Perturbative correlation functions from homotopy algebras

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arXiv:2203.05366, arXiv:2305.11634 with Keisuke Konosu

See also arXiv:2305.13103 by Keisuke Konosu

April 15, 2024

at Graduate School of Mathematics, Nagoya University

1. Introduction

String theory has provided us with clues to quantum gravity.

When we explore quantum aspects of string theory such as mass renormalization and vacuum shift, we need to go beyond the world-sheet perturbation theory based on the integration of on-shell vertex operators over the moduli space of Riemann surfaces.

String field theory provides such a framework.

String field theory may also be useful when we attempt to define string theory nonperturbatively.

While closed string theory contains quantum gravity, it would not be promising to use closed string field theory for a nonperturbative definition of closed string theory.

This is because gauge invariance in the classical theory is anomalous and we need correction terms at every loop order to recover gauge invariance.

The most promising approach to the nonperturbative definition of closed string theory would be the AdS/CFT correspondence, but the world-sheet picture is gone in the strict low-energy limit of the gauge theory on D-branes.

It might be useful to consider the theory on D-branes before taking the low-energy limit.

We may think that such a theory would be open-closed string field theory, but my claim is that it can be described by open string field theory with the source term for gauge-invariant operators.

This seems to be the case at least for the bosonic string as a consequence of a few nontrivial facts.

1. Unlike closed string field theory, gauge invariance of open bosonic string field theory is not anomalous, and we do not need correction terms to the classical action.

2. It is in general difficult to construct gauge-invariant operators in string field theory, but a class of gauge-invariant operators are constructed in open bosonic string field theory.

hep-th/0111092, Hashimoto and Itzhaki
hep-th/0111129, Gaiotto, Rastelli, Sen and Zwiebach

We can construct a gauge-invariant operator for each on-shell closed string state, and peculiarly it is linear in the open string field.

3. Open string field theory with the source term for gauge-invariant operators can be obtained in a special limit of open-closed string field theory, and it generates all Feynman diagrams which contain at least one boundary.

hep-th/9202015, Zwiebach

Purely closed-string diagrams without boundaries are not generated, but their contributions vanish in the low-energy limit we are interested in.

It is crucially important whether or not this scenario can be extended to open superstring field theory.

The long-standing problem of constructing an action involving the Ramond sector has been overcome in superstring field theory.

Kunitomo and Okawa, arXiv:1508.00366

Sen, arXiv:1508.05387

While the formulations of open superstring field theory need to be developed further, we consider that we are in a position to discuss how we use open superstring field theory to understand the mechanism which realizes the AdS/CFT correspondence.

So what should we do?

Instead of scattering amplitudes, we should consider correlation functions of gauge-invariant operators in open string field theory.

We evaluate correlation functions in the $1/N$ expansion and turn it into the genus expansion of closed string theory.

This step would be the most difficult part and we need to generalize the world-sheet derivation of the large N duality of the topological string by Ooguri and Vafa to the superstring.

hep-th/0205297, Ooguri and Vafa

While string field theory in the classical theory has been useful in describing nonperturbative physics such as tachyon condensation, now we need to study quantum aspects of string field theory.

String field theory is a space-time field theory involving infinitely many fields, and conceptually it is the same as ordinary field theory to some extent. However, string field theory is highly complicated compared to ordinary field theory, and we need efficient tools to study quantum aspects of string field theory.

Homotopy algebras can be useful for this purpose.

We have used homotopy algebras such as A_∞ algebras and L_∞ algebras in the construction of gauge-invariant actions of string field theory.

However, we might not have fully appreciated the power of homotopy algebras, and they can be also useful in solving the theory.

Furthermore, the description in terms of homotopy algebras is universal.

Before working on string field theory, we should apply the description in terms of homotopy algebras to simpler theories to gain insight.

With this motivation, we are currently developing technologies of homotopy algebras for simpler quantum field theories.

In the first talk, I will explain a formula for correlation functions of scalar field theories based on A_∞ algebras. It was originally proposed in arXiv:2203.05366, and then it was refined to a form which is analogous to string field theory and extended to Dirac fields in arXiv:2305.11634 with Konosu and in arXiv:2305.13103 by Konosu.

It was shown in perturbation theory that correlation functions based on this formula satisfy the Schwinger-Dyson equations.

In the second talk, I will present evidence that the formula based on homotopy algebras describes nonperturbative correlation functions!

The plan of the talk

1. Introduction
2. A_∞ algebra
3. Formula for correlation functions
4. Renormalization group
5. Summary

2. A_∞ algebra

Open bosonic string field theory is described in terms of string field, which is a state of the boundary conformal field theory.

The Hilbert space $\mathcal{H}$ can be decomposed based on the ghost number as

$$\mathcal{H} = \ldots \oplus \mathcal{H}_{-1} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots ,$$

and the classical action is written in terms of Φ in $\mathcal{H}_1$.

Consider an action of the form:

$$S = -\frac{1}{2} \langle\, \Phi, V_1(\Phi) \,\rangle - \frac{g}{3} \langle\, \Phi, V_2(\Phi, \Phi) \,\rangle - \frac{g^2}{4} \langle\, \Phi, V_3(\Phi, \Phi, \Phi) \,\rangle + O(g^3) \,,$$

where $\langle\, A_1, A_2 \,\rangle$ is the BPZ inner product of A_1 and A_2, V_n is an n-string product “多弦積,” and g is the string coupling constant.

This action is invariant up to $O(g^3)$ under the gauge transformation with the gauge parameter Λ in $\mathcal{H}_0$ given by

$$\begin{aligned}\delta_\Lambda \Phi = {} & V_1(\Lambda) + g\,(\,V_2(\Phi,\Lambda) - V_2(\Lambda,\Phi)\,) \\ & + g^2\,(\,V_3(\Phi,\Phi,\Lambda) - V_3(\Phi,\Lambda,\Phi) + V_3(\Lambda,\Phi,\Phi)\,) + O(g^3)\end{aligned}$$

if the multi-string products satisfy the following relations:

$$\begin{aligned}& V_1(V_1(A_1)) = 0\,, \\ & V_1(V_2(A_1,A_2)) - V_2(V_1(A_1),A_2) - (-1)^{A_1} V_2(A_1,V_1(A_2)) = 0\,, \\ & V_1(V_3(A_1,A_2,A_3)) + V_3(V_1(A_1),A_2,A_3) \\ & \;+ (-1)^{A_1} V_3(A_1,V_1(A_2),A_3) + (-1)^{A_1+A_2} V_3(A_1,A_2,V_1(A_3)) \\ & \;- V_2(V_2(A_1,A_2),A_3) + V_2(A_1,V_2(A_2,A_3)) = 0\,.\end{aligned}$$

These relations can be extended to higher orders and called A_∞ relations. (In this talk all the discussions on cyclic properties are omitted.)

Let us simplify the description of A_∞ relations in three steps.

Step 1: Degree

We introduce *degree* defined by

$$\deg(A) = \epsilon(A) + 1 \mod 2\,,$$

where $\epsilon(A)$ is the Grassmann parity of A, and we define

$$\begin{aligned}
\omega(A_1, A_2) &= (-1)^{\deg(A_1)} \langle\, A_1, A_2 \,\rangle\,, \\
M_1(A_1) &= V_1(A_1)\,, \\
M_2(A_1, A_2) &= (-1)^{\deg(A_1)}\, V_2(A_1, A_2)\,, \\
M_3(A_1, A_2, A_3) &= (-1)^{\deg(A_2)}\, V_3(A_1, A_2, A_3)\,, \\
&\;\;\vdots
\end{aligned}$$

Step 2: Tensor products of $\mathcal{H}$

We denote the tensor product of n copies of $\mathcal{H}$ by $\mathcal{H}^{\otimes n}$. For an n-string product $D_n(A_1, A_2, \ldots, A_n)$ we define a corresponding operator D_n which maps $\mathcal{H}^{\otimes n}$ into $\mathcal{H}$ by

$$D_n\left(A_1 \otimes A_2 \otimes \ldots \otimes A_n\right) \equiv D_n(A_1, A_2, \ldots, A_n)\,.$$

We also introduce the vector space for the zero-string space denoted by $\mathcal{H}^{\otimes 0}$. It is a one-dimensional vector space given by multiplying a single basis vector $\mathbf{1}$ by complex numbers. The vector $\mathbf{1}$ satisfies

$$\mathbf{1} \otimes A = A\,, \qquad A \otimes \mathbf{1} = A$$

for any string field A.

The A_∞ relations are written as

$$
\begin{aligned}
& M_1\, M_1 = 0\,, \\
& M_1\, M_2 + M_2\,(\, M_1 \otimes \mathbb{I} + \mathbb{I} \otimes M_1\,) = 0\,, \\
& M_1\, M_3 + M_3\,(\, M_1 \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes M_1 \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes M_1\,) \\
& \; + M_2\,(\, M_2 \otimes \mathbb{I} + \mathbb{I} \otimes M_2\,) = 0\,, \\
& \qquad\qquad\qquad \vdots
\end{aligned}
$$

where we denoted the identity map from $\mathcal{H}$ to $\mathcal{H}$ by $\mathbb{I}$.

Step 3: Coderivations

It is convenient to consider linear operators acting on the vector space $T\mathcal{H}$ defined by

$$T\mathcal{H} = \mathcal{H}^{\otimes 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \mathcal{H}^{\otimes 3} \oplus \ldots .$$

We denote the projection operator onto $\mathcal{H}^{\otimes n}$ by π_n.

For a map D_n from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}$, we define an associated operator $\boldsymbol{D}_n$ acting on $T\mathcal{H}$ as follows.

$$\begin{aligned}
\boldsymbol{D}_n \, \pi_m &= 0 \quad \text{for} \quad m < n \, , \\
\boldsymbol{D}_n \, \pi_n &= D_n \, \pi_n \, , \\
\boldsymbol{D}_n \, \pi_{n+1} &= (\, D_n \otimes \mathbb{I} + \mathbb{I} \otimes D_n \,) \, \pi_{n+1} \, , \\
\boldsymbol{D}_n \, \pi_{n+2} &= (\, D_n \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes D_n \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes D_n \,) \, \pi_{n+2} \, , \\
&\;\;\vdots
\end{aligned}$$

An operator acting on $T\mathcal{H}$ of this form is called a *coderivation*.

We define $\mathbf{M}$ by

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \ldots$$

for coderivations $\mathbf{M}_n$ associated with M_n. Then the A_∞ relations can be compactly expressed as

$$\mathbf{M}^2 = 0\,.$$

When we consider projections onto subspaces of $\mathcal{H}$, homotopy algebras have turned out to provide useful tools.

- Projection onto on-shell states $\rightarrow$ on-shell scattering amplitudes
  Kajiura, math/0306332

- Projection onto the physical sector
  $\rightarrow$ mapping between covariant and light-cone string field theories
  Erler and Matsunaga, arXiv:2012.09521

- Projection onto the massless sector $\rightarrow$ the low-energy effective action
  Sen, arXiv:1609.00459
  Erbin, Maccaferri, Schnabl and Vošmera, arXiv:2006.16270
  Koyama, Okawa and Suzuki, arXiv:2006.16710

Let us decompose $\mathbf{M}$ as

$$\mathbf{M} = \mathbf{Q} + \boldsymbol{m}\,,$$

where $\mathbf{Q}$ describes the free theory and $\boldsymbol{m}$ is for interactions. We consider projections which commute with Q.

We denote the projection operator by P:

$$P^2 = P\,, \qquad P\,Q = Q\,P\,.$$

We then promote P on $\mathcal{H}$ to $\mathbf{P}$ on $T\mathcal{H}$ as follows:

$$\begin{aligned}
\mathbf{P}\,\pi_0 &= \pi_0\,,\\
\mathbf{P}\,\pi_1 &= P\,\pi_1\,,\\
\mathbf{P}\,\pi_2 &= (\,P \otimes P\,)\,\pi_2\,,\\
\mathbf{P}\,\pi_3 &= (\,P \otimes P \otimes P)\,\pi_3\,,\\
&\;\;\vdots
\end{aligned}$$

The operators $\mathbf{Q}$ and $\mathbf{P}$ satisfy

$$\mathbf{P}^2 = \mathbf{P}\,, \qquad \mathbf{Q}\,\mathbf{P} = \mathbf{P}\,\mathbf{Q}\,.$$

In the context of the projection onto the massless sector, the propagator h for massive fields is given by

$$h = \frac{b_0}{L_0} \, (\mathbb{I} - P \,) \, .$$

In general we consider h satisfying the following relations:

$$Q \, h + h \, Q = \mathbb{I} - P \, , \qquad h \, P = 0 \, , \qquad P \, h = 0 \, , \qquad h^2 = 0 \, .$$

We then promote h on $\mathcal{H}$ to $\boldsymbol{h}$ on $T\mathcal{H}$ as follows:

$$\begin{aligned}
\boldsymbol{h} \, \pi_0 &= 0 \, , \\
\boldsymbol{h} \, \pi_1 &= h \, \pi_1 \, , \\
\boldsymbol{h} \, \pi_2 &= (\, h \otimes P + \mathbb{I} \otimes h \,) \, \pi_2 \, , \\
\boldsymbol{h} \, \pi_3 &= (\, h \otimes P \otimes P + \mathbb{I} \otimes h \otimes P + \mathbb{I} \otimes \mathbb{I} \otimes h \,) \, \pi_3 \, , \\
&\vdots
\end{aligned}$$

The relations involving Q, P, and h are promoted to the following relations

$$\mathbf{Q}\,\boldsymbol{h} + \boldsymbol{h}\,\mathbf{Q} = \mathbf{I} - \mathbf{P}\,, \qquad \boldsymbol{h}\,\mathbf{P} = 0\,, \qquad \mathbf{P}\,\boldsymbol{h} = 0\,, \qquad \boldsymbol{h}^2 = 0\,,$$

where $\mathbf{I}$ is the identity operator on $T\mathcal{H}$.

The important point is that the theory after the projection inherits the A_∞ structure from the theory before the projection as follows:

$$\mathbf{Q} + \boldsymbol{m} \quad \to \quad \mathbf{P}\,\mathbf{Q}\,\mathbf{P} + \mathbf{P}\,\boldsymbol{m}\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\,\mathbf{P}\,,$$

which is known as the *homological perturbation lemma*.

On-shell scattering amplitudes at the tree level can be calculated from this formula with the projection onto on-shell states.

On-shell scattering amplitudes at the loop level can also be calculated by extending A_∞ algebras to quantum A_∞ algebras (to be discussed later).

The formula from quantum A_∞ algebras has not been explored much.

In addition to scattering amplitudes we are also interested in correlation functions.

Actually, when actions are written in terms of homotopy algebras, expressions of on-shell scattering amplitudes are universal for both string field theories and ordinary field theories.

Let us study scalar field theories in terms of quantum A_∞ algebras to gain insights into quantum aspects of string field theories.

We also find that correlation functions of scalar field theories can also be described in terms of homotopy algebras.

Okawa, arXiv:2203.05366

3. Formula for correlation functions

Let us consider φ^3 theory in d dimensions:

$$S = \int d^d x \left[-\frac{1}{2}\,\partial_\mu \varphi(x)\,\partial^\mu \varphi(x) - \frac{1}{2}\,m^2\,\varphi(x)^2 + \frac{1}{6}\,g\,\varphi(x)^3 \right].$$

When we use A_∞ algebras to describe theories without gauge symmetries, we only need two sectors for the vector space $\mathcal{H}$, and in our convention that is analogous to open string field theory they are denoted by $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\,.$$

In my previous paper arXiv:2203.05366, I chose each of $\mathcal{H}_1$ and $\mathcal{H}_2$ to be the vector space of functions of x.

In my paper arXiv:2305.11634 with Keisuke Konosu, we denoted the basis vector of $\mathcal{H}_1$ by $c(x)$, where the label x represents coordinates of Minkowski spacetime in d dimensions, and we define $c(x)$ to be degree even.

The important point is that we do *not* identify $c(x)$ with the scalar field which appears in the action. The element Φ of $\mathcal{H}_1$ can be expanded as

$$\Phi = \int d^d x \, \varphi(x) \, c(x) \, ,$$

and we identify $\varphi(x)$ in this expansion with the scalar field which appears in the action. We define $\varphi(x)$ to be degree even.

The relation between these two descriptions is analogous to the relation between wave functions and states in quantum mechanics.

The state $|\,\Psi\,\rangle$ in quantum mechanics can be expanded in terms of position eigenstates $|\,x\,\rangle$ as

$$|\,\Psi\,\rangle = \int dx\,\psi(x)\,|\,x\,\rangle\,,$$

and the wave function $\psi(x)$ appears as coefficients in this expansion. The basis vector $c(x)$ plays the role of $|\,x\,\rangle$ in this analogy.

The two descriptions are not so different when we only consider a single scalar field $\varphi(x)$.

However, the description in terms of Φ can be extended to incorporate a Dirac field $\Psi_\alpha(x)$ as follows:

$$\Phi = \int d^d x \, \varphi(x) \, c(x) + \int d^d x \, (\, \overline{\theta}_\alpha(x) \, \Psi_\alpha(x) + \overline{\Psi}_\alpha(x) \, \theta_\alpha(x) \,) \, ,$$

where $\theta_\alpha(x)$ and its Dirac adjoint $\overline{\theta}_\alpha(x)$ are *degree-odd* basis vectors.

We expect that the description in terms of Φ can be further generalized to gauge theories including string field theories.

For the vector space $\mathcal{H}_2$, we denote the basis vector by $d(x)$, and we define $d(x)$ to be degree odd. We then define Q by

$$Q\,c(x) = (\,-\,\partial^2 + m^2\,)\,d(x)\,, \qquad Q\,d(x) = 0\,.$$

The symplectic form ω is defined by

$$\begin{pmatrix} \omega\,(\,c(x_1)\,,c(x_2)\,) & \omega\,(\,c(x_1)\,,d(x_2)\,) \\ \omega\,(\,d(x_1)\,,c(x_2)\,) & \omega\,(\,d(x_1)\,,d(x_2)\,) \end{pmatrix} = \begin{pmatrix} 0 & \delta^d(x_1 - x_2) \\ -\,\delta^d(x_1 - x_2) & 0 \end{pmatrix}.$$

The cubic interaction can be described by m in the following form:

$$m\,(\,c\,(x_1) \otimes c\,(x_2)\,) = \,-\,\frac{g}{2} \int d^d x\, \delta^d(x - x_1)\,\delta^d(x - x_2)\,d\,(x)\,,$$

$$m\,(\,c\,(x_1) \otimes d\,(x_2)\,) = 0\,, \quad m\,(\,d\,(x_1) \otimes c\,(x_2)\,) = 0\,, \quad m\,(\,d\,(x_1) \otimes d\,(x_2)\,) = 0\,.$$

The A_∞ structure of the classical action is described by $\mathbf{Q} + \boldsymbol{m}$. The A_∞ relations are trivially satisfied for this theory without gauge symmetries.

When we consider on-shell scattering amplitudes, we use the projection onto on-shell states. In the case of the projection onto on-shell states, $\mathbf{P}\,\mathbf{Q}\,\mathbf{P}$ vanishes and on-shell scattering amplitudes at the tree level can be calculated from

$$\mathbf{P}\,\boldsymbol{m}\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\,\mathbf{P}\,.$$

When we discuss the quantum theory, we need to include counterterms, and the counterterms are included in $\boldsymbol{m}$. On-shell scattering amplitudes including loop diagrams can be calculated from

$$\mathbf{P}\,\boldsymbol{m}\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P}\,.$$

The operator $\mathbf{U}$ is defined by

$$\mathbf{U} = \int d^d x \, \boldsymbol{c}(x) \, \boldsymbol{d}(x) \,,$$

where $\boldsymbol{c}(x)$ is a degree-even coderivation with $\pi_1 \, \boldsymbol{c}(x)$ given by

$$\pi_1 \, \boldsymbol{c}(x) \, \mathbf{1} = c(x) \,, \quad \pi_1 \, \boldsymbol{c}(x) \, \pi_n = 0 \,,$$

for $n > 0$ and $\boldsymbol{d}(x)$ is a degree-odd coderivation with $\pi_1 \, \boldsymbol{d}(x)$ given by

$$\pi_1 \, \boldsymbol{d}(x) \, \mathbf{1} = d(x) \,, \quad \pi_1 \, \boldsymbol{d}(x) \, \pi_n = 0$$

for $n > 0$. The two coderivations $\boldsymbol{c}(x)$ and $\boldsymbol{d}(x)$ commute so that their order in $\mathbf{U}$ does not matter.

The operator $\mathbf{U}$ is normalized such that

$$(\omega \otimes \mathbb{I}) (\mathbb{I} \otimes U) = \mathbb{I}$$

is satisfied, where U is a map from $\mathcal{H}^{\otimes 0}$ to $\mathcal{H}^{\otimes 2}$ given by

$$U = \pi_2 \, \mathbf{U} \, \pi_0$$

and ω is a map from $\mathcal{H}^{\otimes 2}$ to $\mathcal{H}^{\otimes 0}$ with

$$\omega \, (\, \Phi_1 \otimes \Phi_2 \,) = \omega \, (\, \Phi_1 \, , \, \Phi_2 \,) \, \mathbf{1}$$

for Φ_1 and Φ_2 in $\mathcal{H}$.

A_∞ algebras are extended to quantum A_∞ algebras in the quantum theory. The quantum A_∞ relations are again trivially satisfied for this theory without gauge symmetries.

If we recall that the projection onto the massless sector corresponds to integrating out massive fields, carrying out the path integral *completely* should correspond to the projection with

$$P = 0\,.$$

The associated operator $\mathbf{P}$ corresponds to the projection onto $\mathcal{H}^{\otimes 0}$:

$$\mathbf{P} = \pi_0\,.$$

This may result in a trivial theory in the classical case, but it can be nontrivial for the quantum case and in fact it is exactly what we do when we calculate correlation functions.

When $P=0$, the conditions for h are given by

$$Q\,h+h\,Q=\mathbb{I}\,,\qquad h^2=0\,.$$

To define the path integral of the free theory in Minkowski space, we use the $i\epsilon$ prescription and as a result we obtain the Feynman propagator given by

$$\Delta(x-y)=\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{ik\,(x-y)}}{k^2+m^2-i\epsilon}$$

As we define correlation functions in Minkowski space as vacuum expectation values associated with the unique vacuum in the quantum theory, we use the Feynman propagation to define the operator h as follows:

$$h\,c(x)=0\,,\qquad h\,d(x)=\int d^dy\,\Delta(x-y)\,c(y)\,.$$

Since $P=0$, the associated operator $\boldsymbol{h}$ is given by

$$\boldsymbol{h}=h\,\pi_1+\sum_{n=2}^{\infty}(\,\mathbb{I}^{\otimes(n-1)}\otimes h\,)\,\pi_n\,.$$

We define $\boldsymbol{f}$ by

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,,$$

and we write

$$\mathbf{P}\,\boldsymbol{m}\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P} = \mathbf{P}\,\boldsymbol{m}\,\boldsymbol{f}\,\mathbf{P}\,.$$

While $\mathbf{P}\,\boldsymbol{m}\,\boldsymbol{f}\,\mathbf{P}$ vanishes, $\boldsymbol{f}$ is nonvanishing and this operator plays a central role in generating Feynman diagrams.

We claim that information on correlation functions is encoded in $\boldsymbol{f}\,\mathbf{1}$ associated with the case where $P = 0$.

More explicitly, we claim that correlation functions are given by

$$\langle \Phi^{\otimes n} \rangle = \pi_n \, \boldsymbol{f} \, \mathbf{1}$$

with

$$\Phi^{\otimes n} = \underbrace{\Phi \otimes \Phi \otimes \ldots \otimes \Phi}_{n}$$

and

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}} \,.$$

Since the left-hand side can be expanded as

$$\langle\, \Phi^{\otimes n} \,\rangle = \langle\, \underbrace{\Phi \otimes \Phi \otimes \ldots \otimes \Phi}_{n} \,\rangle$$

$$= \int d^d x_1 d^d x_2 \ldots d^d x_n \,\langle\, \varphi(x_1)\, \varphi(x_2)\, \ldots\, \varphi(x_n) \,\rangle\, c(x_1) \otimes c(x_2) \otimes \ldots \otimes c(x_n)\,,$$

the formula states that the correlation functions appear as coefficients when we expand the right-hand side:

$$\pi_n\, \boldsymbol{f}\, \mathbf{1}$$

$$= \int d^d x_1 d^d x_2 \ldots d^d x_n \,\langle\, \varphi(x_1)\, \varphi(x_2)\, \ldots\, \varphi(x_n) \,\rangle\, c(x_1) \otimes c(x_2) \otimes \ldots \otimes c(x_n)\,.$$

Since

$$\omega\,(\,c(x')\,,d(x)\,)=\delta^d(x'-x)\,,$$

the correlation functions $\langle\,\varphi(x_1)\,\varphi(x_2)\,\ldots\,\varphi(x_n)\,\rangle$ can be extracted from $\langle\,\Phi^{\otimes n}\,\rangle$ as

$$\langle\,\varphi(x_1)\,\varphi(x_2)\,\ldots\,\varphi(x_n)\,\rangle=\omega_n\,(\,\langle\,\Phi^{\otimes n}\,\rangle\,,d(x_1)\otimes d(x_2)\otimes\ldots\otimes d(x_n)\,)\,,$$

where

$$\omega_n\,(\,\Phi_1\otimes\Phi_2\otimes\ldots\otimes\Phi_n\,,\widetilde{\Phi}_1\otimes\widetilde{\Phi}_2\otimes\ldots\otimes\widetilde{\Phi}_n\,)=\prod_{i=1}^{n}\omega\,(\,\Phi_i\,,\widetilde{\Phi}_i\,)\,.$$

Then the formula can be expressed as

$$\langle\,\varphi(x_1)\,\varphi(x_2)\,\ldots\,\varphi(x_n)\,\rangle=\omega_n\,(\,\pi_n\,\boldsymbol{f}\,\mathbf{1}\,,d(x_1)\otimes d(x_2)\otimes\ldots\otimes d(x_n)\,)\,.$$

Let us first demonstrate that correlation functions of the free theory are correctly reproduced. We denote correlation functions of the free theory by $\langle\, \varphi(x_1)\, \varphi(x_2)\, \ldots\, \varphi(x_n)\, \rangle^{(0)}$. The two-point function can be calculated from $\pi_2\, \boldsymbol{f}\, \mathbf{1}$:

$$\pi_2\, \boldsymbol{f}\, \mathbf{1} \;=\; -\, i\hbar\, \pi_2\, \boldsymbol{h}\, \mathbf{U}\, \mathbf{1}\,.$$

The operator $\mathbf{U}$ acting on $\mathbf{1}$ generates the following element of $\mathcal{H} \otimes \mathcal{H}$:

$$\mathbf{U}\, \mathbf{1} = \int d^d x\, (\, c(x) \otimes d(x) + d(x) \otimes c(x)\,)\,.$$

The action of $\boldsymbol{h}$ on $\mathcal{H} \otimes \mathcal{H}$ is given by

$$\boldsymbol{h}\, \pi_2 = (\, \mathbb{I} \otimes h\,)\, \pi_2\,,$$

so we have

$$\boldsymbol{h}\, \mathbf{U}\, \mathbf{1} = \int d^d x\, c(x) \otimes h\, d(x)\,.$$

We thus find

$$\begin{aligned} \pi_2 \, \boldsymbol{f} \, \mathbf{1} &= \; - \, i\hbar \, \pi_2 \, \boldsymbol{h} \, \mathbf{U} \, \mathbf{1} \\ &= \; - \, i\hbar \int d^d x \int d^d y \, [\, c(x) \otimes \Delta(x-y) \, c(y) \,] \, , \end{aligned}$$

and $\omega_2 \, (\, \pi_2 \, \boldsymbol{f} \, \mathbf{1} \, , d(x_1) \otimes d(x_2) \,)$ is given by

$$\omega_2 \, (\, \pi_2 \, \boldsymbol{f} \, \mathbf{1} \, , d(x_1) \otimes d(x_2) \,) = \; - \, i\hbar \, \Delta \, (x_1 \, - \, x_2 \,) \, .$$

This correctly reproduces the two-point function of the free theory:

$$\langle \, \varphi(x_1) \, \varphi(x_2) \, \rangle^{(0)} = \frac{\hbar}{i} \, \Delta \, (x_1 \, - \, x_2 \,) \, .$$

The four-point function can be calculated from $\pi_4\, \boldsymbol{f}\, \mathbf{1}$:

$$
\begin{aligned}
\pi_4\, \boldsymbol{f}\, \mathbf{1} &= -\,\hbar^2\, \pi_4\, \boldsymbol{h}\, \mathbf{U}\, \boldsymbol{h}\, \mathbf{U}\, \mathbf{1} \\
&= -\,\hbar^2 \int d^d x \int d^d x' \,(\, c(x') \otimes c(x) \otimes h\, d(x) \otimes h\, d(x') \\
&\qquad\qquad + c(x) \otimes c(x') \otimes h\, d(x) \otimes h\, d(x') \\
&\qquad\qquad + c(x) \otimes h\, d(x) \otimes c(x') \otimes h\, d(x')\,) \\
&= -\,\hbar^2 \int d^d x \int d^d x' \int d^d y \int d^d y' \; \mathcal{F}\,(x, y, x', y')\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{F}\,(x, y, x', y') = {} & c(x') \otimes c(x) \otimes \Delta(x-y)\, c(y) \otimes \Delta(x'-y')\, c(y') \\
& + c(x) \otimes c(x') \otimes \Delta(x-y)\, c(y) \otimes \Delta(x'-y')\, c(y') \\
& + c(x) \otimes \Delta(x-y)\, c(y) \otimes c(x') \otimes \Delta(x'-y')\, c(y')\,.
\end{aligned}
$$

The symplectic form $\omega_4\,(\,\pi_4\,\boldsymbol{f}\,\mathbf{1}\,, d(x_1)\otimes d(x_2)\otimes d(x_3)\otimes d(x_4)\,)$ is

$$\begin{aligned}
&\omega_4\,(\,\pi_4\,\boldsymbol{f}\,\mathbf{1}\,, d(x_1)\otimes d(x_2)\otimes d(x_3)\otimes d(x_4)\,) \\
&\;=\;-\,\hbar^2\,[\,\Delta(x_2-x_3)\,\Delta(x_1-x_4)+\Delta(x_1-x_3)\,\Delta(x_2-x_4) \\
&\qquad\qquad +\Delta(x_1-x_2)\,\Delta(x_3-x_4)\,]\,,
\end{aligned}$$

so the four-point function is given by

$$\begin{aligned}
&\langle\,\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4)\,\rangle^{(0)} \\
&\;=\langle\,\varphi(x_2)\,\varphi(x_3)\,\rangle^{(0)}\,\langle\,\varphi(x_1)\,\varphi(x_4)\,\rangle^{(0)}+\langle\,\varphi(x_1)\,\varphi(x_3)\,\rangle^{(0)}\,\langle\,\varphi(x_2)\,\varphi(x_4)\,\rangle^{(0)} \\
&\qquad+\langle\,\varphi(x_1)\,\varphi(x_2)\,\rangle^{(0)}\,\langle\,\varphi(x_3)\,\varphi(x_4)\,\rangle^{(0)}\,.
\end{aligned}$$

We have thus reproduced Wick's theorem for four-point functions, and it is not difficult to extend the analysis to six-point functions and further.

Let us next consider φ^3 theory. The action including counterterms is given by

$$S = \int d^d x \left[-\frac{1}{2} Z_\varphi \, \partial_\mu \varphi(x) \, \partial^\mu \varphi(x) - \frac{1}{2} Z_m \, m^2 \, \varphi(x)^2 + \frac{1}{6} Z_g \, g \, \varphi(x)^3 + Y \varphi(x) \right],$$

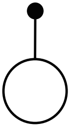
where Y, Z_φ, Z_m, and Z_g are constants. We expand Y, Z_φ, Z_m, and Z_g in g as follows:

$$\begin{aligned} Y &= g\hbar \, Y^{(1)} + O(g^3) \,, \\ Z_\varphi &= 1 + g^2\hbar \, Z_\varphi^{(1)} + O(g^4) \,, \\ Z_m &= 1 + g^2\hbar \, Z_m^{(1)} + O(g^4) \,, \\ Z_g &= 1 + g^2\hbar \, Z_g^{(1)} + O(g^4) \,. \end{aligned}$$

The one-point function is given by

$$\begin{aligned} \langle \, \varphi(x_1) \, \rangle &= \langle \, \varphi(x_1) \, \rangle^{(1)} + O(g^2) \,, \\ \langle \, \varphi(x_1) \, \rangle^{(1)} &= \frac{g\hbar}{m^2} \left[-\frac{i}{2} \int \frac{d^d p}{(2\pi)^d} \, \frac{1}{p^2 + m^2 - i\epsilon} + Y^{(1)} \right]. \end{aligned}$$

We have reproduced the contribution from the one-loop tadpole diagram:



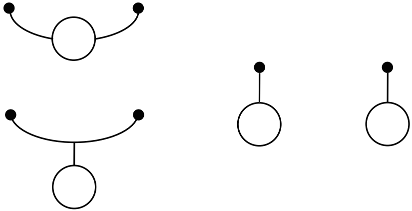
Note that the correct symmetry factor appeared.

The two-point function is given by

$$\begin{aligned}\langle\, \varphi(x_1)\, \varphi(x_2)\,\rangle &= \omega_2\, (\, \pi_2\, \boldsymbol{f}\, \mathbf{1}\,,\, d(x_1) \otimes d(x_2)\,) \\ &= \langle\, \varphi(x_1)\, \varphi(x_2)\,\rangle^{(0)} + \langle\, \varphi(x_1)\, \varphi(x_2)\,\rangle_C^{(1)} \\ &\qquad + \langle\, \varphi(x_1)\,\rangle^{(1)}\, \langle\, \varphi(x_2)\,\rangle^{(1)} + O(g^3)\,.\end{aligned}$$

The connected part is given by

$$
\begin{aligned}
&\langle\, \varphi(x_1)\, \varphi(x_2)\, \rangle_C^{(1)} \\
&= i g^2 \hbar^2 \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{-ip\,(x_1 - x_2)}}{(\, p^2 + m^2 - i\epsilon\,)^2} \\
&\qquad \times \left[\, \frac{i}{2} \int \frac{d^d \ell}{(2\pi)^d}\, \frac{1}{(\ell + p)^2 + m^2 - i\epsilon}\, \frac{1}{\ell^2 + m^2 - i\epsilon} + Z_\varphi^{(1)}\, p^2 + Z_m^{(1)}\, m^2 \,\right] \\
&\quad - i g^2 \hbar^2 \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{-ip\,(x_1 - x_2)}}{m^2\,(\, p^2 + m^2 - i\epsilon\,)^2} \left[\, - \frac{i}{2} \int \frac{d^d \ell}{(2\pi)^d}\, \frac{1}{\ell^2 + m^2 - i\epsilon} + Y^{(1)} \,\right].
\end{aligned}
$$



Through the process of renormalization, we found the description of the 1PI effective action in terms of A_∞ algebras.

Okawa and Shibuya, *in preparation*

We can show that correlation functions from our formula satisfy the Schwinger-Dyson equations as an immediate consequence of the structure

$$(\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U})\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{1} = \mathbf{1}\,.$$

We can extend the proof to correlation functions involving Dirac fields.

Konosu and Okawa, arXiv:2305.11634

Konosu, arXiv:2305.13103

4. Renormalization group

The construction of $\boldsymbol{h}$ from h is not unique. In addition to $\mathbf{P}$ for $P = 0$, let us introduce $\mathbf{P}_\Lambda$ for the projection onto modes below the energy scale Λ, and use $\boldsymbol{h}$ given by

$$\boldsymbol{h} = \boldsymbol{h}_H + \boldsymbol{h}_L\,,$$

where the propagator $\boldsymbol{h}_H$ for high-energy modes satisfy

$$\mathbf{Q}\,\boldsymbol{h}_H + \boldsymbol{h}_H\,\mathbf{Q} = \mathbf{I} - \mathbf{P}_\Lambda\,, \qquad \boldsymbol{h}_H\,\mathbf{P}_\Lambda = 0\,, \qquad \mathbf{P}_\Lambda\,\boldsymbol{h}_H = 0\,, \qquad \boldsymbol{h}_H^2 = 0$$

and the propagator $\boldsymbol{h}_L$ for low-energy modes satisfy

$$\mathbf{Q}\,\boldsymbol{h}_L + \boldsymbol{h}_L\,\mathbf{Q} = \mathbf{P}_\Lambda - \mathbf{P}\,, \quad \boldsymbol{h}_L\,(\,\mathbf{I} - \mathbf{P}_\Lambda\,) = 0\,, \quad (\,\mathbf{I} - \mathbf{P}_\Lambda\,)\,\boldsymbol{h}_L = 0\,, \quad \boldsymbol{h}_L^2 = 0\,.$$

Then we can write $\boldsymbol{f}\,\mathbf{P}$ as

$$
\begin{aligned}
&\frac{1}{\mathbf{I}+\boldsymbol{h}\,\boldsymbol{m}+i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P} \\
&=\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}+i\hbar\,\boldsymbol{h}_H\,\mathbf{U}}\left(\mathbf{I}+\boldsymbol{h}_L\,(\,\boldsymbol{m}+i\hbar\,\mathbf{U}\,)\,\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}+i\hbar\,\boldsymbol{h}_H\,\mathbf{U}}\right)^{-1}\mathbf{P} \\
&=\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}+i\hbar\,\boldsymbol{h}_H\,\mathbf{U}}\,\mathbf{P}_\Lambda\,\frac{1}{\mathbf{I}+\boldsymbol{h}_L\,\boldsymbol{m}_\Lambda+i\hbar\,\boldsymbol{h}_L\,\mathbf{U}}\,\mathbf{P}\,,
\end{aligned}
$$

where

$$
\boldsymbol{m}_\Lambda=\mathbf{P}_\Lambda\left[\,(\,\boldsymbol{m}+i\hbar\,\mathbf{U}\,)\,\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}+i\hbar\,\boldsymbol{h}_H\,\mathbf{U}}-i\hbar\,\mathbf{U}\right]\mathbf{P}_\Lambda\,.
$$

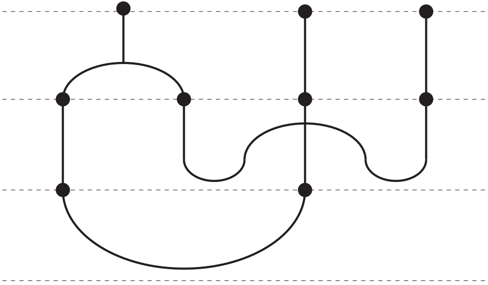
The operator $\boldsymbol{m}_\Lambda$ describes the Wilsonian effective action at the energy scale Λ, and correlation functions are calculated from a product of the operator for high-energy modes and the operator for low-energy modes.

We can further introduce a sequence of projections and write $\boldsymbol{f}\,\mathbf{P}$ as

$$\frac{1}{\mathbf{I}+\boldsymbol{h}\,\boldsymbol{m}+i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P}=\prod_i\frac{1}{\mathbf{I}+\boldsymbol{h}_i\,\boldsymbol{m}_i+i\hbar\,\boldsymbol{h}_i\,\mathbf{U}}\,\mathbf{P}_i$$

with

$$\boldsymbol{h}=\sum_i\boldsymbol{h}_i\,.$$



5. Summary

We proposed the formula for correlation functions using A_∞ algebras:

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n\, \boldsymbol{f}\, \mathbf{1}\,,$$

where

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} + i\hbar\, \boldsymbol{h}\, \mathbf{U}}\,.$$

For a scalar field $\varphi(x)$, Φ is given by

$$\Phi = \int d^d x\, \varphi(x)\, c(x)\,.$$

For a Dirac field $\Psi(x)$, Φ is given by

$$\Phi = \int d^d x\, (\, \overline{\theta}_\alpha(x)\, \Psi_\alpha(x) + \overline{\Psi}_\alpha(x)\, \theta_\alpha(x)\,)\,.$$

We proved that correlation functions from our formula satisfy the Schwinger-Dyson equations as an immediate consequence of the structure

$$(\, \mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} + i\hbar\, \boldsymbol{h}\, \mathbf{U}\,)\, \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} + i\hbar\, \boldsymbol{h}\, \mathbf{U}}\, \mathbf{1} = \mathbf{1}\,.$$

Our ultimate goal is to provide a framework to prove the AdS/CFT correspondence using open string field theory with source terms for gauge-invariant operators. The quantum treatment of open string field theory must be crucial for this program, and we hope that quantum A_∞ algebras will provide us with powerful tools in this endeavor.

<u>Future directions</u>

- 1PI effective action
- LSZ reduction formula
- duality