Into multiplier Hopf $*$-graph algebras, and beyond

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From non-unital to unital algebras

- For any field $\mathbb{K}$ and a finite group G, take $A = \mathbb{K}(G)$. Then we have

$$\begin{aligned} &A \otimes A \xrightarrow{\cong} \mathbb{K}(G \times G) \\ &f \otimes g \mapsto (f \otimes g)(p, q) = f(p) \otimes g(q) := f(p)g(q), \end{aligned}$$

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- In 1994, A. Van Daele came up with a solution to this issue as follows:

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- hence by defining

$$\begin{aligned} \Delta : &\mathbb{K}_f(G) \to \mathbb{K}_f(G) \otimes \mathbb{K}_f(G) \cong \mathbb{K}_f(G \times G) \subseteq \mathbb{K}(G \times G) \\ &f \longmapsto \Delta(f)(g_1, g_2) := f(g_1 g_2) \end{aligned}$$

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- we can have the following definition

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- For A a unital or non-unital algebra over $\mathbb{C}$ with a non-degenerate product, and $\Delta : A \to M(A \otimes A)$ a homomorphism, assume that $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\Delta(b)$ belong to $A \otimes A$ for all a and b in A.

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- Then we say that Δ is co-associative if for all a, b and c in A, and $i : A \to A$ and 1 the unit element of $M(A)$, we have

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- then Δ will be called a comultiplication on A,
- and A will be called a multiplier Hopf algebra if the linear maps $T_1, T_2 : A \otimes A \to A \otimes A$, defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b), \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

are bijective.

Quantum permutation group S_n^+

- A compact quantum group G is a pair (A, Δ), for A a C^*-algebra and Δ a unital $*$-homomorphism from A to $A \otimes A$, called comultiplication, satisfying the coassociativity relation

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- and the cancellation properties

$$\Delta(A)(1 \otimes A) = \{\Delta(a)(1 \otimes b) | a, b \in A\}$$
$$\Delta(A)(A \otimes 1) = \{\Delta(a)(b \otimes 1) | a, b \in A\},$$

dense in $A \otimes A$, due to Woronowicz.

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- For the compact group G, we can see $\mathbb{C}G$ as the group C^*-algebra associated with G, consisting of the set of finite linear combinations $\sum_{g \in G} c_g g$, for $c_g \in \mathbb{C}$, with the multiplication adopted from the group multiplication and equipped with the involution $(\sum c_g g)^* := \sum \overline{c}_g g^{-1}$

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- isomorphic with the universal C^*-algebra

$$C^* \left(c_g | c_g \text{ unitary}, \; c_g c_h = c_{gh}, c_g^* = c_{g^{-1}} \right).$$

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- in the late nineties, Wang came with an answer, saying that

  “*the quantum permutation group* S_n^+

  could be defined as the largest compact

  quantum group acting on the set $\{1, \ldots, N\}$”

Quantum permutation group S_n^+, continuation

- by looking at it as the compact set $X_N := \{x_1, \ldots, x_N\}$ consisting of a finite set of points (pointwise isomorphic) and studying its function space $C(X_N) \equiv C^*\left(p_1, \cdots, p_N \text{ projections} \mid \sum_{i=1}^N p_i = 1\right)$.

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- This has led him to define $C(S_n^+)$ as the following universal C^*-algebra

$$C^*\left(u_{ij}, i, j = 1, \cdots, n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{kj} = \sum_{k=1}^n u_{ik} = 1\right)$$

Quantum permutation group S_n^+, continuation

- and calling $S_n^+ = (C(S_n^+), u)$ the quantum symmetric (permutation) group as the quantum automorphism group of X_N, and proving that it satisfies the relations of being a compact (matrix) quantum group in the sense of Woronowicz.

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- The main ingredients in defining $C(S_n^+)$, meaning that the u_{ij}s, are very important in our construction of the ($*$-)multiplier Hopf graph algebras, and we have the following definition:

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- The main ingredients in defining $C(S_n^+)$, meaning that the u_{ij}s, are very important in our construction of the (∗-)multiplier Hopf graph algebras, and we have the following definition:
- Matrix $u = (u_{ij})_{i,j}$ with entries u_{ij}s from a non-trivial unital C^*-algebra satisfying relations $u_{ij} = u_{ij}^* = u_{ij}^2$ and $\sum_{k=1}^n u_{kj} = \sum_{k=1}^n u_{ik} = 1$, will be called a magic unitary.

Quantum permutation group S_n^+, continuation

- Recently, Rollier and Vaes put one step forward and applied the above constructions to the connected locally finite graphs, and more than that, they used this association in order to make a bridge between the already known abstract concept of the multiplier Hopf algebras, introduced and studied by Van Daele, to a more intuitive field of Graph Theory, in the form of the following theorem

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- Recently, Rollier and Vaes put one step forward and applied the above constructions to the connected locally finite graphs, and more than that, they used this association in order to make a bridge between the already known abstract concept of the multiplier Hopf algebras, introduced and studied by Van Daele, to a more intuitive field of Graph Theory, in the form of the following theorem
- For Π a locally connected finite graph with vertex set I, there exist a (necessarily unique) universal nondegenerate $*$-algebra $\mathcal{A}$ generated by the elements u_{ij} satisfying the relations of the magic unitary matrix in definition of the quantum permutation group, and a unique nondegenerate $*$-homomorphism $\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A})$ taking u_{ij} to $\sum_{k \in I}(u_{ik} \otimes u_{kj})$ for all $i, j \in I$, such that the pair $(\mathcal{A}, \Delta)$ is a multiplier Hopf $*$-algebra in the sense of Van Daele.

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- $$\begin{aligned} X_{ri}X_{rj} &= q^{-1}X_{rj}X_{ri}, && \forall i < j; \\ X_{ri}X_{si} &= q^{-1}X_{si}X_{ri}, && \forall r < s; \\ X_{ri}X_{sj} &= X_{sj}X_{ri}, && \text{if } r < s \text{ and } i > j; \\ X_{ri}X_{sj} - X_{sj}X_{ri} &= \widehat{q}X_{si}X_{rj}, && \text{if } r < s \text{ and } i < j, \end{aligned} \tag{1}$$

  where we have $\widehat{q} = q^{-1} - q$.

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  where we have $\widehat{q} = q^{-1} - q$.
- Now, let us associate a directed structure to the above set of relations by using the following rules:

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  3. $i < k$ and $j > \ell$,
- and we have $u_{ij} \overset{\leftrightarrow}{\sim} u_{k\ell}$ if and only if $i > k$ and $j < \ell$.

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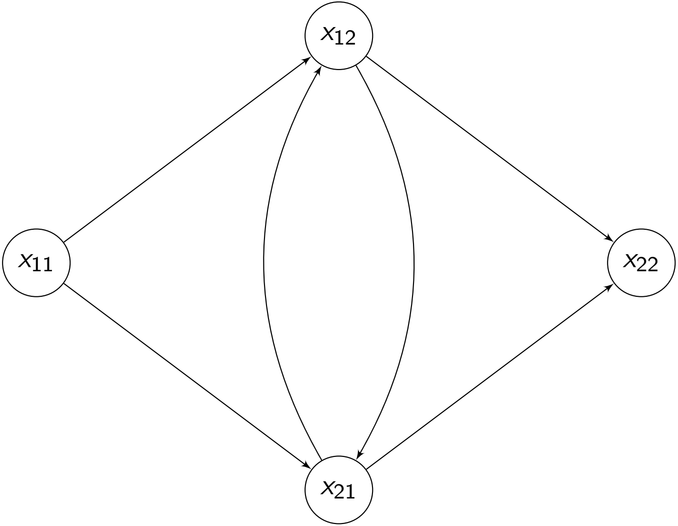
- 

Figure 1: **Directed locally connected graph related to $\mathbb{K}\left(M_q(2)\right)$**

- Note that the only commuting matrix with Π_2 satisfying relations of being a magic unitary matrix will be

$$\pi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- and by continuing of finding the commuting matrices, the algorithm will be as follows, for any n, that, the entries associated with the row related to x_{ij}, will be 1 in $(ij)(ji)$ position and 0 elsewhere, and the associated graph, for example for π_2 will be

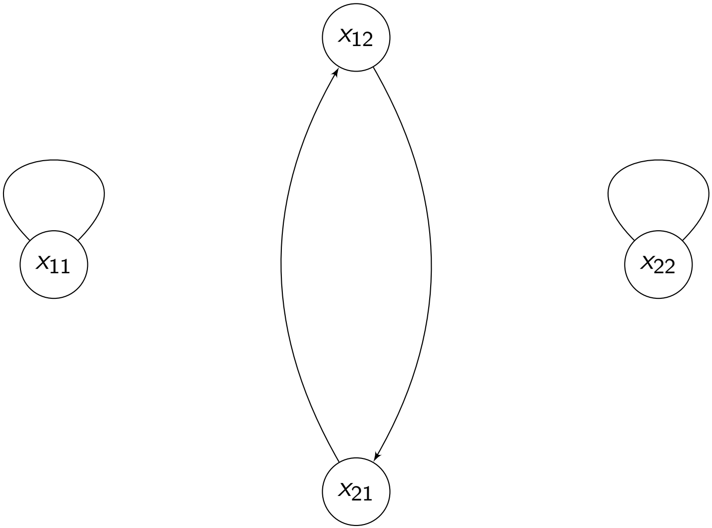

Figure 2: **Directed 2-connected graph related to π_2**

(Quantum) Graph Algebra

- Following the theorem by Rollier and Vaes, and by using the (n^2-2)-connected locally finite graphs $\mathcal{G}_i = \{\mathcal{G}(\pi_i) \mid i \in \{1, \cdots n\}\}$ associated with the adjacency matrices Π_i of $\mathbb{K}[M_q(n)]$ and their commuting matrices π_i, for $i \in I = \{1, \cdots, n\}$,

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- recently we showed that the set $\mathcal{G}_i$ possesses a nondegenerate $*$-monoid algebra structure equiped with the following binary operations

$$\begin{aligned} \pi_i + \pi_j &:= (V_i \cup V_j, E_i \cup E_j) \\ \pi_i \to \pi_j &:= (V_i \cup V_j, E_i \cup E_j)\,, \end{aligned} \tag{2}$$

and the identity element π_2.

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- The third approach is to directly work on the graph structures and put an algebra structure on the set of specific graphs, as we have done on the set of $\mathcal{G}_i$s, illustrated bellow

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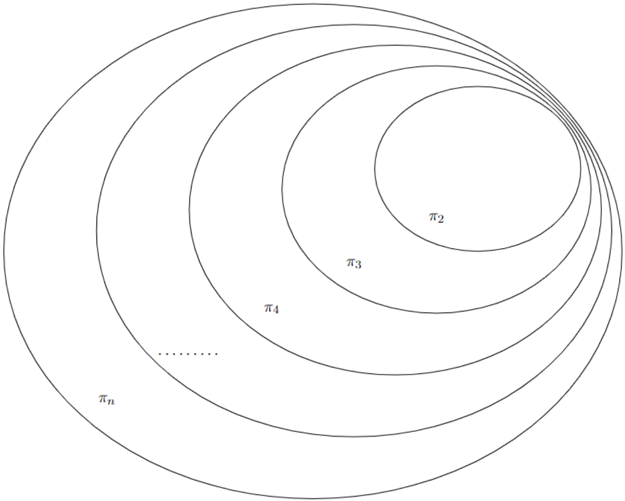

Figure 3: Illustration of the set of $n-1$ graphs $\mathcal{G}_i$

Cuntz-Krieger Algebras

- Cuntz algebras $\mathcal{O}_n$ have been introduced by Cuntz in 1977 as a class of simple purely infinite C^*-algebras generated by isometries (S is an isometry if $S^*S = \mathrm{id}$) $S_1, S_2, \cdots, S_n$ satisfying in the following relations

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- Following the above definition, to a directed graph Γ, one can associate a C^*-algebra $C^*(\Gamma^0, \Gamma^1) := C^*(\Gamma)$ by associating to its set of edges Γ^1 a set of partial isometries and to its set of vertices Γ^0 a set of pairwise orthogonal projections (Hilbert spaces) satisfying in some specific relations, studied first by Cuntz and Krieger in 1980, as a generalization of the Cuntz algebras.

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- Any unitary matrix is a partial isometry, and any invertible partial isometry is unitary.

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- A is a partial isometry if and only if A^*A and AA^* are orthogonal projections.

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- 1. $s_e^* s_e = p_{r(e)}$ for all edges $e \in \Gamma^1$,
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- For finite directed graph $\Gamma = (\Gamma^0, \Gamma^1)$, the graph C^*-algebra $\mathcal{C}^*(\Gamma)$ is the universal C^*-algebra generated by a Cuntz-Krieger Γ-family $\{P_v, S_e\}$.

Graph C^*-algebra of a finite directed graph

- Now let us consider those $n \times n$ matrices with entries in $\{0, 1\}$ by considering
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- For $n \times n$ matrix $\Pi \in M_n(0, 1)$, the Cuntz-Krieger algebra $\mathcal{K}_\Pi$ will be defined as the (nondegenerate) C^*-algebra generated by a universal Cuntz-Krieger Γ-family S_i for $i \in \{1, \cdots, n\}$ satisfying in $s_i^* s_i = \sum_{j=1}^{n} a_{ij} s_j s_j^*$.

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- For any finite locally connected (directed) graph $E = (E^0, E^1)$, it is well known that we have $P_v\mathcal{H} = \left(\sum_{\{e\in E^1 | r(e)=v\}} S_e S_e^*\right)\mathcal{H} = \oplus_{\{e\in E^1 | r(e)=v\}} S_e\mathcal{H}$.

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- Now let us consider the very initial part of our toy example.

Graph C^*-algebra of a finite directed graph

- For graph $\mathcal{G}(\Pi_2)$ associated with $\mathbb{K}[M_q(2)]$, consider its set of vertices and edges as $\mathcal{G}^0 = \{x_{11} := u, x_{12} := v, x_{22} := k, x_{21} := w\}$ and $\mathcal{G}^1 = \{x_{11} \overrightarrow{\sim} x_{12} := e, x_{11} \overrightarrow{\sim} x_{21} := f, x_{12} \overrightarrow{\sim} x_{22} := h, x_{21} \overrightarrow{\sim} x_{22} := g, x_{12} \overrightarrow{\sim} x_{21} := i, x_{21} \overrightarrow{\sim} x_{12} := j\}$, we have the following Proposition.

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- For Π_2 as before, and $\mathcal{G}(\Pi_2) = (\mathcal{G}^0, \mathcal{G}^1)$ the associated adjacency matrix, and let $\mathcal{H} := \ell^2(\mathbb{N})$ be the underlying infinite dimensional Hilbert space. Then the set

$$S = \{S_e := \sum_{n=1}^{\infty} E_{6n,3n-2}, S_f := \sum_{n=1}^{\infty} E_{6n-4,3n-2}, S_h := \sum_{n=1}^{\infty} E_{6n-3,3n}, \\ S_g := \sum_{n=1}^{\infty} E_{6n-4,3n-1}, S_i := \sum_{n=1}^{\infty} E_{6n-1,3n}, S_j := \sum_{n=1}^{\infty} E_{6n-3,3n-1}\} \tag{4}$$

is a Cuntz-Krieger $\mathcal{G}$-family and gives us a graph C^*-algebra structure $\mathcal{C}^*(\Pi_2)$.

Graph C^*-algebra of a finite directed graph

- For graph $\mathcal{G}(\pi_2)$ associated with π_2, as before consider its set of vertices and edges as $\mathcal{G}^0 = \{x_{11} := v_1, x_{12} := v_2, x_{22} := v_3, x_{21} := v_4\}$ and $\mathcal{G}^1 = \{x_{11} \overrightarrow{\rightsquigarrow} x_{11} := e_{11}, x_{12} \overrightarrow{\rightsquigarrow} x_{21} := e_{24}, x_{21} \overrightarrow{\rightsquigarrow} x_{12} := e_{42}, x_{22} \overrightarrow{\rightsquigarrow} x_{22} := e_{33}\}$, we have the following Proposition.

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- For $\mathcal{G}_2 := \mathcal{G}(\pi_2) = (\mathcal{G}^0, \mathcal{G}^1)$ as above, consider $\mathcal{H}$ be the underlying Hilbert space, that can be finite or infinite. Then the set

$$\begin{aligned} S = \{S_{e_{11}} &:= E_{2,1}, S_{e_{24}} := E_{4,1}, \\ S_{e_{42}} &:= E_{1,4}, S_{e_{33}} := E_{3,1}\} \end{aligned} \tag{5}$$

is a Cuntz-Krieger $\mathcal{G}_2$-family and gives us a graph C^*-algebra structure $\mathcal{C}^*(\pi_2) := M_4(\mathbb{C})$.

Multiplier Hopf $*$-graph algebras

- As I already said, from the different kind of graph algebras, the one that we are interested should be nondegenerate, and the $*$-monoid algebra with an identity element considered by the graph associated with the matrix $\pi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and illustrated above, and extendable to $\mathbb{K}[M_q(n)]$ for any $n \geq 2$, by the rule of assigning 1 to the entries located in the $(ij)(ji)$ position and 0 elsewhere, in the row related to x_{ij}, is the one which we are looking for, and we have the following proposition.

Multiplier Hopf $*$-graph algebras

- For Π, a locally finite connected graph associated with coordinate algebra $\mathbb{K}(M_q(n))$ with vertex set $\{x_{11}, x_{12}, \cdots, x_{ij}\}$ for $i, j \in \{1, 2, \cdots, n\}$ and the index set $I := \{11, 12, \cdots, ij\}$, there exists a unique universal nondegenerate $*$-algebra $\mathcal{A}$ generated by elements $(u_{hh'})_{h,h' \in I}$, satisfying the relations of quantum permutation groups, and a unique nondegenerate $*$-homomorphism $\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A})$ satisfying $\Delta(u_{hh'}) = \sum_{k \in I}(u_{hk} \otimes u_{kh'})$ for all $h, h' \in I$, such that the pair $(\mathcal{A}, \Delta)$ is a multiplier Hopf $*$-algebra in the sense of Van Daele, and since it admits a positive faithful left-invariant (resp. right-invariant) functional, it is an algebraic quantum group in the sense of Van Daele.

Multiplier Hopf $*$-graph algebras

- Let us call the algebra structure on the vector space of the (n^2-1)-connected locally finite graphs $\mathcal{G}_i$, simply $\mathcal{G}$. This is a unital $*$-algebra, and in order to have a $*$-multiplier Hopf algebra, we need to define a map Δ on $\mathcal{G}$ to $M(\mathcal{G}\otimes\mathcal{G})$, resembling the co-product and satisfying the co-associativity condition

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- $$(\pi_i \otimes 1 \otimes 1)(\Delta \otimes \mathrm{id})(\Delta(\pi_j)(1\otimes \pi_k))$$
$$=$$
$$(\mathrm{id} \otimes \Delta)((\pi_i \otimes 1)\Delta(\pi_j))(1 \otimes 1 \otimes \pi_k),$$

Multiplier Hopf $*$-graph algebras

- in a way that $\Delta(\pi_i)(1 \otimes \pi_j)$ and $(\pi_i \otimes 1)\Delta(\pi_j)$ belong in $\mathcal{G} \otimes \mathcal{G}$, for any $\pi_i, \pi_j, \pi_k \in \mathcal{G}$.

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- But now the question is that
- How can we define such a map for a graph algebra consisting of graphs?
- As the previous constructions suggest, the idea is to implement the graph C^*-algebra $C^*(S, P)$ associated with π_is. In this case we will obtain n different multiplier Hopf $*$-graph algebras.

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Thank You For Your Time!

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