Multidimensional integrable systems from contact geometry

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Noncommutative Integrable Systems

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Integrable *nonlinear* systems from *linear* Lax pairs

A nonlinear partial differential system $\mathcal{S}$ is *(Lax) integrable* if $\mathcal{S} \Leftrightarrow [L, M] = 0$ for a pair of 'nice' linear partial differential operators L and M.

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$L\psi = 0$, $M\psi = 0$ a *Lax pair*,
and $[L, M] = 0$ a *Lax-type representation* for $\mathcal{S}$

If $[L, M] = 0$ only yields differential consequences of $\mathcal{S}$ but not $\mathcal{S}$ itself, so $\mathcal{S} \Rightarrow [L, M] = 0$ but not the other way around, we have *weak* Lax pairs or Lax-type representations.

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Under further technical assumptions an $\mathcal{S}$ admitting even a weak Lax pair has infinitely many conservation laws and symmetries and plethoras of exact solutions.

KdV equation: the prototypic integrable system

Let $n\mathrm{D}$ indicate n independent variables a.k.a. n dimensions: $2\mathrm{D}$ or $(1+1)\mathrm{D}$ for $n=2$ etc.

The $2\mathrm{D}$ Korteweg–de Vries equation for $u = u(x,t)$,

$$u_t + 6uu_x + u_{xxx} = 0, \qquad (1)$$

has a Lax-type representation $[L, M] = 0$ with

$$L = -\partial_x^2 - u - \lambda, \quad M = \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x.$$

$[L, M] = 0 \Rightarrow$ compatibility of Lax pair for $\psi(x, t, \lambda)$:

$$Q\psi = \lambda\psi, \qquad M\psi = 0, \qquad (2)$$

where $Q = -\partial_x^2 - u$ and λ is the *spectral parameter*

Nonisospectral Lax pairs: An example

Lax operators may contain derivatives w.r.t. variables not present in the associated nonlinear system

Example. The dKP eqn $(u_t + uu_x)_x + u_{yy} = 0$ is known to admit a Lax-type rep with the Lax operators

$$L = \partial_y + p\partial_x - u_x\partial_p, \quad M = \partial_t + (p^2 + u)\partial_x + (u_y - pu_x)\partial_p$$

containing derivatives w.r.t. p, so they, as well as the associated Lax pair $L\chi = 0, M\chi = 0$ for $\chi = \chi(x, y, t, p)$, are *nonisospectral* and p is the *variable spectral parameter*.

The isomonodromic representations for the Painlevé equations are apparently the first known examples of nonisospectral Lax pairs.

Integrable systems in three independent variables

Many integrable systems for $\boldsymbol{u} = \boldsymbol{u}(x, y, t)$ of general form

$$A_1(\boldsymbol{u})\boldsymbol{u}_x + A_2(\boldsymbol{u})\boldsymbol{u}_y + A_0(\boldsymbol{u})\boldsymbol{u}_t = 0$$

admit weak Lax pairs with the Lax operators of the form

$$L = \partial_y - \mathcal{X}_f, \quad M = \partial_t - \mathcal{X}_g \qquad (*)$$

where $f = f(p, \boldsymbol{u})$, $g = g(p, \boldsymbol{u})$ are the *Lax functions*; $\mathcal{X}_h = h_p\partial_x - h_x\partial_p$ formally looks like a Hamiltonian vector field in one d.o.f. with the Hamiltonian $h(p, \boldsymbol{u})$.

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Lax operators $(*)$ involve $\partial_p \Rightarrow$ are *nonisospectral*, so p is called the *variable spectral parameter* (recall that $\boldsymbol{u}_p \equiv 0$).

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Example. For $f = p^2/2 + u$, $g = -p^3 - up - v$ we get the dispersionless KP system $u_y = v_x$, $v_y = -u_t - uu_x$ (which implies the dKP equation for u from previous slide).

Integrable systems in three independent variables II

Many examples with the Lax operators

$$L = \partial_y - \mathscr{A}, \quad M = \partial_t - \mathscr{B},$$

where $\mathscr{A}$ and $\mathscr{B}$ are diff. operators of the general form

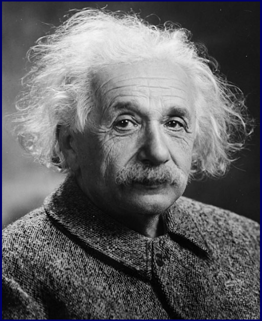
$$\mathscr{A} = \sum_{j=0}^{n} u_j \partial_x^j, \quad \mathscr{B} = \sum_{k=0}^{m} v_k \partial_x^k$$

Example. For the KP system

$$u_t + 6uu_x + u_{xxx} + 3\sigma^2 v_x = 0, \quad v_x - u_y = 0, \quad \sigma^2 = \pm 1$$

$$L = \partial_y + \sigma^{-1}\left(\partial_x^2 + u\right), \quad M = \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x - 3\sigma v$$

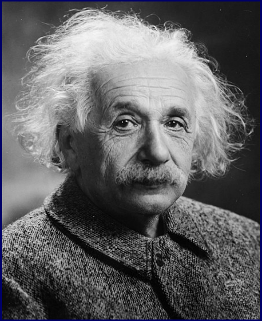
Integrable systems in four independent variables



Let nD indicate n independent variables a.k.a. n dimensions: 3D or $(2+1)$D for $n=3$ etc.

Einstein's GR $\Rightarrow$ our spacetime is 4D, so 4D partial differential systems are of particular relevance for applications

Integrable systems in four independent variables



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Einstein's GR $\Rightarrow$ our spacetime is 4D, so 4D partial differential systems are of particular relevance for applications

For a long time it appeared that, unlike 2D and 3D, integrable 4D systems are scarce, and there is no effective construction for them.

Integrable systems in 4D: What was known so far

The most important ones are (anti-)self-dual vacuum Einstein equations and (anti-)self-dual Yang–Mills equations; some other examples are related to them, e.g. the Przanowski equation or the general heavenly equation.

There also is a number of other examples, e.g. the $4\mathrm{D}$ Martínez Alonso–Shabat equation and its modified version, the Dunajski equation etc.

The overwhelming majority of the known integrable $4\mathrm{D}$ systems can be written in dispersionless form, i.e., as quasi-linear homogeneous first-order partial differential systems.

Self-dual Yang–Mills eqs on a matrix Lie group

They boil down to a single equation for the Yang matrix J:

$$(J_{y^-}J^{-1})_{y^+} + (J_{z^-}J^{-1})_{z^+} = 0,$$

and can be rewritten in dispersionless form as

$$J_{z^-}J^{-1} - W_{y^+} = 0, \quad J_{y^-}J^{-1} + W_{z^+} = 0$$

The associated Lax pair reads

$$(\partial_{y^+} + \lambda(\partial_{z^-} - A_{z^-}))\psi = 0, \ (\partial_{z^+} - \lambda(\partial_{y^-} - A_{y^-}))\psi = 0,$$

where $A_{y^-} = J_{y^-}J^{-1}$ and $A_{z^-} = J_{z^-}J^{-1}$.

How it appeared

3D effective constructions
(central extension,
Hamiltonian vec. fields)
+ sporadic examples

4D sporadic examples

Integrable systems: 3D vs 4D

	How it appeared	How it really is
3D	effective constructions (central extension, Hamiltonian vec. fields) + sporadic examples	effective constructions (central extension, Hamiltonian vec. fields) + sporadic examples
4D	sporadic examples	*effective construction* (*contact vec. fields*) + sporadic examples

New kind of Lax pairs for 4D systems

Let $L = \partial_y - X_f$ and $M = \partial_t - X_g$, where

- $f = f(p, \boldsymbol{u})$, $g = g(p, \boldsymbol{u})$ are the *Lax functions*;
- $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$ is the vector of unknown functions for the associated nonlinear system
- p is the *variable spectral parameter* ($\boldsymbol{u}_p \equiv 0$)
- $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$ formally looks exactly like the $3\mathrm{D}$ contact vector field w.r.t. $dz + pdx$ with the contact Hamiltonian h

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The Lax pair $L\chi = 0$, $M\chi = 0$ can be rewritten as

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$

where $\chi = \chi(x, y, z, t, p)$.

Infinitely many new integrable 4D systems

Theorem *For all natural* m *and* n *and all* (f,g) *given by*

i) $$f = p^{n+1} + \sum_{i=0}^{n} u_i p^i, \; g = p^{m+1} + \frac{m}{n} u_n p^m + \sum_{j=0}^{m-1} v_j p^j$$

with $\boldsymbol{u} = (u_0, \dots, u_n, v_0, \dots, v_{m-1})^{\mathrm{T}}$*, and*

ii) $$f = \sum_{i=1}^{m} \frac{a_i}{(p-u_i)}, \quad g = \sum_{j=1}^{n} \frac{b_j}{(p-v_j)}$$

with $\boldsymbol{u} = (a_1, \dots, a_m, u_1, \dots, u_m, b_1, \dots, b_n, v_1, \dots, v_n)^{\mathrm{T}}$;

Lax pairs $\chi_y = X_f(\chi)$*,* $\chi_t = X_g(\chi)$ *for* $\chi = \chi(x,y,z,t,p)$ *with* $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$ *yield* 4D *integrable systems for* $\boldsymbol{u} = \boldsymbol{u}(x,y,z,t)$ *transformable into Cauchy–Kowalevski form.*

A simple example

Let $f = p^2 + wp + u$, $g = p^3 + 2wp^2 + rp + v$,
i.e. $m = 2$, $n = 1$, $u_0 \equiv u, u_1 \equiv w, v_0 \equiv v, v_1 \equiv r$,
in class i) of the above thm.

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Let $f = p^2 + wp + u$, $g = p^3 + 2wp^2 + rp + v$,
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in class i) of the above thm.

The Lax pair $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$ then reads

$$\begin{aligned} \chi_y &= (2p + w)\chi_x + (-p^2 + u)\chi_z \\ &\quad +(w_z p^2 + (u_z - w_x)p - u_x)\chi_p, \\ \chi_t &= (r + 4wp + 3p^2)\chi_x + (v - 2wp^2 - 2p^3)\chi_z \\ &\quad +(2w_z p^3 + (r_z - 2w_x)p^2 + (v_z - r_x)p - v_x)\chi_p. \end{aligned}$$

Recap : $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$

A simple example II

For $f = p^2 + wp + u$ and $g = p^3 + 2wp^2 + rp + v$ the above Lax pair $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$ yields a system

$$\begin{aligned}
&u_t - vu_z - ru_x + uv_z + wv_x - v_y = 0,\\
&2u_z + w_x + 2ww_z - r_z = 0,\\
&2r_x - 3u_x - 2w_y + 2wu_z - v_z - 2ww_x + 2uw_z = 0,\\
&w_t - r_y + 2v_x - 4wu_x + wr_x - rw_x - vw_z + ur_z = 0.
\end{aligned} \tag{3}$$

A simple example II

For $f = p^2 + wp + u$ and $g = p^3 + 2wp^2 + rp + v$ the above Lax pair $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$ yields a system

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&w_t - r_y + 2v_x - 4wu_x + wr_x - rw_x - vw_z + ur_z = 0.
\end{aligned} \tag{3}$$

Proposition *System (3) is, up to a simple change of variables, an* integrable *generalization to the case of* four *independent variables for the well-known dKP equation*

$$(u_t + 6uu_x)_x - 3u_{yy} = 0.$$

Another example (A.J. Pan-Collantes, QTDS, to appear)

For $f = u + v/p, g = rp^2 + wp + s + q/p + cv^2/p^2$, where c is a constant the compatibility condition for the Lax pair $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$ leads to a system

$$
\begin{aligned}
u_t &= 2rv_x + vr_x + su_z - us_z + wu_x - 2vw_z + s_y,\\
v_t &= 2qu_z - uq_z + sv_z - 2vs_z + vw_x + wv_x + q_y,\\
w_y &= -2ru_x + rv_z + 2vr_z + uw_z,\\
r_y &= ru_z + ur_z,\\
q_x &= 2cvu_x + cvv_z + (q/v)v_x,\\
s_x &= (q/v)u_x - 3cvu_z - 2cv_y + (2cuv - 2q)v_z/v + 2q_z,
\end{aligned}
$$

which is a (3+1)-dimensional integrable generalization of (2+1)-dimensional dispersionless Davey–Stewartson system.

Compatibility condition for the Lax pairs

Proposition *For* $L = \partial_y - X_f$ *and* $M = \partial_t - X_g$ *the condition* $[L, M] = 0$ *holds iff*

$$f_t - g_y + \{f, g\} = 0,$$

where $\{,\}$ *is the* contact bracket

$$\{f, g\} \stackrel{\mathrm{df}}{=} f_p g_x - g_p f_x - p\,(f_p g_z - g_p f_z) + f g_z - g f_z.$$

In turn, $[L, M] = 0$ implies compatibility of the Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi)$$

Reminder: $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$

The function χ in the Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi)$$

has a straightforward interpetation: it is a joint integral of motion for the following pair of contact dynamical systems

$$\begin{aligned} dx/dy &= -f_p, & dx/dt &= -g_p, \\ dz/dy &= pf_p - f, & dz/dt &= pg_p - g, \\ dp/dy &= f_x - pf_z, & dp/dt &= g_x - pg_z, \end{aligned}$$

which are compatible if we substitute there a sufficiently smooth solution $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$ of the associated nonlinear system

Reminder: $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$

Relation to previously known 3D construction

Consider an integrable nonlinear $4\mathrm{D}$ system with a Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi) \qquad (*)$$

and impose a reduction $\boldsymbol{u}_z = 0$ and $\chi_z = 0$.

Then $(*)$ boils down to a $3\mathrm{D}$ Lax pair of a well-known type,

$$\chi_y = \mathcal{X}_f(\chi),\ \chi_t = \mathcal{X}_g(\chi),$$

where $\mathcal{X}_h = h_p\partial_x - h_x\partial_p$ formally looks like a Hamiltonian vector field with one degree of freedom (recall that $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$).

Lax functions polynomial in p

Let m and n be arbitrary natural numbers,

$\boldsymbol{u} = (u_0, \ldots, u_n, v_0, \ldots, v_{m-1})^{\mathrm{T}}$,

$$f = p^{n+1} + \sum_{i=0}^{n} u_i p^i, \quad g = p^{m+1} + \frac{m}{n} u_n p^m + \sum_{j=0}^{m-1} v_j p^j.$$

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The associated Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$

where $X_h = h_p \partial_x + (p h_z - h_x)\partial_p + (h - p h_p)\partial_z$,
yields a system shown at the next slide.

Lax functions polynomial in p: Part II

$$
\begin{aligned}
&(u_k)_t - (v_k)_y + m\,(u_{k-m-1})_z - n\,(v_{k-n-1})_z \\
&+(n+1)\,(v_{k-n})_x - (m+1)\,(u_{k-m})_x \\
&+\sum_{i=0}^{n}\Big\{(k-i-1)v_{k-i}\,(u_i)_z - (i-1)u_i\,(v_{k-i})_z \\
&-(k+1-i)v_{k+1-i}\,(u_i)_x + iu_i\,(v_{k+1-i})_x\Big\} = 0.
\end{aligned}
$$

Here $k = 0, \ldots, n+m$, $u_i \stackrel{\text{def}}{=} 0$ for $i > n$ and $i < 0$, $v_j \stackrel{\text{def}}{=} 0$ for $j > m$ and $j < 0$; $v_m \stackrel{\text{def}}{=} (m/n)u_n$.

This is an evolution system in disguise: it can be solved w.r.t. the z-derivatives $(u_i)_z$ and $(v_j)_z$ for all i and j.

Lax functions rational in p

$$\forall m, n \in \mathbb{N} \ \text{ let } \ f = \sum_{i=1}^{m} \frac{a_i}{(p - u_i)}, \quad g = \sum_{j=1}^{n} \frac{b_j}{(p - v_j)},$$

$\boldsymbol{u} = (a_1, \dots, a_m, u_1, \dots, u_m, b_1, \dots, b_n, v_1, \dots, v_n)^{\mathrm{T}}.$

The associated Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$

where, as before, $X_h = h_p \partial_x + (p h_z - h_x)\partial_p + (h - p h_p)\partial_z$, yields a system for $\boldsymbol{u}$ shown at the next slide that can be brought into Cauchy–Kowalevski form e.g. by passing from t to $T = y + t$ with all other variables intact

Lax functions rational in p: Part II

$$(u_i)_t + \sum_{j=1}^{n} \left\{ \left(\frac{b_j}{v_j - u_i} \right)_x - \left(\frac{b_j u_i}{v_j - u_i} \right)_z - \frac{2b_j (u_i)_z}{v_j - u_i} \right\} = 0, \quad i = 1, \ldots, m,$$

$$(v_j)_y + \sum_{i=1}^{m} \left\{ - \left(\frac{a_i}{v_j - u_i} \right)_x + \left(\frac{a_i v_j}{v_j - u_i} \right)_z + \frac{2a_i (v_j)_z}{v_j - u_i} \right\} = 0, \quad j = 1, \ldots, n,$$

$$(a_i)_t + \sum_{j=1}^{n} \left\{ \left(\frac{a_i b_j}{(v_j - u_i)^2} \right)_x + \left(\frac{a_i b_j (v_j - 2u_i)}{(v_j - u_i)^2} \right)_z + \frac{3a_i (b_j)_z}{v_j - u_i} + \frac{3a_i b_j (v_j)_z}{(v_j - u_i)^2} \right\} = 0, \quad i = 1, \ldots, m,$$

$$(b_j)_y + \sum_{i=1}^{m} \left\{ \left(\frac{a_i b_j}{(v_j - u_i)^2} \right)_x + \left(\frac{a_i b_j (v_j - 2u_i)}{(v_j - u_i)^2} \right)_z + \frac{3a_i (b_j)_z}{v_j - u_i} + \frac{3a_i b_j (v_j)_z}{(v_j - u_i)^2} \right\} = 0, \quad j = 1, \ldots, n.$$

Lax functions algebraic in p: an example

Let $\boldsymbol{u} = (u, v, a, b, r, s)^{\mathrm{T}}$,

$$f = \sqrt{p^2 + 2up + 2v},$$
$$g = a + bp + (r + sp)\sqrt{p^2 + 2up + 2v}.$$

The compatibility condition for the associated Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$

where $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$, yields a system shown at the next slide.

Lax functions algebraic in p: an example

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where $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$, yields a system shown at the next slide.

This is the first known example of a $4\mathrm{D}$ integrable system with a nonisospectral Lax pair whose Lax operators are algebraic in the spectral parameter p.

Lax functions algebraic in p: an example cont'd

$$
\begin{aligned}
a_y &= -sv_x + ur_x + 2vr_z, \\
b_y &= -su_x + sv_z + r_x + ur_z + us_x + 2vs_z, \\
r_y &= -2wu_x - su_y - 2uwu_z + wv_z - uw_x \\
&\quad +2(v - u^2)w_z + b_x + ub_z, \\
s_y &= wu_z + w_x + uw_z, \\
u_t &= bu_x - 4uwu_x + ru_y - 2usu_y \\
&\quad +(-4u^2w + 2vw + a)u_z + 2wv_x + sv_y \\
&\quad +2uwv_z + 2vw_x - 2u^2w_x + (-4u^3 + 6uv)w_z \\
&\quad -a_x - ua_z + ub_x + (2u^2 - 2v)b_z, \\
v_t &= -4vwu_x - 2vsu_y - 4uvwu_z + bv_x + rv_y \\
&\quad +(2vw + a)v_z - 2uvw_x + 4v(v - u^2)w_z - ua_x \\
&\quad -2va_z + 2vb_x + 2uvb_z.
\end{aligned}
$$

More details in AS, Appl. Math. Lett. 92 (2019), 196–200, arXiv:1812.02263

Open questions

? Find examples of Lax functions $f(p, \boldsymbol{u})$ and $g(p, \boldsymbol{u})$ *transcendental* in p such that the associated Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi), \qquad (*)$$

where $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$, yields a $4\mathrm{D}$ integrable system for $\boldsymbol{u}(x, y, z, t)$

Open questions

- Find examples of Lax functions $f(p, \boldsymbol{u})$ and $g(p, \boldsymbol{u})$ *transcendental* in p such that the associated Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi), \qquad (*)$$

where $X_h = h_p \partial_x + (p h_z - h_x)\partial_p + (h - p h_p)\partial_z$, yields a $4\mathrm{D}$ integrable system for $\boldsymbol{u}(x, y, z, t)$

- For a given natural N, where $\boldsymbol{u} = (u^1, \ldots, u^N)^{\mathrm{T}}$), classify all pairs of Lax functions $f = f(p, \boldsymbol{u})$ and $g = g(p, \boldsymbol{u})$ such that $(*)$ yield $4\mathrm{D}$ integrable systems

- Can we find noncommutative generalizations of integrable systems with Lax pairs $(*)$?

Summary of main results

- Far more integrable $4\mathrm{D}$ systems than it appeared before: infinitely many new ones with Lax pairs of the form
$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$
where $\chi = \chi(x, y, z, t, p)$, $f = f(p, \boldsymbol{u})$, $g = g(p, \boldsymbol{u})$, $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$, $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$

- The first known example of a $4\mathrm{D}$ integrable system with a nonisospectral Lax pair whose Lax operators are algebraic in the spectral parameter p

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どうもありがとうございます