Qusideterminants. Lecture 3

Various applications

Orthogonal polynomials

Recall that if $\mu(t)$ is a non-decreasing function on the real numbers. If $\int f(t)d\mu(t)$ is finite for any polynomial $f(t)$, one can define an inner product on pairs of polynomials

$$\langle f, g\rangle = \int f(t)g(t)d\mu(t)$$

The sequence of orthogonal polynomials $(P_n)_{n\geq 0}$ is defined by the relations

$\deg P_n = n$ and $\langle P_n, P_m\rangle = 0$ if $m \neq n$.

Such polynomials can be constructed via determinants of matrices of moments

$c_n = \int t^n d\mu(t), \quad n \geq 0.$

Noncommutative generalization.

Let $C_0, C_1, \ldots$ be elements of a ring R. Define orthogonal polynomials $P_n(t) \in R[t]$ as

$$P_n(t) = \begin{vmatrix} C_0 & C_1 & \ldots & C_{n-1} & 1 \\ C_1 & C_2 & \ldots & C_n & t \\ & & \ldots & & \\ C_n & C_{n+1} & \ldots & C_{2n-1} & \boxed{t^n} \end{vmatrix} .$$

In this definition elements C_i play a role of abstract (noncommutative) moments.

Polynomials $P_n(t)$ are polynomilas of degree n.

They are orthogonal in the following sense.

Let $r \mapsto \overline{r}$ be an anti-involution on R such that $\overline{C}_i = C_i$.

Define the scalar product on $R[t]$ by formula

$$\langle at^i, bt^j \rangle = a \cdot C_{i+j} \cdot \overline{b}$$

Then

$$\langle t^i, P_n(t) \rangle = \langle P_n(t), t^i \rangle = 0 \text{ for } i \leq n-1$$

Therefore,

$$\langle P_m(t), P_n(t)\rangle = 0 \text{ for } m \neq n$$

Recurrence relations

Let $H(n)$ be the Hankel matrix with the first row $C_0, \dots, C_n$. Let q_k be the South-East quasideterminant of $H(k)$ and let p_k be the South-East quasideterminant of the submatrix of $H(k+1)$ with k-th row and $(k+1)$-th column removed.

Set $a_n = p_n q_n^{-1} - p_{n-1} q_{n-1}^{-1}$, $b_n = p_n p_{n-2}^{-1}$

Then $P_{n+1}(t) = (t - a_n)P_n(t) - b_n P_{n-1}(t)$.

This is the classical recurrence relations for orthogonal polynomials if R is a field.

Iterated Darboux transformations.

Let R be an algebra with a derivation $D : R \to R$ and $\phi \in R$ be an invertible element. Recall that we denote $D(g) = g'$ and $D^k(g) = g^{(k)}$.

Define Darboux transformation of $f \in R$ as

$$\mathcal{D}(\phi; f) = f' - \phi'\phi^{-1} f = \begin{vmatrix} f & \phi \\ \boxed{f'} & \phi' \end{vmatrix}$$

Define inductively the iterated Darboux transformation $\mathcal{D}(\phi_k, \ldots, \phi_1; f)$ inductively by formula

$$\mathcal{D}(\mathcal{D}(\phi_k, \ldots, \phi_2; f); \mathcal{D}(\phi_1; f))$$

(provided all appropriate expressions are defined and invertible).

In this case

$$\mathcal{D}(\phi_k, \ldots, \phi_1; f) = \begin{vmatrix} f & \phi_1 & \ldots & \phi_k \\ f' & \phi_1' & \ldots & \phi_k' \\ \ldots & \ldots & \ldots & \ldots \\ \boxed{f^{(k)}} & \phi_1^{(k)} & \ldots & \phi_k^{(k)} \end{vmatrix}$$

In commutative case, the iterated Darboux transformation is a ratio of two Wronskians:

$$\mathcal{D}(\phi_k, \ldots, \phi_1; f) = \frac{W(\phi_k, \ldots, \phi_1, f)}{W(\phi_k, \ldots, \phi_1)}$$

Noncommutative Toda lattice.

In the previous notations set

$$\phi_n = \begin{vmatrix} \phi & D\phi & \dots & D^{n-1}\phi \\ D\phi & D^2\phi & \dots & D^n\phi \\ \dots & \dots & \dots & \dots \\ D^{n-1}\phi & D^n\phi & \dots & \boxed{D^{2n-2}\phi} \end{vmatrix}$$

Elements ϕ_n satisfy the following system of equations:

$$D((D\phi_1)\phi_1^{-1}) = \phi_2\phi_1^{-1}$$

$$D((D\phi_n)\phi_n^{-1}) = \phi_{n+1}\phi_n^{-1} - \phi_n\phi_{n-1}^{-1}, \ n \geq 2$$

Determinants and cyclic vectors

Let R be a unital algebra and $A : R^m \to R^m$ be a linear map of right vector spaces.

A vector $v \in R^m$ is an A-*cyclic vector* if $v, Av, \ldots, A^{m-1}v$ form a basis in R^m regarded as a right R-module. In this case there exist $\Lambda_i(v, A) \in R$, $i = 1, \ldots, m$ such that

$$(-1)^m v \cdot \Lambda_m(v, A) + (-1)^{m-1}(Av) \cdot \Lambda_{m-1}(v, A) +$$

$$+ \cdots - (A^{m-1}v) \cdot \Lambda_1(v, A) + A^m v = 0$$

We call $\Lambda_m(v, A)$ the *determinant* of (v, A) and $\Lambda_1(v, A)$ the *trace* of (v, A).

When R is commutative $\Lambda_m(v, A)$ is the determinant of A and $\Lambda_1(v, A)$ is the trace of A.

When R is noncommutative, the expressions $\Lambda_i(v, A) \in R$ depend on vector v. However, they provide some information about A. For example, if the determinant $\Lambda(v, A) = 0$ for a cyclic vector v, then the map A is not invertible.

Example: Computation of $\Lambda_m(v, A)$. Let $A = (a_{ij})$ be $m \times m$-matrix and $v = e_1 = (1, 0, \ldots, 0)^t$. Denote by $a_{ij}^{(k)}$ the corresponding entry of A^k. Then

$$\Lambda_m(v, A) = (-1)^{m-1} \begin{vmatrix} \boxed{a_{11}^{(m)}} & a_{12}^{(m)} & \ldots & a_{1m}^{(m)} \\ a_{11}^{(m-1)} & a_{12}^{(m-1)} & \ldots & a_{1m}^{(m-1)} \\ & \ldots & \ldots & \\ a_{11} & a_{12} & \ldots & a_{1m} \end{vmatrix}$$

For $m = 2$ "noncommutative trace":

$\Lambda_1(e_1, A) = a_{11} + a_{12}a_{22}a_{12}^{-1}$

and "noncommutative determinant":

$\Lambda_2(e_1, A) = a_{12}a_{22}a_{12}^{-1}a_{11} - a_{12}a_{21}$

When A is a quantum matrix then $\Lambda_m(e_i, A)$ for every i equals, up to a power of q, to quantum determinant $\det_q(A)$.

Quasideterminants and characteristic functions of graphs.

Let $A = (a_{ij}), 1 \leq i, j \leq n$ where a_{ij} are free variables. Fix $p, q \in \{1, \dots, n\}$ and $J \subset \{1, \dots, \hat{p}, \dots, n\} \times \{1, \dots, \hat{q}, \dots, n\}$ such that $|J| = n - 1$ and projections of J onto $\{1, \dots, \hat{p}, \dots, n\}$ and $\{1, \dots, \hat{q}, \dots, n\}$ are surjective.

Introduce new variables $b_{k\ell}$ by setting

$$b_{k\ell} = a_{k\ell} \text{ for } (k, \ell) \notin J$$
$$b_{k\ell} = a_{k\ell}^{-1} \text{ for } (k, \ell) \in J$$

Quasideterminant $|A|_{ij}$ is defined in the ring of formal series in variables $b_{k\ell}$ and is given by the formula

$$|A|_{ij} = b_{ij} - \sum (-1)^s b_{ii_1} b_{i_1 i_2} \dots b_{i_s j}$$

the sum is taken over all sequences $i_1, \dots, i_s$ such that $i_k \neq i, j$ for all k.

The inverse to $|A|_{ij}$ is given by the same formula where the sum is taken over all sequences $i_1, \dots, i_s$.

All relations between quasideterminants, including the noncommutative Sylvester identity,, can be deduced from above formulas.

The above formal series can be interpreted in terms of graphs. Let Γ_n be a complete oriented graph with vertices $1, \dots, n$ and edges $e_{k\ell}$. Introduce a bijective correspondence between edges of the graph and elements $b_{k\ell}$ by formula $e_{k\ell} \mapsto b_{k\ell}$.

Then there exist a bijective correspondence between the monomials $b_{ii_1} b_{i_1 i_2} \dots b_{i_s j}$ and paths from the vertex i to the vertex j.

Noncommutative Catalan numbers
(joint with A. Berenstein)

Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$, $n \geq 0$ are important combinatorial objects which satisfy a number of remarkable properties such as:

• recursion $c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$ for all $n \geq 0$ (with $c_0 = c_1 = 1$).

• determinantal identities

$$\begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ & & \cdots & \\ c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n} \end{vmatrix} = 1$$

for $n \geq 0$ $m \in \{0, 1\}$.

Introduce formal variables x_k, $k \geq 0$ and define *noncommutative Catalan numbers* as solutions of the quasideterminant equations

$$\begin{vmatrix} C_m & C_{m+1} & \dots & C_{m+n} \\ C_{m+1} & C_{m+2} & \dots & C_{m+n+1} \\ & & \dots & \\ C_{m+n} & C_{m+n+1} & \dots & \boxed{C_{m+2n}} \end{vmatrix} = 1$$

for $n \geq 0$ $m \in \{0, 1\}$.

It turns out that solutions of the equations are Laurent polynomials:

$$C_0 = x_0,\ C_1 = x_1,\ C_2 = x_2 + x_1 x_0^{-1} x_1,$$

$$C_3 = x_3 + x_2 x_1^{-1} x_2 +$$

$$+ x_2 x_0^{-1} x_1 + x_1 x_0^{-1} x_2 + x_1 x_0^{-1} x_1 x_0^{-1} x_1$$

and so on.

Let F be the free group generated by x_k, $k \geq 0$ and F_n be the subgroup of F generated by $x_0, \dots, x_n$. Noncommutative Catalan number C_n is an element of the group ring $\mathbf{Z}F_n$. Also, $\overline{C}_n = C_n$ for the canonical anti-involution on F.

Recursion. For $n \geq 0$

$$C_{n+1} = \sum_{k=0}^{n} C_k x_0^{-1} T(C_{n-k}) = \sum_{k=0}^{n} T(C_k) x_0^{-1} C_{n-k}$$

where T is an endomorphism of the group ring given by $T(x_k) = x_{k+1}$.

Combinatorial description of non-commutative Catalan numbers

Let P be a monotonic lattice path in $[0, n] \times [0, n]$ from $(0, 0)$ to (n, n).

We say that P is Catalan if for each point $p = (p_1, p_2) \in P$ one has $c(p) \geq 0$, where $c(p) := p_1 - p_2$. The number of such paths is exactly the Catalan number c_n.

We say that a point $p = (p_1, p_2)$ of P is a *southeast* (resp. *northwest*) corner of P if $(p_1 - 1, p_2) \in P$ and $(p_1, p_2 + 1) \in P$ (resp. $(p_1, p_2 - 1) \in P$ and $(p_1 + 1, p_2) \in P$).

To each Catalan path P from $(0,0)$ to (n,n) we assign an element $M_P \in F_n$ by

$$M_P = \overrightarrow{\prod} x_{c(p)}^{\operatorname{sgn}(p)} \ ,$$

where the product is over all corners $p \in P$ (taken in the natural order) and

$$\operatorname{sgn}(p) = \begin{cases} 1 & \text{if } p \text{ is southeast} \\ -1 & \text{if } p \text{ is northwest} \end{cases}$$

Then

$$C_n = \sum_P M_P$$

where the sum is taken over all Catalan paths P from $(0,0)$ to (n,n).

Under the counit homomorphism $\varepsilon : \mathbf{Z}F \to \mathbf{Z}$ where $x_k \mapsto 1$ the image $\varepsilon(C_n)$ is the ordinary Catalan number.

Other papers related to this talk

Generalized adjoint actions
(Berenstein, R.); arXiv:1506.07071

The reciprocal of $\sum_{n\geq 0} a^n b^n$ for non-commuting a and b, Catalan numbers and non-commutative quadratic equations
(Berenstein, R., Reutenauer, Zeilberger)
arXiv:1206.4225

"Lie algebras and Lie groups over noncommutative rings" (Berenstein, R)
arxiv: math/0701399