Qusideterminants. Lecture 1

Determinants in the commutative case:

1) solving systems of linear equations (Cramer rules);

2) multiplicativity, i.e. $\det(AB) = \det(A) \cdot \det(B)$.

We cannot keep both properties in the non-commutative case, so one choose the multiplicativity (Dieudonne) or Cramer rules (Gelfand and R.).

Basic example

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_1 = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1}(b_1 - a_{12}a_{22}^{-1}b_2) =$$
$$= (a_{21} - a_{22}a_{12}^{-1}a_{11})^{-1}(b_2 - a_{22}a_{12}^{-1}b_1)$$

Advantage: “categorical property”

Disadvantage: rational expressions

General definition

Let $A = (a_{ij})$, $i, j = 1, \dots, n$ be a matrix over an associative ring. Denote by A^{pq} the submatrix of A obtained by removing the p-th row and q-th column.
Let $r_p = (a_{p1}, \dots, \widehat{a}_{pq}, \dots, a_{pn})$,
$c_q = (a_{1q}, \dots, \widehat{a}_{pq}, \dots, a_{nq})^t$.
The quasideterminant (a special case of Schur complement) $|A|_{pq}$ is defined if submatrix A^{pq} is invertible:

$$|A|_{pq} = a_{pq} - r_p (A^{pq})^{-1} c_q$$

Example: $|A|_{12} = a_{12} - a_{11} a_{21}^{-1} a_{22}$.
If $B = A^{-1}$, $B = (b_{ij})$ then $b_{qp} = |A|_{pq}^{-1}$.
In the commutative case
$|A|_{pq} = (-1)^{p+q} \det A / \det A^{pq}$.
We can also write

$$|A|_{pq} = a_{pq} - \sum_{k \neq p,\ \ell \neq q} a_{pk} |A^{pq}|_{\ell k}^{-1} a_{\ell q}$$

which leads to branching continuous fractions.

Two applications.

Quasi-Wronskins

Let A be an algebra over a field F of characteristic zero with a F-linear derivation $D : A \to A$. Denote $D^k f$ as $f^{(k)}$. Set

$$w_i = \begin{vmatrix} f_1 & f_2 & \dots & f_i \\ f_1' & f_2' & \dots & f_i' \\ & \dots & \dots & \\ f_1^{(i-1)} & f_2^{(i-1)} & \dots & f_i^{(i-1)} \end{vmatrix}_{ii}$$

Let $L(D) = D^n + a_1 D^{n-1} + \dots + a_n,\ a_i \in A$.

Theorem (EGR) Let f_i, $i = 1, \dots, n$ be solutions of $L(D)$. Set $b_i = (Dw_i)w_i^{-1}$. Then

$$L(D) = (D - b_n)(D - b_{n-1}) \dots (D - b_1) \ .$$

Quasi-Vandermonds

Let R be a ring, $P(t) \in R[t]$ (t is central) and $x_1, \ldots, x_n \in R$ be right roots of $P(t)$, i.e. $P(t) = Q(t)(t - x_i)$. Set

$$v_i = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_i \\ & \ldots & \ldots & \\ x_1^{i-1} & x_2^{i-1} & \ldots & x_i^{i-1} \end{vmatrix}_{ii}$$

Theorem (GR) Set $y_i = v_i x_i v_i^{-1}$. Then

$$P(t) = (t - y_n)(t - y_{n-1}) \ldots (t - y_1) \ .$$

$y_1 = x_1$, $y_2 = (x_2 - x_1)x_2(x_2 - x_1)^{-1}$, ...

If $P(t) = t^n - a_1 t^{n-1} + \cdots + (-1)^n a_n$ then

$$a_1 = \sum_i y_i,$$

$$a_2 = \sum_{i>j} y_i y_j$$

$$\ldots$$

$$a_n = y_n y_{n-1} \ldots y_1$$

i.e. a_i's are **elementary symmetric functions** in free variables $x_1, \ldots, x_n$.

Complete symmetric functions:

$$\sum_{i_1 \le i_2 \le \cdots \le i_k} y_{i_1} y_{i_2} \ldots y_{i_k}$$

Power symmetric function in x_1, x_2:

$$y_1^2 + y_2^2 + [y_1, y_2] \ .$$

Theorem (R.L. Wilson) Let polynomial $P(y_1, \ldots, y_n)$ be symmetric in $x_1, \ldots, x_n$ as a rational function. Then

$$P(y_1, \ldots, y_n) = Q(a_1, \ldots, a_n)$$

where Q is a polynomial.

Main Tool: Heredity principle

New notation:

$$|A|_{ij} = \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ & \ldots & \\ \ldots & \boxed{a_{ij}} & \ldots \\ & \ldots & \\ a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

Let $A = (a_{ij})$, $1 \le i, j \le n$,
$A_0 = (a_{ij})$, $1 \le i, j \le k$, $k < n$.
For $p, q > k$ set

$$b_{pq} = \begin{vmatrix} a_{11} & \dots & a_{1k} & a_{1q} \\ & \dots & & \dots \\ a_{k1} & \dots & a_{kk} & a_{kq} \\ a_{p1} & \dots & a_{pk} & \boxed{a_{pq}} \end{vmatrix}$$

Define $B = (b_{pq})$, $k + 1 \le p, q \le n$.

Theorem (GR) $|B|_{rs} = |A|_{rs}$.

Commutative version:

$$\tilde{b}_{pq} = \begin{vmatrix} a_{11} & \dots & a_{1k} & a_{1q} \\ & \dots & & \dots \\ a_{k1} & \dots & a_{kk} & a_{kq} \\ a_{p1} & \dots & a_{pk} & a_{pq} \end{vmatrix}$$

Set $\tilde{B} = (\tilde{b}_{pq})$. Then (Sylvester)

$$\det \tilde{B} = \det A \cdot (\det A_0)^{n-k-1}$$

Special case: Dodgson condensation

Let $A = (a_{ij})$, $1 \le i, j \le n$. Define quasideterminants of "corner" submatrices of A:

$c_{11} = |A^{nn}|_{11}$, $c_{1n} = |A^{n1}|_{1n}$,

$c_{n1} = |A^{1n}|_{n1}$, $c_{nn} = |A^{11}|_{nn}$.

Then $|A|_{nn} = \begin{vmatrix} c_{11} & c_{1n} \\ c_{n1} & \boxed{c_{nn}} \end{vmatrix} = c_{nn} - c_{n1} c_{11}^{-1} c_{1n}$

Determinants and quasideterminanats

Let $A = (a_{ij})$, $1 \leq i, j \leq n$ over a ring R.

Commutative case.

$$\det A = |A|_{11} |A^{1,1}|_{22} |A^{12,12}|_{33} \dots a_{nn}$$

Quantum matrices. Let $q \in Z(R)$ and

$$q a_{ik} a_{i\ell} = a_{i\ell} a_{ik}, \ k < \ell$$
$$q a_{ik} a_{jk} = a_{jk} a_{ik}, \ i < j$$
$$a_{il} a_{jk} = a_{jk} a_{i\ell}, \ i < j, k < \ell,$$
$$a_{ik} a_{j\ell} - a_{jl} a_{ik} = (q^{-1} - q) a_{i\ell} a_{jk}, \ i < j, k < \ell$$

Recall

$$det_q(A) = \sum_{\sigma \in S_n} (-q)^{i(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

where $i(\sigma)$ is the number of inversions in σ.

Then

$$det_q(A) = |A|_{11} |A^{1,1}|_{22} |A^{12,12}|_{33} \dots a_{nn}$$

and the factors commute.

Etingof and R. generalized this result for matrices defined by FRT-relations.

Moore determinant.

Write $\sigma \in S_n$ as a product of disjoint cycles such that each cycle starts with a smallest number (disjoint cycles commute).

$\sigma = (k_{11} \dots k_{1j_1})(k_{21} \dots k_{2j_2}) \dots (k_{m1} \dots k_{mj_m})$ where $k_{11} > k_{21} \dots > k_{m1}$. Let $p(\sigma)$ be a parity of σ. Denote by C_s the cyclic product of entries of A with indices from s-th cycle, $1 \leq s \leq m$.

$$\det(A) := \sum_{\sigma \in S_n} p(\sigma) \cdot C_1 C_2 \dots C_m$$

Moore determinant is a multiplicative function on quaternionic Hermitian matrices. It admits the above factorization as a product of commuting quasideterminants.

Capelli determinant: factorizations.

Let $X = (x_{ij})$, $1 \le i, j \le n$ be a matrix of formal commuting variables and X^t the transposed matrix. Let $D = (\partial_{ij}/\partial x_{ij})$, be the matrix of the corresponding derivations. Then

$$X^t D = (f_{ij}),\ \ f_{ij} = \sum_k x_{ki}\partial/\partial x_{kj}$$

Let $W = \mathrm{diag}(0, 1, 2, \ldots, n)$. The Capelli determinant $\det_C$ is the row-determinant of matrix $X^t D + W = (g_{ij})$:

$$det_C = \sum_{\sigma \in S_n} (-1)^{i(\sigma)} g_{1,\sigma(1)} g_{2,\sigma(2)} \cdots g_{n,\sigma(n)}$$

Capelli identity : $\mathrm{d}et_C = \det X \cdot \det D$.

Set $Z = X^t D$. Set

$$\mathbf{z}_k = |Z^{1\ldots k;\ 1\ldots k} + kI_{n-k}|_{kk}$$

where I_ℓ is the identity $\ell \times \ell$- matrix.

In particular, $\mathbf{z}_0 = |Z|_{11}$, $\mathbf{z}_{n-1} = z_{nn} + n - 1$.

Theorem (GR). Determinant Capelli is the product of commuting factors:

$$det_C = \mathbf{z}_0 \cdot \mathbf{z}_1 \ldots \mathbf{z}_{n-1}$$

Similar result is valid for Yangians $Y(gl_n)$.

Applications of the factorization of Capelli determinant to a structure of $U(gl_n)$ are discussed in [GKKLRT].