Solutions to the SU($\mathcal{N}$) self-dual Yang-Mills equation

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Outline

Introduction

Basic knowledge of Yang-Mills theory

Yang–Mills theory is one of the most important development in quantum physics, which plays essential role in describing the interactions between elementary particles.

The establishment of Yang-Mills theory origins from the concept of gauge field, where the gauge group is non-abelian.

The definition of field strength $F_{\mu v}$

Let G be the gauge group, g the Lie algebra of G, then field strength can be determined as

$$F_{\mu v} \doteq [\mathcal{D}_v, \mathcal{D}_\mu] = \partial_v B_\mu - \partial_u B_v - [B_\mu, B_v], \tag{1}$$

where $\mathcal{D}_\mu = \partial_\mu + B_\mu$ is the covariant derivative, ∂_μ is differential operator with respect to x^μ, $(x^0, x^1, x^2, x^3) \in \mathbb{R}^4$, B_μ's are gauge potentials, $B_\mu \in g$, $[A, B] = AB - BA$, $\mu, v = 0, 1, 2, 3$ and $\mu \neq v$.

Basic knowledge of Yang-Mills theory

The equation that used to describe the motion in Yang-Mills field is called the Yang-Mills equation, which is given by

$$\partial_\mu F_{\mu\nu} + [B_\mu, F_{\mu\nu}] = [\mathcal{D}_\mu, F_{\mu\nu}] = 0. \tag{2}$$

Usually it is very hard to solve this equation, yet under the self-dual condition, it is possible to investigate this equation by means of some integrable methods.

The self-dual condition

$$F_{\mu v} = *F_{\mu\nu} \doteq \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{3}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor, $*F$ is the duality of field strength F, and μ, ν, α, β run over $\{0, 1, 2, 3\}$.

The self-dual Yang-Mills equation

For equation (3), one can verify the self-dual condition must satisfies the Yang-Mills equation. This is the reason that why we call (3) the self-dual Yang-Mills (SDYM) equation.

In this paper, we mainly focus on the SU($\mathcal{N}$) SDYM equation ($\mathcal{N} \geq 2$), i.e. $G = \mathrm{SU}(\mathcal{N})$. In 1977, Yang [1] gave a coordinate transformation:

$$y = \frac{\sqrt{2}}{2}(x^0 - \mathrm{i}x^3), \qquad \bar{y} = \frac{\sqrt{2}}{2}(x^0 + \mathrm{i}x^3), \tag{4a}$$

$$z = \frac{\sqrt{2}}{2}(x^2 - \mathrm{i}x^1), \qquad \bar{z} = \frac{\sqrt{2}}{2}(x^2 + \mathrm{i}x^1), \tag{4b}$$

where $\mathrm{i}^2 = -1$. By the new coordinates, the self-dual condition can be expanded as

$$F_{yz} = F_{\bar{y}\bar{z}} = 0, \qquad F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{5}$$

[1] C.N. Yang, Condition of self-duality for SU(2) gauge fields on euclidean four-dimensional space, Phys. Rev. Lett., 38 (1977) 1377–1379.

The self-dual Yang-Mills equation

The definition of field strength in this case still follow (1), while the gauge potentials are determined by:

$$B_y = \frac{\sqrt{2}}{2}(B_0 + \mathrm{i}B_3), \qquad B_{\bar{y}} = \frac{\sqrt{2}}{2}(B_0 - \mathrm{i}B_3), \tag{6a}$$

$$B_z = \frac{\sqrt{2}}{2}(B_2 + \mathrm{i}B_1), \qquad B_{\bar{z}} = \frac{\sqrt{2}}{2}(B_2 - \mathrm{i}B_1). \tag{6b}$$

Transformation (6) along with the definition of field strength shows

$$[\mathcal{D}_y, \mathcal{D}_z] = 0, \qquad [\mathcal{D}_{\bar{y}}, \mathcal{D}_{\bar{z}}] = 0, \tag{7}$$

which implies there exist two $\mathcal{N} \times \mathcal{N}$ invertible matrices h and $\tilde{h}$ such that

$$\mathcal{D}_y(h) = \mathcal{D}_z(h) = 0, \qquad \mathcal{D}_{\bar{y}}(\tilde{h}) = \mathcal{D}_{\bar{z}}(\tilde{h}) = 0. \tag{8}$$

The self-dual Yang-Mills equation

Then we have

$$B_y = h\partial_y(h^{-1}), \qquad B_z = h\partial_z(h^{-1}), \tag{9a}$$

$$B_{\bar{y}} = \tilde{h}\partial_{\bar{y}}(\tilde{h}^{-1}), \qquad B_{\bar{z}} = \tilde{h}\partial_{\bar{z}}(\tilde{h}^{-1}). \tag{9b}$$

In a SU($\mathcal{N}$) group, the Lie algebra condition is

$$B_\mu^\dagger = -B_\mu, \qquad \mathrm{tr}(B_\mu) = 0, \qquad \mu = 0, 1, 2, 3. \tag{10}$$

According to (6), the Lie algebra condition can be rewritten as

$$B_\chi^\dagger = -B_{\bar{\chi}}, \qquad \mathrm{tr}(B_\chi) = \mathrm{tr}(B_{\bar{\chi}}) = 0, \qquad \chi = y, z. \tag{11}$$

Thus h and $\tilde{h}$ will satisfy

$$\tilde{h} = (h^\dagger)^{-1}, \qquad |h| = 1. \tag{12}$$

The self-dual Yang-Mills equation

Let $J = h\tilde{h}^{-1} = hh^\dagger$, the second equation of (5) can be rewritten as a matrix equation, which can be regarded as an equivalent form of the self-dual condition (3).

J-formulation of the self-dual Yang-Mills equation

The J-formulation of the SDYM equation in SU($\mathcal{N}$) group is

$$(J_{\bar{y}}J^{-1})_y + (J_{\bar{z}}J^{-1})_z = 0, \tag{13}$$

or alternatively

$$(J^{-1}J_y)_{\bar{y}} + (J^{-1}J_z)_{\bar{z}} = 0, \tag{14}$$

where J is a Hermitian, determinant-one and positive-definite $\mathcal{N} \times \mathcal{N}$ matrix function.

The self-dual Yang-Mills equation

The self-dual Yang-Mills equation is an integrable system, it possesses a Lax pair.

Lax pair of the SDYM equation

The Lax pair of the self-dual Yang-Mills equation is given by

$$(\partial_y - \lambda\partial_{\bar{z}})\Phi = -A\Phi, \qquad (\partial_z + \lambda\partial_{\bar{y}})\Phi = -B\Phi, \tag{15}$$

where $A = J^{-1}J_y$ and $B = J^{-1}J_z$, λ is arbitrary complex parameter. The compatible condition of (15) is listed as below:

$$\partial_{\bar{y}}A + \partial_{\bar{z}}B = 0, \tag{16a}$$

$$\partial_z A - \partial_y B - [A, B] = 0, \tag{16b}$$

where the first equation is exactly the self-dual Yang-Mills equation, the second equation is just an identity.

The Cauchy matrix approach

The Cauchy matrix approach is an algebra method that used to construct and study integrable equations by means of the Sylvester equation.

The Sylvester equation

A general form of the Sylvester equation is usually given by

$$\boldsymbol{A}\boldsymbol{X} - \boldsymbol{X}\boldsymbol{B} = \boldsymbol{C}, \tag{17}$$

where $\boldsymbol{A} \in \mathbb{C}_{N\times N}$, $\boldsymbol{B} \in \mathbb{C}_{M\times M}$ and $\boldsymbol{C} \in \mathbb{C}_{N\times M}$ are known matrices. Sylvester [a] had proved that when $\boldsymbol{A}$ and $\boldsymbol{B}$ do not share eigenvalues, the solution $\boldsymbol{X} \in \mathbb{C}_{N\times M}$ will be unique.

[a] J. Sylvester, Sur l'equation en matrices $px = xq$, C. R. Acad. Sci. Paris, 99 (1884) 67–76.

The Cauchy matrix approach

A special case of the Sylvester equation (17) is when $\boldsymbol{A}$ and $\boldsymbol{B}$ are both diagonal matrices, where

$$\boldsymbol{A} = \mathrm{diag}(a_1, \ldots, a_N), \qquad \boldsymbol{B} = \mathrm{diag}(b_1, \ldots, b_M), \tag{18}$$

and $\boldsymbol{C}$ satisfies $\boldsymbol{C} = \boldsymbol{a}\boldsymbol{b}^T$, where

$$\boldsymbol{a} = (\alpha_1, \ldots, \alpha_N)^T, \qquad \boldsymbol{b} = (\beta_1, \ldots, \beta_M)^T. \tag{19}$$

Then $\boldsymbol{X}$ will have the form [2]:

$$\boldsymbol{X} = (X_{i,j})_{N\times M}, \qquad X_{i,j} = \frac{\alpha_i \beta_j}{a_i - b_j}, \tag{20a}$$

$$a_i \neq b_j, \qquad i = 1, \ldots, N, \qquad j = 1, \ldots, M, \tag{20b}$$

which is called the Cauchy-like matrix.

[2] https://en.wikipedia.org/wiki/Cauchy_matrix

The Cauchy matrix approach

The Cauchy matrix approach was first proposed by F.W. Nijhoff et al. to investigate integrable quadrilateral equations [3]. By now, this method has been developed to study many different cases.

- To obtain the multi-pole solution [4].
- Continuous integrable systems [5].
- The 2×2 Ablowitz-Kaup-Newell-Segur system [6].

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[3] F.W. Nijhoff, J. Atkinson, J. Hietarinta, Soliton solutions for ABS lattice equations: I. Cauchy matrix approach, J Phys A: Math Theor. 42 (2009) 404005 (34pp).

[4] D.J. Zhang, S.L. Zhao, Solutions to ABS lattice equations via generalized Cauchy matrix approach, Stud. Appl. Math., 131 (2013) 72-103.

[5] D.D. Xu, D.J. Zhang, S.L. Zhao, The Sylvester equation and integrable equations: I. The Korteweg-de Vries system and sine-Gordon equation, J. Nonl. Math. Phys., 21 (2014) 382-406.

[6] S.L. Zhao, The Sylvester equation and integrable equations: The Ablowitz-Kaup-Newell-Segur system, Rep. Math. Phys., 82 (2018) 241-263.

The Cauchy matrix approach

In one of our recent work, we have built up the connection between the 2×2 AKNS system with SU(2) SDYM equation, where they share the same Cauchy matrix scheme.

Lemma 1

Suppose there exist $\boldsymbol{S}^{(i,j)} \in \mathbb{C}_{\mathcal{N}\times\mathcal{N}}[\mathbf{x}]$ that satisfy the evolution.

$$\boldsymbol{S}^{(i,j)}_{x_n} = \boldsymbol{S}^{(i+n,j)}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{S}^{(i,j+n)} - \sum_{l=0}^{n-1} \boldsymbol{S}^{(n-1-l,j)}\boldsymbol{a}\boldsymbol{S}^{(i,l)}, \qquad (n \in \mathbb{Z}^+), \tag{21a}$$

$$\boldsymbol{S}^{(i,j)}_{x_0} = \boldsymbol{S}^{(i,j)}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{S}^{(i,j)} = [\boldsymbol{S}^{(i,j)}, \boldsymbol{a}], \tag{21b}$$

$$\boldsymbol{S}^{(i,j)}_{x_n} = \boldsymbol{S}^{(i+n,j)}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{S}^{(i,j+n)} + \sum_{l=-1}^{n} \boldsymbol{S}^{(n-1-l,j)}\boldsymbol{a}\boldsymbol{S}^{(i,l)}, \qquad (n \in \mathbb{Z}^-), \tag{21c}$$

The Cauchy matrix approach

Lemma 1

and a difference relation

$$\boldsymbol{S}^{(i+1,j)} - \boldsymbol{S}^{(i,j+1)} = \boldsymbol{S}^{(0,j)}\boldsymbol{S}^{(i,0)}, \tag{22}$$

where $i, j \in \mathbb{Z}$, $\mathbf{x} = (\cdots, x_{-1}, x_0, x_1, \cdots)$, $\boldsymbol{a} = \mathrm{diag}(a^{(1)}, \cdots, a^{(\mathcal{N})})$ with $a^{(i)} \in \mathbb{C}$, and $[A, B] = AB - BA$.

By defining $\boldsymbol{U} \doteq \boldsymbol{S}^{(0,0)}$ and $\boldsymbol{V} = \boldsymbol{I}_{\mathcal{N}} - \boldsymbol{S}^{(-1,0)}$, there will be a differential recurrence relation

$$\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1} = -\boldsymbol{U}_{x_n}, \quad (n \in \mathbb{Z}), \tag{23}$$

and consequently it leads to the unreduced SDYM equation:

$$(\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1})_{x_m} - (\boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1})_{x_n} = 0, \quad (n, m \in \mathbb{Z}). \tag{24}$$

Cauchy matrix structure for the SDYM equation

Cauchy matrix structure for the SDYM equation

- The Sylvester equation:

$$\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L} = \boldsymbol{r}\boldsymbol{s}^T, \tag{25}$$

where $\boldsymbol{M} \in \mathbb{C}_{N\times N}[\mathbf{x}]$, $\boldsymbol{K}, \boldsymbol{L} \in \mathbb{C}_{N\times N}$, $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{C}_{N\times \mathcal{N}}[\mathbf{x}]$ [7].

- The dispersion relations:

$$\boldsymbol{r}_{x_n} = \boldsymbol{K}^n \boldsymbol{r}\boldsymbol{a}, \quad \boldsymbol{s}_{x_n} = -(\boldsymbol{L}^T)^n \boldsymbol{s}\boldsymbol{a}, \quad (n \in \mathbb{Z}), \tag{26}$$

where $\boldsymbol{a}$ is the $\mathcal{N}$-th order diagonal matrix defined as in Lemma 1.

- The master function:

$$\boldsymbol{S}^{(i,j)} \doteq \boldsymbol{s}^T \boldsymbol{L}^j (\boldsymbol{C} + \boldsymbol{M})^{-1} \boldsymbol{K}^i \boldsymbol{r}, \quad i, j \in \mathbb{Z}. \tag{27}$$

where $\boldsymbol{C}$ is an arbitrary $N \times N$ complex matrix independent of $\mathbf{x}$.

[7] One should be aware of the difference between N and $\mathcal{N}$; Here N indicates the number of solitons

Cauchy matrix structure for the SDYM equation

In our research, we have found the construction of (27) will always satisfy the derivative relations (21). To make $\boldsymbol{S}^{(i,j)}$ meet the difference relation, the following condition holds for $\boldsymbol{K}, \boldsymbol{L}$ and $\boldsymbol{C}$:

$$\boldsymbol{K}\boldsymbol{C} - \boldsymbol{C}\boldsymbol{L} = \mathbf{0}, \tag{28}$$

which indicates two different cases:

- When $\boldsymbol{K}$ and $\boldsymbol{L}$ do not share eigenvalues, both (25) and (28) have a unique solution, and in particular, $\boldsymbol{C} = \mathbf{0}$.
- When $\boldsymbol{K} = \boldsymbol{L}$, condition (28) holds if $\boldsymbol{K}$ and $\boldsymbol{C}$ commute. For convenience, we can take $\boldsymbol{C} = \boldsymbol{I}$.

The first case leads to the asymmetric Sylvester equation (the matrix KP hierarchy) and the second one can is related to the symmetric Sylvester equation (the matrix $2\mathcal{M} \times 2\mathcal{M}$ AKNS hierarchy).

Cauchy matrix structure for the SDYM equation

According to Lemma 1, the explicit formula of the solution is

$$\boldsymbol{V} = \boldsymbol{I}_{\mathcal{N}} - \boldsymbol{s}^T(\boldsymbol{C}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}. \tag{29}$$

Recalling the Weinstein–Aronszajn formula

$$|\boldsymbol{I}_{\mathcal{N}} + \boldsymbol{B}\boldsymbol{A}| = |\boldsymbol{I}_N + \boldsymbol{A}\boldsymbol{B}|, \tag{30}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $N \times \mathcal{N}$ and $\mathcal{N} \times N$ matrices. Then we have

$$\begin{aligned}
|\boldsymbol{V}| &= |\boldsymbol{I}_{\mathcal{N}} - \boldsymbol{s}^T(\boldsymbol{C}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}| = |\boldsymbol{I}_N - (\boldsymbol{C}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{s}^T| \\
&= |(\boldsymbol{C}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}(\boldsymbol{C}+\boldsymbol{M})\boldsymbol{L} + (\boldsymbol{C}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}(\boldsymbol{K}\boldsymbol{C}-\boldsymbol{C}\boldsymbol{L})| \\
&= |(\boldsymbol{C}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}(\boldsymbol{C}+\boldsymbol{M})\boldsymbol{L}| = |\boldsymbol{L}|/|\boldsymbol{K}|. \qquad (31)
\end{aligned}$$

Especially when $\boldsymbol{K} = \boldsymbol{L}$, we have $|\boldsymbol{V}| = 1$.

Cauchy matrix structure for the SDYM equation

The unreduced SDYM equation:

$$(\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1})_{x_m} - (\boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1})_{x_n} = 0, \qquad (n, m \in \mathbb{Z}).$$

The coordinate condition of the SU($\mathcal{N}$) SDYM equation in Euclidean space is

$$x_{n+1} = \bar{x}_m, \qquad x_{m+1} = -\bar{x}_n. \tag{32}$$

Hence one can take $m = -n-1$ and define

$$z_n = x_n = \xi_n + \mathrm{i}\eta_n, \qquad \bar{z}_n = (-1)^{n+1}x_{-n} = \xi_n - \mathrm{i}\eta_n, \qquad n = 1, 2, \cdots,$$

where $\xi_n, \eta_n \in \mathbb{R}$. The unreduced SDYM equation reduces to

$$(\boldsymbol{V}_{z_{n+1}}\boldsymbol{V}^{-1})_{\bar{z}_{n+1}} + (\boldsymbol{V}_{\bar{z}_n}\boldsymbol{V}^{-1})_{z_n} = 0, \qquad (n \in \mathbb{Z}). \tag{33}$$

Asymmetric Sylvester equation formulation

For the asymmetric case, one can expand (26) in terms of

$$\boldsymbol{r} = (\boldsymbol{r}^{(1)}, \boldsymbol{r}^{(2)}, \cdots, \boldsymbol{r}^{(\mathcal{N})}), \quad \boldsymbol{s} = (\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)}, \cdots, \boldsymbol{s}^{(\mathcal{N})}), \tag{34}$$

which leads to

$$\boldsymbol{r}^{(j)}_{x_n} = a^{(j)} \boldsymbol{K}^n \boldsymbol{r}^{(j)}, \quad \boldsymbol{s}^{(j)}_{x_n} = -a^{(j)} \boldsymbol{L}^n \boldsymbol{s}^{(j)}, \quad j = 1, 2, \cdots, \mathcal{N}. \tag{35}$$

This relation shows a matrix exponential representation of $\boldsymbol{r}^{(j)}$ and $\boldsymbol{s}^{(j)}$:

$$\boldsymbol{r}^{(j)} = \exp\left(a^{(j)} \sum_{n\in\mathbb{Z}} \boldsymbol{K}^n x_n\right) \boldsymbol{r}^{(0)}, \quad \boldsymbol{s}^{(j)} = \exp\left(-a^{(j)} \sum_{n\in\mathbb{Z}} \boldsymbol{L}^n x_n\right) \boldsymbol{s}^{(0)}, \tag{36}$$

where $\boldsymbol{r}^{(0)}$ and $\boldsymbol{s}^{(0)}$ are initial values that independent of x_n.

Asymmetric Sylvester equation formulation

We introduce a Lemma to solve out the Sylvester equation with rank-$\mathcal{N}$ condition. It was first announced in [[8]].

Lemma 2

The rank-$\mathcal{N}$ Sylvester equation can be expanded as

$$\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L} = \boldsymbol{r}^{(1)}(\boldsymbol{s}^{(1)})^T + \boldsymbol{r}^{(2)}(\boldsymbol{s}^{(2)})^T + \cdots + \boldsymbol{r}^{(\mathcal{N})}(\boldsymbol{s}^{(\mathcal{N})})^T. \tag{37}$$

Now we consider such a Sylvester equation, where

$$\boldsymbol{K}\boldsymbol{M}^{(j)} - \boldsymbol{M}^{(j)}\boldsymbol{L} = \boldsymbol{r}^{(j)}(\boldsymbol{s}^{(j)})^T. \tag{38}$$

Suppose $\boldsymbol{K} = \mathrm{diag}(k_1, \cdots, k_N)$, $\boldsymbol{L} = \mathrm{diag}(l_1, \cdots, l_N)$ and

$$\boldsymbol{r}^{(j)} = (\rho^{(j)}(k_1), \cdots, \rho^{(j)}(k_N))^T, \quad \boldsymbol{s}^{(j)} = (\sigma^{(j)}(l_1), \cdots, \sigma^{(j)}(l_N))^T.$$

[8] Z.H. Shi, S.S. Li, D.J. Zhang, Cauchy matrix approach to the noncommutative Kadomtsev-Petviashvili equation with self-consistent sources, Theor. Math. Phys., 213 (2022) 1686-1697.

Asymmetric Sylvester equation formulation

Lemma 2

Then we can write down the explicit formula of $\boldsymbol{M}^{(j)}$ as follows:

$$\boldsymbol{M}^{(j)} = \left(M^{(j)}_{m,n}\right)_{N\times N}, \quad M^{(j)} = \frac{\rho^{(j)}(k_m)\sigma^{(j)}(l_n)}{k_m - l_n}, \quad m,n = 1,\cdots,N.$$

It is easy to see that

$$\boldsymbol{M} = \boldsymbol{M}^{(1)} + \boldsymbol{M}^{(2)} + \cdots + \boldsymbol{M}^{(\mathcal{N})}, \tag{39}$$

will be a solution of (37) and it is also unique.

Hence the solution of the SDYM equation and its determinant will be constructed as

$$\boldsymbol{V} = \boldsymbol{I}_{\mathcal{N}} - \boldsymbol{s}^T\boldsymbol{M}^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}, \qquad |\boldsymbol{V}| = \frac{|\boldsymbol{L}|}{|\boldsymbol{K}|} = \prod_{i=1}^{N}\frac{l_i}{k_i}. \tag{40}$$

Asymmetric Sylvester equation formulation

By applying the conjugate reduction

$$\boldsymbol{L}^T = -(\bar{\boldsymbol{K}})^{-1}, \quad \bar{\boldsymbol{a}} = \boldsymbol{a}, \quad \boldsymbol{s} = \bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{r}}, \tag{41}$$

the Sylvester equation reduces to

$$\boldsymbol{K}\boldsymbol{M}\boldsymbol{K}^\dagger + \boldsymbol{M} = \boldsymbol{r}\boldsymbol{r}^\dagger, \tag{42}$$

which indicates $\boldsymbol{K}\boldsymbol{M}^\dagger\boldsymbol{K}^\dagger + \boldsymbol{M}^\dagger = \boldsymbol{r}\boldsymbol{r}^\dagger$ and hence $\boldsymbol{M} = \boldsymbol{M}^\dagger$. Then the solution and determinant (40) becomes

$$\boldsymbol{V} = \boldsymbol{I}_{\mathcal{N}} - \boldsymbol{r}^\dagger(\boldsymbol{K}^\dagger)^{-1}\boldsymbol{M}^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}, \quad |\boldsymbol{V}| = (-1)^N \prod_{i=1}^{N} \frac{1}{|k_i|^2}. \tag{43}$$

So $\boldsymbol{V}$ is a Hermitian matrix and the determinant will be a constant. By a scale transformation $\boldsymbol{V}' = (\sqrt[\mathcal{N}]{|\boldsymbol{V}|})^{-1}\boldsymbol{V}$, we can have a determinant-one solution.

Symmetric Sylvester equation formulation

In this case, we have $\boldsymbol{K} = \boldsymbol{L}$ and $\boldsymbol{C} = \boldsymbol{I}$, hence one can take

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_1 & \\ & \boldsymbol{K}_2 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} & \boldsymbol{M}_1 \\ \boldsymbol{M}_2 & \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} \boldsymbol{a}_1 & \\ & \boldsymbol{a}_2 \end{pmatrix}, \tag{44}$$

where $\boldsymbol{K}_1, \boldsymbol{K}_2, \in \mathbb{C}_{M\times M}$, $\boldsymbol{M}_1, \boldsymbol{M}_2 \in \mathbb{C}_{M\times M}[\mathbf{x}]$ and $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{C}_{\mathcal{M}\times\mathcal{M}}$, $N = 2M$, $\mathcal{N} = 2\mathcal{M}$, and $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbb{C}_{M\times\mathcal{M}}[\mathbf{x}]$.

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}_1 & \\ & \boldsymbol{r}_2 \end{pmatrix}, \qquad \boldsymbol{s} = \begin{pmatrix} & \boldsymbol{s}_1 \\ \boldsymbol{s}_2 & \end{pmatrix}. \tag{45}$$

Then the Sylvester equation $\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{K} = \boldsymbol{r}\boldsymbol{s}^T$ reduces to

$$\boldsymbol{K}_1\boldsymbol{M}_1 - \boldsymbol{M}_1\boldsymbol{K}_2 = \boldsymbol{r}_1\boldsymbol{s}_2^T, \tag{46a}$$

$$\boldsymbol{K}_2\boldsymbol{M}_2 - \boldsymbol{M}_2\boldsymbol{K}_1 = \boldsymbol{r}_2\boldsymbol{s}_1^T. \tag{46b}$$

The dispersion relations reduce to

$$\boldsymbol{r}_{1,x_n} = \boldsymbol{K}_1^n\boldsymbol{r}_1\boldsymbol{a}_1, \qquad \boldsymbol{s}_{2,x_n} = -\boldsymbol{K}_2^n\boldsymbol{s}_2\boldsymbol{a}_1, \tag{47a}$$

$$\boldsymbol{r}_{2,x_n} = \boldsymbol{K}_2^n\boldsymbol{r}_2\boldsymbol{a}_2, \qquad \boldsymbol{s}_{1,x_n} = -\boldsymbol{K}_1^n\boldsymbol{s}_1\boldsymbol{a}_2. \tag{47b}$$

Symmetric Sylvester equation formulation

Let $\boldsymbol{a}_1 = \mathrm{diag}(a_1^{(1)}, \ldots, a_1^{(\mathcal{M})})$, $\boldsymbol{a}_2 = \mathrm{diag}(a_2^{(1)}, \ldots, a_2^{(\mathcal{M})})$, similar like the asymmetric case, the vectors will have a representation of matrix exponential function:

$$\boldsymbol{r}_1^{(j)} = \exp\left(a_1^{(j)} \sum_{n\in\mathbb{Z}} \boldsymbol{K}_1^n x_n\right) \boldsymbol{r}_1^{(0)}, \quad \boldsymbol{s}_2^{(j)} = \exp\left(-a_1^{(j)} \sum_{n\in\mathbb{Z}} \boldsymbol{K}_2^n x_n\right) \boldsymbol{s}_1^{(0)}, \tag{48a}$$

$$\boldsymbol{r}_2^{(j)} = \exp\left(a_2^{(j)} \sum_{n\in\mathbb{Z}} \boldsymbol{K}_2^n x_n\right) \boldsymbol{r}_2^{(0)}, \quad \boldsymbol{s}_1^{(j)} = \exp\left(-a_2^{(j)} \sum_{n\in\mathbb{Z}} \boldsymbol{K}_1^n x_n\right) \boldsymbol{s}_2^{(0)}, \tag{48b}$$

where $\boldsymbol{r}_1^{(0)}, \boldsymbol{r}_2^{(0)}, \boldsymbol{s}_1^{(0)}, \boldsymbol{s}_2^{(0)}$ are constant vectors, $j = 1, \ldots, \mathcal{M}$.

Symmetric Sylvester equation formulation

The expression of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ can be constructed according to Lemma 2, so we skip it in this part. Then the solution of the SDYM equation and its determinant will be

$$\boldsymbol{V} = \boldsymbol{I}_{2\mathcal{M}} - \boldsymbol{s}^T(\boldsymbol{I}_{2M} + \boldsymbol{M})^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}, \quad |\boldsymbol{V}| = 1. \tag{49}$$

We can take the following conjugate reduction

$$\boldsymbol{K}_2 = -\bar{\boldsymbol{K}}_1^{-1}, \;\; \boldsymbol{a}_2 = -\bar{\boldsymbol{a}}_1, \;\; \boldsymbol{r}_2 = \bar{\boldsymbol{K}}_1^{-1}\bar{\boldsymbol{r}}_1, \;\; \boldsymbol{s}_2 = -(\boldsymbol{K}_1^{\dagger})^{-1}\bar{\boldsymbol{s}}_1, \tag{50}$$

so that $\boldsymbol{V}$ will be a Hermitian matrix (Explaining it will take lots of time, one can see section 4.2 in our paper [9] for details).

[9] S.S. Li, C.Z. Qu, D.J. Zhang, Solutions to the SU(N) self-dual Yang-Mills equation, Physica D, 453 (2023) 133828 (17pp).

Conclusion and future investigations

Conclusion and future investigations

- In this paper, we have constructed the explicit solution to the self-dual Yang-Mills equation and make it satisfy the SU(N) gauge condition, i.e., J is a Hermitian and determinant-one matrix. (Those who are interested in them can check section 3.3.1, 3.3.2, 4.3.1 and 4.3.2 in our paper for explicit formulae and the singularity of the solutions).
- The symmetric Sylvester equation $\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{K} = \boldsymbol{r}\boldsymbol{s}^T$ and its expansion:

$$\boldsymbol{K}_1\boldsymbol{M}_1 - \boldsymbol{M}_1\boldsymbol{K}_2 = \boldsymbol{r}_1\boldsymbol{s}_2^T,$$
$$\boldsymbol{K}_2\boldsymbol{M}_2 - \boldsymbol{M}_2\boldsymbol{K}_1 = \boldsymbol{r}_2\boldsymbol{s}_1^T.$$

  This type of the Sylvester equation corresponds to the AKNS system, by adding coordinates x_n we can extend it to investigate the AKNS hierarchy, then use it to derive the SDYM equation.

Conclusion and future investigations

- The asymmetric Sylvester equation $\boldsymbol{KM} - \boldsymbol{ML} = \boldsymbol{rs}^T$ is usually applied to investigate the KP-type equations. One can check [[10]] for a matrix KP equation with self-consistent sources, where we have to use the asymmetric Sylvester equation. In this paper, we also use the asymmetric Sylvester equation to construct the Cauchy matrix structure for the SDYM equation.

- We have checked by direct calculations that the one-soliton and two-soliton solutions have singularity in symmetric Sylvester equation case. For the asymmetric Sylvester equation, the one-soliton solution is non-singular, but it is not positive-definite, the two-soliton solution also has singularity.

[10] Z.H. Shi, S.S. Li, D.J. Zhang, Cauchy matrix approach to the noncommutative Kadomtsev-Petviashvili equation with self-consistent sources, Theor. Math. Phys., 213 (2022) 1686-1697.

Conclusion and future investigations

- In this paper, we only consider the SDYM equation in Euclidean space:
$$(J_{\bar{y}}J^{-1})_y + (J_{\bar{z}}J^{-1})_z = 0. \tag{51}$$
There are other SDYM equations in different spaces. In Minkowski space, the equation is
$$(J_{\bar{y}}J^{-1})_y - (J_{\tilde{z}}J^{-1})_z = 0, \qquad \tilde{z}, z \in \mathbb{R}. \tag{52}$$
In ultrahyperbolic space, there will be two equations:
$$(J_{\bar{y}}J^{-1})_y - (J_{\bar{z}}J^{-1})_z = 0, \tag{53a}$$
$$(J_{\tilde{y}}J^{-1})_y - (J_{\tilde{z}}J^{-1})_z = 0, \qquad \tilde{y}, y, \tilde{z}, z \in \mathbb{R}. \tag{53b}$$
In the future, we may use the Cauchy matrix approach to investigate them.

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Thanks for Listening!