{-1} \end{pmatrix}, \qquad (\omega_3)_{\mu\nu} = \begin{pmatrix} 0 & H & 0 & 0 \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

T-duality transformation from the hyperkähler (J_a, ω_a) to the bi-hypercomplex structure $(J_{a,\pm}, \omega_{a,\pm})$:

[Hassan '95][Kimura-Sasaki-KS '22]
[Blair-Hulik-Sevrin-Thompson '22]

$$(J'_{a,\pm})^i{}_j = (J_a)^i{}_j - \frac{(J_a)^i{}_y g_{yj}}{g_{yy}}, \qquad (J'_{a,\pm})^i{}_y = \mp\frac{(J_a)^i{}_y}{g_{yy}}, \qquad (J'_{a,\pm})^y{}_j = \pm(\omega_a)_{yj}, \qquad (J'_{a,\pm})^y{}_y = 0,$$

$$(\omega'_{a,\pm})_{ij} = (\omega_a)_{ij} - \frac{(\omega_a)_{iy}g_{yj} + g_{iy}(\omega_a)_{yj}}{g_{yy}}, \qquad (\omega'_{a,\pm})_{iy} = \mp\frac{(\omega_a)_{iy}}{g_{yy}}.$$

Bi-hypercomplex structure on T-fold

- Using the Buscher-like rule, we obtain the **bi-hypercomplex structure** of the 5^2_2-brane from the hyperkähler structure of the KK-vortex [Kimura-Sasaki-KS, to appear]

bi-hypercomplex structure

$$
\left\{
\begin{aligned}
J_{1,+} &= \begin{pmatrix} 0 & 0 & A_3K^{-1} & -HK^{-1} \\ 0 & 0 & HK^{-1} & A_3K^{-1} \\ -A_3 & -H & 0 & 0 \\ H & -A_3 & 0 & 0 \end{pmatrix}, \quad
J_{1,-} = \begin{pmatrix} 0 & 0 & -A_3K^{-1} & -HK^{-1} \\ 0 & 0 & -HK^{-1} & A_3K^{-1} \\ A_3 & H & 0 & 0 \\ H & -A_3 & 0 & 0 \end{pmatrix}, \\
J_{2,+} &= \begin{pmatrix} 0 & 0 & -HK^{-1} & -A_3K^{-1} \\ 0 & 0 & A_3K^{-1} & -HK^{-1} \\ H & -A_3 & 0 & 0 \\ A_3 & H & 0 & 0 \end{pmatrix}, \quad
J_{2,-} = \begin{pmatrix} 0 & 0 & HK^{-1} & -A_3K^{-1} \\ 0 & 0 & -A_3K^{-1} & -HK^{-1} \\ -H & A_3 & 0 & 0 \\ A_3 & H & 0 & 0 \end{pmatrix}, \\
J_{3,+} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad
J_{3,-} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned}
\right.
$$

$$K = H^2 + A_3^2$$

associated fundamental 2-forms

$$
\left\{
\begin{aligned}
\omega_{1,+} &= HK^{-1}\begin{pmatrix} 0 & 0 & -A_3 & H \\ 0 & 0 & -H & -A_3 \\ A_3 & H & 0 & 0 \\ -H & A_3 & 0 & 0 \end{pmatrix}, \quad
\omega_{1,-} = HK^{-1}\begin{pmatrix} 0 & 0 & A_3 & H \\ 0 & 0 & H & -A_3 \\ -A_3 & -H & 0 & 0 \\ -H & A_3 & 0 & 0 \end{pmatrix}, \\
\omega_{2,+} &= HK^{-1}\begin{pmatrix} 0 & 0 & H & A_3 \\ 0 & 0 & -A_3 & H \\ -H & A_3 & 0 & 0 \\ -A_3 & -H & 0 & 0 \end{pmatrix}, \quad
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\omega_{3,-} = \begin{pmatrix} 0 & H & 0 & 0 \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & HK^{-1} \\ 0 & 0 & -HK^{-1} & 0 \end{pmatrix}
\end{aligned}
\right.
$$

Bi-hypercomplex structure on T-fold

- We focus on the six complex structures on T-fold (the 5^2_2-brane)
- The complex structures also differ between $\theta = 0$ and $\theta = 2\pi$, similar to the metric, B-field, and dilaton

six complex structures

at $\theta = 0$:

$$
J_{1,+} = \begin{pmatrix} 0 & 0 & 0 & -H^{-1} \\ 0 & 0 & H^{-1} & 0 \\ 0 & -H & 0 & 0 \\ H & 0 & 0 & 0 \end{pmatrix}, \quad
J_{1,-} = \begin{pmatrix} 0 & 0 & 0 & -H^{-1} \\ 0 & 0 & -H^{-1} & 0 \\ 0 & H & 0 & 0 \\ H & 0 & 0 & 0 \end{pmatrix},
$$

$$
J_{2,+} = \begin{pmatrix} 0 & 0 & -H^{-1} & 0 \\ 0 & 0 & 0 & -H^{-1} \\ H & 0 & 0 & 0 \\ 0 & H & 0 & 0 \end{pmatrix}, \quad
J_{2,-} = \begin{pmatrix} 0 & 0 & H^{-1} & 0 \\ 0 & 0 & 0 & -H^{-1} \\ -H & 0 & 0 & 0 \\ 0 & H & 0 & 0 \end{pmatrix},
$$

$$
J_{3,+} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad
J_{3,-} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

$\longrightarrow$ $J^{(0)}_{a,\pm} \neq J^{(2\pi)}_{a,\pm}$ (where $a \neq 3$)

at $\theta = 2\pi$:

$$
J_{1,+} = \begin{pmatrix} 0 & 0 & -\frac{2\pi\sigma}{H^2+(2\pi\sigma)^2} & -\frac{H}{H^2+(2\pi\sigma)^2} \\ 0 & 0 & \frac{H}{H^2+(2\pi\sigma)^2} & -\frac{2\pi\sigma}{H^2+(2\pi\sigma)^2} \\ 2\pi\sigma & -H & 0 & 0 \\ H & 2\pi\sigma & 0 & 0 \end{pmatrix}, \quad
J_{1,-} = \begin{pmatrix} 0 & 0 & \frac{2\pi\sigma}{H^2+(2\pi\sigma)^2} & -\frac{H}{H^2+(2\pi\sigma)^2} \\ 0 & 0 & -\frac{H}{H^2+(2\pi\sigma)^2} & -\frac{2\pi\sigma}{H^2+(2\pi\sigma)^2} \\ -2\pi\sigma & H & 0 & 0 \\ H & 2\pi\sigma & 0 & 0 \end{pmatrix},
$$

$$
J_{2,+} = \begin{pmatrix} 0 & 0 & -\frac{H}{H^2+(2\pi\sigma)^2} & \frac{2\pi\sigma}{H^2+(2\pi\sigma)^2} \\ 0 & 0 & -\frac{2\pi\sigma}{H^2+(2\pi\sigma)^2} & -\frac{H}{H^2+(2\pi\sigma)^2} \\ H & 2\pi\sigma & 0 & 0 \\ -2\pi\sigma & H & 0 & 0 \end{pmatrix}, \quad
J_{2,-} = \begin{pmatrix} 0 & 0 & \frac{H}{H^2+(2\pi\sigma)^2} & \frac{2\pi\sigma}{H^2+(2\pi\sigma)^2} \\ 0 & 0 & \frac{2\pi\sigma}{H^2+(2\pi\sigma)^2} & -\frac{H}{H^2+(2\pi\sigma)^2} \\ -H & -2\pi\sigma & 0 & 0 \\ -2\pi\sigma & H & 0 & 0 \end{pmatrix},
$$

$$
J_{3,+} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad
J_{3,-} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

Monodromies of geometric strcs. on T-fold

- The monodromies of $J_{a,\pm}$ and $\omega_{a,\pm}$ are clearly evaluated in the doubled formalism
- $J_{a,\pm}$, $\omega_{a,\pm}$ and B are combined into an $O(D,D)$ covariant quantity called the generalized hyperkähler structure

$$\mathcal{J}_{a,\pm} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} J_{a,+} \pm J_{a,-} & -(\omega_{a,+}^{-1} \mp \omega_{a,-}^{-1}) \\ \omega_{a,+} \mp \omega_{a,-} & -(J_{a,+}^{*} \pm J_{a,-}^{*}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

- We explicitly obtain the monodromies of the bi-hypercomplex structure of the 5_2^2-brane as follows

[Kimura-Sasaki-KS, to appear]

$$\mathcal{J}_{1,+}^{(2\pi)} = \Omega_{-2\pi}\mathcal{J}_{1,+}^{(0)}\Omega_{2\pi}, \quad \mathcal{J}_{2,+}^{(2\pi)} = \Omega_{-2\pi}\mathcal{J}_{2,+}^{(0)}\Omega_{2\pi}, \quad \mathcal{J}_{3,+}^{(2\pi)} = \Omega_{-2\pi}\mathcal{J}_{3,+}^{(0)}\Omega_{2\pi},$$
$$\mathcal{J}_{1,-}^{(2\pi)} = \Omega_{-2\pi}\mathcal{J}_{1,-}^{(0)}\Omega_{2\pi}, \quad \mathcal{J}_{2,-}^{(2\pi)} = \Omega_{-2\pi}\mathcal{J}_{2,-}^{(0)}\Omega_{2\pi}, \quad \mathcal{J}_{3,-}^{(2\pi)} = \mathcal{J}_{3,-}^{(0)}.$$

$$\Omega_{2\pi} = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\pi\sigma \\ 0 & 0 & -2\pi\sigma & 0 \end{pmatrix}$$

where $\Omega_{2\pi}$ is the $O(D,D)$ matrix for the β-shift $\longrightarrow$ T-duality monodromy

Worldsheet instantons in T-fold

- We consider the worldsheet instantons on the T-fold
- Recall: the worldsheet instanton equations require a complex structure
- The complex structures of T-fold have *monodromies*, so the worldsheet instantons will be *multivalued* ⇒ **ill-defined**

$$\boxed{\mathrm{d}X^\mu \pm (J^{(0)})^\mu{}_\nu * \mathrm{d}X^\nu = 0} \neq \boxed{\mathrm{d}X^\mu \pm (J^{(2\pi)})^\mu{}_\nu * \mathrm{d}X^\nu = 0}$$

- The monodromies on T-fold appear as the T-duality $O(D,D)$ transformation
- We reformulate the worldsheet instantons in the $O(D,D)$-covariant doubled formalism

Doubled instantons in Born sigma models

Born sigma model

[Tseytlin '90][Hull '07][Copland '11][Arvanitakis-Blair '18]
[Sakatani-Uehara '20][Marotta-Szabo '22] &c.

- **Born sigma model** : string sigma model with manifest T-duality
- Target space : 2D-dim. "hypercomplex" geometry

$$S = \frac{1}{4}\int_\Sigma \Big(\underbrace{\mathcal{H}_{MN}}_{\text{generalized metric}} d\mathbb{X}^M \wedge * d\mathbb{X}^N - \underbrace{\Omega_{MN}\, d\mathbb{X}^M \wedge d\mathbb{X}^N}_{\text{topological term}} \Big)$$

$$\mathbb{X}^M = (X^\mu,\ \tilde{X}_\mu)$$

$$\Omega_{MN} = -\Omega_{NM}$$

- Structures of "hypercomplex" geometry :

generalized metric

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

neutral metric

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

fundamental two-form

$$\omega = \begin{pmatrix} 2B & -1 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{I} = \mathcal{H}^{-1}\omega \qquad \mathcal{J} = \eta^{-1}\mathcal{H} \qquad \mathcal{K} = \eta^{-1}\omega \qquad -\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = 1$$

with

generalized hyperkähler structure

$$\mathcal{J}_{a,\pm}$$

$$\mathcal{J}_{a,+}\mathcal{J}_{b,+} = -\delta_{ab}\mathbf{1}_{2D} + \varepsilon_{abc}\mathcal{J}_{c,+}, \qquad \mathcal{J}_{a,-}\mathcal{J}_{b,-} = -\delta_{ab}\mathbf{1}_{2D} + \varepsilon_{abc}\mathcal{J}_{c,+},$$

$$\mathcal{J}_{a,+}\mathcal{J}_{b,-} = \delta_{ab}\mathcal{G} + \varepsilon_{abc}\mathcal{J}_{c,-}, \qquad \mathcal{J}_{a,-}\mathcal{J}_{b,+} = \delta_{ab}\mathcal{G} + \varepsilon_{abc}\mathcal{J}_{c,-},$$

Born sigma model

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$$\Omega_{MN} = -\Omega_{NM}$$

By imposing the *chiral condition*, a **D**-dim. subspace of the **2D**-dim. target space is selected

$$d\mathbb{X}^M \pm \big(\eta^{MP}\mathcal{H}_{PN}\big) * d\mathbb{X}^N = 0$$

chiral condition

$$\mathbb{X}^M = (X^\mu,\ \cancel{\tilde{X}_\mu})$$

choosing T-dual frame

- The Born sigma model is then reduced to a *string sigma model*

$$S = \frac{1}{2}\int \Big(g_{\mu\nu}\, dX^\mu \wedge *dX^\nu + B_{\mu\nu}\, dX^\mu \wedge dX^\nu\Big)$$

Instantons in Born sigma model

[Kimura-Sasaki-KS '22]

The Bogomol'nyi completion of the Born sigma model action is as follows.

generalized (hyper)Kähler strc.

$$
\begin{aligned}
S_{\mathrm{E}} &= \frac{1}{8}\int \mathcal{H}_{MN}\Big(\underline{d\mathbb{X}^M \pm \mathcal{J}_{\pm}{}^{M}{}_{P} * d\mathbb{X}^P}\Big) \wedge *\Big(\underline{d\mathbb{X}^N \pm \mathcal{J}_{\pm}{}^{N}{}_{Q} * d\mathbb{X}^Q}\Big) \\
&\quad \pm \frac{1}{4}\int (\omega_{\pm})_{MN}\, d\mathbb{X}^M \wedge d\mathbb{X}^N \\
&\geq \pm\frac{1}{4}\int (\omega_{\pm})_{MN}\, d\mathbb{X}^M \wedge d\mathbb{X}^N
\end{aligned}
$$

The following instanton eq. is obtained as a cond. for saturating this bound.

$$
\underbrace{d\mathbb{X}^M \pm \mathcal{J}_{\pm}{}^{M}{}_{P} * d\mathbb{X}^P = 0}_{\text{doubled instanton equation}}
$$

Since the Born sigma model is an $O(D,D)$ covariant formulation, this instanton eq. is also *T-duality covariant*.

Consistency check

$$d\mathbb{X}^M \pm \mathcal{J}_{\pm}^M{}_P * d\mathbb{X}^P = 0$$

doubled instantons

$$dX^\mu \pm J^\mu{}_\nu * dX^\nu = 0$$

worldsheet instantons

$$S^{\text{Born}}_{\text{inst.}} = \frac{1}{4}\left|\int \omega_\pm\right| + \frac{i}{4}\int \Omega$$

action bound

$$S_{\text{inst.}} = \frac{1}{2}\left|\int_{C_2} \omega\right| + \frac{i}{2}\int_{C_2} B$$

action bound

imposing chiral condition

choosing a polarization

Examples of polarization :

$$\mathbb{X}^M = (X^\mu, \tilde{X}_\mu) = \begin{cases} (X^1, X^2, X^3, X^4; \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4) & : \text{NS5 frame polarization} \\ (X^1, X^2, X^3, \tilde{X}_4; \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, X^4) & : \text{TN frame polarization} \end{cases}$$

The doubled instantons include the worldsheet instantons in different T-duality frames

From doubled inst. to worldsheet inst.

Indeed, changing the polarization of the T-duality frame yields the following instanton actions from the doubled instanton action

$$S^{\text{Born}}_{\text{inst.}} = \frac{1}{4}\left|\int \omega_{\pm}\right| + \frac{i}{4}\int \Omega$$

$$\downarrow$$

TN frame X^{μ}

$$S^{\text{TN}}_{\text{inst.}} = \frac{1}{2}\left|\int_{C_2} \omega_{\mu\nu} dX^{\mu} \wedge dX^{\nu}\right| + \frac{i}{2}\int_{C_2} B_{\mu\nu} dX^{\mu} \wedge dX^{\nu}$$

$\|$

NS5 frame X'^{μ}

$$S^{\text{NS5}}_{\text{inst.}} = \frac{1}{2}\left|\int_{C_2} \omega'_{\mu\nu} dX'^{\mu} \wedge dX'^{\nu}\right| + \frac{i}{2}\int_{C_2} B'_{\mu\nu} dX'^{\mu} \wedge dX'^{\nu}$$

$\|$

T-fold frame X''^{μ}

$$S^{\text{T-fold}}_{\text{inst.}} = \frac{1}{2}\left|\int_{C_2} \omega''_{\mu\nu} dX''^{\mu} \wedge dX''^{\nu}\right| + \frac{i}{2}\int_{C_2} B''_{\mu\nu} dX''^{\mu} \wedge dX''^{\nu}$$

partition function

$$Z = \int \mathcal{D}\mathbb{X}\; e^{-S^{\text{Born}}(\mathbb{X})}$$

$$\downarrow$$

$$Z = \underbrace{\int \mathcal{D}\tilde{X}}_{\text{vol.}} \underbrace{\int \mathcal{D}X\; e^{-S^{\text{TN}}(X)}}_{\text{D-dim.}}$$

$\|$

$$Z = \underbrace{\int \mathcal{D}\tilde{X}'}_{\text{vol.}} \underbrace{\int \mathcal{D}X'\; e^{-S^{\text{NS5}}(X')}}_{\text{D-dim.}}$$

$\|$

$$Z = \underbrace{\int \mathcal{D}\tilde{X}''}_{\text{vol.}} \underbrace{\int \mathcal{D}X''\; e^{-S^{\text{T-fold}}(X'')}}_{\text{D-dim.}}$$

Doubled instantons on T-fold

- The six complex structures on the T-fold are embedded in the generalized hyperkähler structure on the doubled space

- The monodromy of the doubled instanton equation on the T-fold is as follows

$(\mathcal{J}^{(0)}_{\pm}$ is at $\theta = 0$ and $\mathcal{J}^{(2\pi)}_{\pm}$ is at $\theta = 2\pi)$

$$(\mathrm{d}\mathbb{X}^{(2\pi)})^M \pm (\mathcal{J}^{(2\pi)}_{\pm})^M{}_P * (\mathrm{d}\mathbb{X}^{(2\pi)})^P = 0$$
$$\Leftrightarrow \quad (\mathrm{d}\mathbb{X}^{(2\pi)})^M \pm (\Omega_{-2\pi})^M{}_K (\mathcal{J}^{(0)}_{\pm})^K{}_L (\Omega_{2\pi})^L{}_P * (\mathrm{d}\mathbb{X}^{(2\pi)})^P = 0$$

- The doubled instantons $\mathbb{X}^M$ have the monodromy

$$(\mathrm{d}\mathbb{X}^{(2\pi)})^M = (\Omega_{-2\pi})^M{}_N (\mathrm{d}\mathbb{X}^{(0)})^N$$

- Then the doubled instanton equation is covariant to the non-trivial T-duality monodromy, and the worldsheet instanton on the T-fold is well-defined

$$(\Omega_{-2\pi})^M{}_N \Big[\underbrace{(\mathrm{d}\mathbb{X}^{(0)})^N \pm (\mathcal{J}^{(0)}_{\pm})^N{}_Q * (\mathrm{d}\mathbb{X}^{(0)})^Q}_{\text{well-defined}} \Big] = 0$$

⟶ The worldsheet instantons on T-fold have to be treated in an $O(D,D)$ covariant doubled formalism

Summary

Summary

- We explicitly showed the non-trivial T-duality monodromies of the complex structures on the T-fold using the doubled formalism
- The doubled instantons in the Born sigma model include the worldsheet instantons in each T-duality frame
- The worldsheet instantons acting on the T-fold are well-defined by using the doubled instanton equation in the Born sigma model

Future directions

- More details on the worldsheet instanton effects in the T-fold geometry
- Finding a concrete solution to the doubled instanton equation on the T-fold
- U-duality extensions — membrane instantons

Backup

Monodromy of codim 2 branes

defect NS5-brane

$$\mathcal{H} = \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & A_3\epsilon \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H\delta & 0 & 0 \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & 0 & 0 & H^{-1}\delta \end{pmatrix} \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & -A_3\epsilon & 0 & \delta \end{pmatrix}$$

$O(D,D)$ matrix known as the B-shift (gauge symmetry)

KK-vortex

$$\mathcal{H} = \begin{pmatrix} \Lambda^{\mathsf{T}} & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} \begin{pmatrix} g_0 & 0 \\ 0 & g_0^{-1} \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-\mathsf{T}} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_3 & 1 \end{pmatrix} \qquad g_0 = \begin{pmatrix} H\delta & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H^{-1} \end{pmatrix}$$

$O(D,D)$ matrix corresponding to diffeomorphism

5^2_2-brane

$$\mathcal{H}(\theta = 2\pi) = \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & -2\pi\sigma\epsilon & 0 & \delta \end{pmatrix} \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H^{-1}\delta & 0 & 0 \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & 0 & 0 & H\delta \end{pmatrix} \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 2\pi\sigma\epsilon \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

$O(D,D)$ matrix known as the β-shift

non-geometry!

Bi-hypercomplex structure

- Let M be a $4n$-dimensional differentiable manifold
- The **bi-hypercomplex structure** on M is $(J_{a,\pm},\ \omega_{a,\pm},\ g)$ satisfying the following conditions
  - Each $J_{a,\pm}$ is an *integrable almost complex structure* on M
  - Each of $\{J_{a,+}\}$ and $\{J_{a,-}\}$ satisfies a *quaternion algebra*
  - $J_{a,+}$ and $J_{b,-}$ are *commutative*: $[J_{a,+}, J_{b,-}] = 0$
  - g is a *metric* where each $J_{a,\pm}$ is preserved
  - $\omega_{a,\pm}$ is a *fundamental 2-form* satisfying condition $\omega_{a,\pm} = -\, gJ_{a,\pm}$

T-duality b/w (J, ω) and $(J_\pm, \omega_\pm)$

T-duality transformation b/w bi-hypercomplex structures $(J_{a,\pm}, \omega_{a,\pm}) \leftrightarrow (J'_{a,\pm}, \omega'_{a,\pm})$:

$$(J'_{a,\pm})^i{}_j = (J_{a,\pm})^i{}_j - \frac{(J_{a,\pm})^i{}_y (g_{yj} \mp B_{yj})}{g_{yy}}, \qquad (J'_{a,\pm})^i{}_y = \mp \frac{(J_{a,\pm})^i{}_y}{g_{yy}},$$

$$(J'_{a,\pm})^y{}_j = \pm (\omega_{a,\pm})_{yj} + B_{yk} \left((J_{a,\pm})^k{}_j - \frac{(J_{a,\pm})^k{}_y (g_{yj} \mp B_{yj})}{g_{yy}} \right), \qquad (J'_{a,\pm})^y{}_y = \mp \frac{B_{yk} (J_{a,\pm})^k{}_y}{g_{yy}},$$

$$(\omega'_{a,\pm})_{ij} = (\omega_{a,\pm})_{ij} - \frac{(\omega_{a,\pm})_{iy} (g_{yj} \mp B_{yj}) + (g_{iy} \pm B_{iy}) (\omega_{a,\pm})_{yj}}{g_{yy}}, \qquad (\omega'_{a,\pm})_{iy} = \mp \frac{(\omega_{a,\pm})_{iy}}{g_{yy}}.$$

Born geometry

Born structure $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ on 2D-dimensional manifold $\mathcal{M}^{2D}$

para-quaternion algebra

$$\mathcal{I}^2 = -\mathcal{J}^2 = -\mathcal{K}^2 = -1 \qquad \mathcal{I}\mathcal{J}\mathcal{K} = -1$$

$$\{\mathcal{I}, \mathcal{J}\} = \{\mathcal{J}, \mathcal{K}\} = \{\mathcal{K}, \mathcal{I}\} = 0$$

$\mathcal{I}$: almost complex structure $\qquad \mathcal{I} = \mathcal{H}^{-1}\Omega = -\Omega^{-1}\mathcal{H}$

$\mathcal{J}$: chiral structure $\qquad \mathcal{J} = \eta^{-1}\mathcal{H} = \mathcal{H}^{-1}\eta$

$\mathcal{K}$: almost para-complex structure $\qquad \mathcal{K} = \eta^{-1}\Omega = \Omega^{-1}\eta$

metrics in Born geometry

$\mathcal{H}$: generalized metric

η : $O(D, D)$ invariant metric

Ω : fundamental two-form

DFT quantities