Nonperturbative correlation functions from homotopy algebras

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1. Introduction

In the first talk, I explained the formula for correlation functions using A_∞ algebras:

$$\langle \, \Phi^{\otimes n} \, \rangle = \pi_n \, \boldsymbol{f} \, \mathbf{1} \, ,$$

where

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h} \, \boldsymbol{m} + i\hbar \, \boldsymbol{h} \, \mathbf{U}} \, .$$

For a scalar field $\varphi(x)$, Φ is given by

$$\Phi = \int d^d x \, \varphi(x) \, c(x) \, .$$

For a Dirac field $\Psi(x)$, Φ is given by

$$\Phi = \int d^d x \, (\, \overline{\theta}_\alpha(x) \, \Psi_\alpha(x) + \overline{\Psi}_\alpha(x) \, \theta_\alpha(x) \,) \, .$$

It was shown in perturbation theory that correlation functions based on this formula satisfy the Schwinger-Dyson equations when the inverse of the operator $\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}$ is defined by

$$\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}} = \mathbf{I} + \sum_{n=1}^{\infty} (-1)^n \left(\boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}\right)^n .$$

It is possible, however, that the inverse of the operator $\mathbf{I}+\boldsymbol{h}\,\boldsymbol{m}+i\hbar\,\boldsymbol{h}\,\mathbf{U}$ exists for finite coupling constants, and in that case our formula may be regarded as a nonperturbative definition of correlation functions for finite coupling constants.

In the second talk, we present evidence that this is indeed the case for scalar field theories in zero dimensions.

The plan of the talk

1. Introduction
2. Scalar field theories in zero dimensions
3. Perturbation theory around a nontrivial solution
4. New formula
5. Evidence for the claim
6. Conclusions and discussion

2. Scalar field theories in zero dimensions

In the Euclidean case, we consider the action S given by

$$S = \frac{1}{2}\, m^2\, \varphi^2 + \frac{1}{3}\, g\, \varphi^3 + \frac{1}{4}\, \lambda\, \varphi^4 \,.$$

In the path integral formalism, the partition function Z is given by

$$Z = \int_{-\infty}^{\infty} d\varphi\, e^{-\frac{S}{\hbar}} \,,$$

and the correlation functions $\langle\, \varphi^n \,\rangle$ are given by

$$\langle\, \varphi^n \,\rangle = \frac{1}{Z} \int_{-\infty}^{\infty} d\varphi\, \varphi^n\, e^{-\frac{S}{\hbar}} \,.$$

In the Lorentzian case, the action differs by a sign and is given by

$$S = \, - \, \frac{1}{2}\, m^2 \, \varphi^2 - \frac{1}{3}\, g \, \varphi^3 - \frac{1}{4}\, \lambda \, \varphi^4 \, .$$

The partition function Z is defined by

$$Z = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d\varphi \, e^{\frac{i}{\hbar} S_\epsilon}$$

with

$$S_\epsilon = \, - \, \frac{1}{2}\, (\, m^2 - i\epsilon \,) \, \varphi^2 - \frac{1}{3}\, g \, \varphi^3 - \frac{1}{4}\, \lambda \, \varphi^4 \, .$$

Correlation functions $\langle \, \varphi^n \, \rangle$ are similarly defined by

$$\langle \, \varphi^n \, \rangle = \frac{1}{Z} \, \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d\varphi \, \varphi^n \, e^{\frac{i}{\hbar} S_\epsilon} \, .$$

In the description in terms of quantum A_∞ algebras, degrees of freedom are described by a vector space which we call $\mathcal{H}$.

For theories without gauge symmetries,

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \,.$$

For gauge theories,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \,.$$

For string field theories,

$$\mathcal{H} = \ldots \oplus \mathcal{H}_{-1} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \ldots \,.$$

The action is described by Φ in $\mathcal{H}_1$.

For scalar field theories,

$$\Phi = \int d^d x \, \varphi(x) \, c(x) \, . \qquad \left(\text{analogous to } | \, \Psi \, \rangle = \int dx \, \psi(x) \, | \, x \, \rangle \right)$$

For theories with a scalar field and a Dirac field,

$$\Phi = \int d^d x \, \varphi(x) \, c(x) + \int d^d x \, (\, \overline{\theta}_\alpha(x) \, \Psi_\alpha(x) + \overline{\Psi}_\alpha(x) \, \theta_\alpha(x) \,) \, .$$

For string field theories,

$$\Phi = \int \frac{d^{26} k}{(2\pi)^{26}} \left[T(k) \, c_1 \, | \, 0; k \, \rangle + A_\mu(k) \, c_1 \, \alpha^\mu_{-1} \, | \, 0; k \, \rangle \; + \ldots \right] .$$

In the case of scalar field theories in zero dimensions, the vector space $\mathcal{H}_1$ is a one-dimensional vector space, and we denote its single basis vector by c. We expand Φ in $\mathcal{H}_1$ as

$$\Phi = \varphi\, c\,.$$

The vector space $\mathcal{H}_2$ is also a one-dimensional vector space, and we denote its single basis vector by d.

We define the symplectic form ω which is a map from $\mathcal{H} \otimes \mathcal{H}$ to a complex number. In the case of scalar field theories in zero dimensions, we define ω by

$$\begin{pmatrix} \omega\,(\,c\,,c\,) & \omega\,(\,c\,,d\,) \\ \omega\,(\,d\,,c\,) & \omega\,(\,d\,,d\,) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\,1 & 0 \end{pmatrix}.$$

The action described by an A_∞ algebra takes a universal form.

In the Euclidean case, the action S is written as

$$S = \frac{1}{2}\,\omega\,(\,\Phi, Q\,\Phi\,) + \sum_{n=2}^{\infty} \frac{1}{n+1}\,\omega\,(\,\Phi\,, m_n\,(\,\Phi \otimes \ldots \otimes \Phi\,)\,)\,.$$

In the Lorentzian case, the action S is written as

$$S = \,-\,\frac{1}{2}\,\omega\,(\,\Phi, Q\,\Phi\,) - \sum_{n=2}^{\infty} \frac{1}{n+1}\,\omega\,(\,\Phi\,, m_n\,(\,\Phi \otimes \ldots \otimes \Phi\,)\,)\,.$$

The kinetic term is described by Q which is a linear map from $\mathcal{H}$ to $\mathcal{H}$, and the cubic interactions are described by m_2 which is a linear map from $\mathcal{H} \otimes \mathcal{H}$ to $\mathcal{H}$. Similarly, the interactions involving $n+1$ fields are described by m_n which is a linear from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}$, where

$$\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}}_{n} \,.$$

For the scalar field theory in zero dimensions, we define Q by

$$Q\, c = m^2\, d\,, \qquad Q\, d = 0\,,$$

and we define m_2 by

$$m_2\,(\, c \otimes c\,) = g\, d\,, \quad m_2\,(\, c \otimes d\,) = 0\,, \quad m_2\,(\, d \otimes c\,) = 0\,, \quad m_2\,(\, d \otimes d\,) = 0\,.$$

Similarly, we define m_3 to be nonvanishing only when it acts on $c \otimes c \otimes c$ and is given by

$$m_3\,(\, c \otimes c \otimes c\,) = \lambda\, d\,.$$

Since the action is quartic, we take m_n to vanish for $n > 3$.

It is convenient to consider linear operators acting on the vector space $T\mathcal{H}$ defined by

$$T\mathcal{H} = \mathcal{H}^{\otimes 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \mathcal{H}^{\otimes 3} \oplus \ldots ,$$

where we also introduced the vector space $\mathcal{H}^{\otimes 0}$. It is a one-dimensional vector space given by multiplying a single basis vector $\mathbf{1}$ by complex numbers. The vector $\mathbf{1}$ satisfies

$$\mathbf{1} \otimes \Phi = \Phi \,, \qquad \Phi \otimes \mathbf{1} = \Phi$$

for any Φ in $\mathcal{H}$.

We denote the projection operator onto $\mathcal{H}^{\otimes n}$ by π_n.

For a map D_n from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}$ with $n = 0, 1, 2, \ldots$, we define an associated operator $\boldsymbol{D}_n$ acting on $T\mathcal{H}$ as follows:

$$\begin{aligned}
\boldsymbol{D}_n \, \pi_m &= 0 \quad \text{for} \quad m < n \,, \\
\boldsymbol{D}_n \, \pi_n &= D_n \, \pi_n \,, \\
\boldsymbol{D}_n \, \pi_{n+1} &= (\, D_n \otimes \mathbb{I} + \mathbb{I} \otimes D_n \,) \, \pi_{n+1} \,, \\
\boldsymbol{D}_n \, \pi_{n+2} &= (\, D_n \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes D_n \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes D_n \,) \, \pi_{n+2} \,, \\
&\;\;\vdots
\end{aligned}$$

Here and in what follows we denote the identity operator on $\mathcal{H}$ by $\mathbb{I}$.

An operator acting on $T\mathcal{H}$ of this form is called a *coderivation*.

In this talk, we often introduce a coderivation $\mathbf{\Phi}$ associated with Φ in $\mathcal{H}$. It is defined by

$$
\begin{aligned}
\mathbf{\Phi}\,\mathbf{1} &= \Phi\,, \\
\mathbf{\Phi}\,\pi_1 &= (\,\Phi \otimes \mathbb{I} + \mathbb{I} \otimes \Phi\,)\,\pi_1\,, \\
\mathbf{\Phi}\,\pi_2 &= (\,\Phi \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \Phi \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \Phi\,)\,\pi_2\,, \\
&\;\;\vdots
\end{aligned}
$$

We define the coderivation $\mathbf{Q}$ associated with Q and the coderivation $\boldsymbol{m}_n$ associated with m_n for each n. We then define $\boldsymbol{m}$ by

$$\boldsymbol{m} = \sum_{n=2}^{\infty} \boldsymbol{m}_n \,,$$

and we define $\mathbf{M}$ by

$$\mathbf{M} = \mathbf{Q} + \boldsymbol{m} \,.$$

When we consider gauge theories, the action described by the coderivation $\mathbf{M}$ is gauge invariant if $\mathbf{M}$ satisfies

$$\mathbf{M}^2 = 0 \,.$$

When we consider projections onto subspaces of $\mathcal{H}$, homotopy algebras have turned out to provide useful tools.

We consider projections which commute with Q, and we denote the projection operator by P:

$$P^2 = P\,, \qquad P\,Q = Q\,P\,.$$

We then promote P on $\mathcal{H}$ to $\mathbf{P}$ on $T\mathcal{H}$ as follows:

$$\begin{aligned}
\mathbf{P}\,\pi_0 &= \pi_0\,,\\
\mathbf{P}\,\pi_1 &= P\,\pi_1\,,\\
\mathbf{P}\,\pi_2 &= (\,P \otimes P\,)\,\pi_2\,,\\
\mathbf{P}\,\pi_3 &= (\,P \otimes P \otimes P)\,\pi_3\,,\\
&\vdots
\end{aligned}$$

The operators $\mathbf{Q}$ and $\mathbf{P}$ satisfy

$$\mathbf{P}^2 = \mathbf{P}\,, \qquad \mathbf{Q}\,\mathbf{P} = \mathbf{P}\,\mathbf{Q}\,.$$

A key ingredient is an operator h satisfying

$$Q\,h + h\,Q = \mathbb{I} - P\,, \qquad h\,P = 0\,, \qquad P\,h = 0\,, \qquad h^2 = 0\,.$$

It is called a contracting homotopy, and physically it describes propagators associated with degrees of freedom which are integrated out.

We then promote h on $\mathcal{H}$ to $\boldsymbol{h}$ on $T\mathcal{H}$ as follows:

$$\begin{aligned}
\boldsymbol{h}\,\pi_0 &= 0\,,\\
\boldsymbol{h}\,\pi_1 &= h\,\pi_1\,,\\
\boldsymbol{h}\,\pi_2 &= (\,h \otimes P + \mathbb{I} \otimes h\,)\,\pi_2\,,\\
\boldsymbol{h}\,\pi_3 &= (\,h \otimes P \otimes P + \mathbb{I} \otimes h \otimes P + \mathbb{I} \otimes \mathbb{I} \otimes h\,)\,\pi_3\,,\\
&\;\;\vdots
\end{aligned}$$

The relations involving Q, P, and h are promoted to the following relations

$$\mathbf{Q}\,\boldsymbol{h} + \boldsymbol{h}\,\mathbf{Q} = \mathbf{I} - \mathbf{P}\,, \qquad \boldsymbol{h}\,\mathbf{P} = 0\,, \qquad \mathbf{P}\,\boldsymbol{h} = 0\,, \qquad \boldsymbol{h}^2 = 0\,,$$

where $\mathbf{I}$ is the identity operator on $T\mathcal{H}$.

When we consider correlation functions, we carry out the path integral *completely*. This should correspond to the projection with

$$P = 0\,.$$

The associated operator $\mathbf{P}$ corresponds to the projection onto $\mathcal{H}^{\otimes 0}$:

$$\mathbf{P} = \pi_0\,.$$

When $P = 0$, the conditions for h are given by

$$Q\,h + h\,Q = \mathbb{I}\,, \qquad h^2 = 0\,.$$

In the case of scalar field theories in zero dimensions, h is given by

$$h\,d = \frac{1}{m^2}\,c\,, \qquad h\,c = 0\,.$$

The associated operator $\boldsymbol{h}$ is given by

$$\boldsymbol{h} = h\,\pi_1 + \sum_{n=2}^{\infty} (\,\mathbb{I}^{\otimes(n-1)} \otimes h\,)\,\pi_n\,.$$

We claim that correlation functions are given by

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n\, \boldsymbol{f}\, \mathbf{1}$$

with

$$\Phi^{\otimes n} = \underbrace{\Phi \otimes \Phi \otimes \ldots \otimes \Phi}_{n},$$

where $\boldsymbol{f}$ for Lorentzian theories is

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} + i\hbar\, \boldsymbol{h}\, \mathbf{U}}$$

and $\boldsymbol{f}$ for Euclidean theories is

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} - \hbar\, \boldsymbol{h}\, \mathbf{U}}\,.$$

In the case of scalar field theories in zero dimensions, the operator $\mathbf{U}$ is defined by

$$\mathbf{U} = \boldsymbol{c}\, \boldsymbol{d}\,,$$

where $\boldsymbol{c}$ and $\boldsymbol{d}$ are coderivations associated with c and d, respectively.

The correlation functions for scalar field theories in d dimensions can be extracted by expanding $\langle\, \Phi^{\otimes n} \,\rangle$ as

$$\langle\, \Phi^{\otimes n} \,\rangle = \langle\, \underbrace{\Phi \otimes \Phi \otimes \ldots \otimes \Phi}_{n} \,\rangle$$

$$= \int d^d x_1 d^d x_2 \ldots d^d x_n \,\langle\, \varphi(x_1)\, \varphi(x_2)\, \ldots\, \varphi(x_n) \,\rangle\, c(x_1) \otimes c(x_2) \otimes \ldots \otimes c(x_n)\,.$$

For scalar field theories in zero dimensions, this simplifies as follows:

$$\langle\, \Phi^{\otimes n} \,\rangle = \langle\, \varphi^n \,\rangle\, \underbrace{c \otimes c \otimes \ldots \otimes c}_{n}\,.$$

We can show that correlation functions from our formula satisfy the Schwinger-Dyson equations as an immediate consequence of the structure

$$(\, \mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} + i\hbar\, \boldsymbol{h}\, \mathbf{U}\,)\, \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} + i\hbar\, \boldsymbol{h}\, \mathbf{U}}\, \mathbf{1} = \mathbf{1}\,.$$

The operator $\boldsymbol{f}$

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}$$

or

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \hbar\,\boldsymbol{h}\,\mathbf{U}}$$

is a linear map from $T\mathcal{H}_1$ to $T\mathcal{H}_1$, where $T\mathcal{H}_1$ is defined by

$$T\mathcal{H}_1 = \mathcal{H}^{\otimes 0} \oplus \mathcal{H}_1 \oplus \mathcal{H}_1^{\otimes 2} \oplus \mathcal{H}_1^{\otimes 3} \oplus \ldots$$

with

$$\mathcal{H}_1^{\otimes n} = \underbrace{\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_1}_{n}$$

for $n > 0$.

For scalar field theories in zero dimensions, a linear map $\mathbf{A}$ from $T\mathcal{H}_1$ to $T\mathcal{H}_1$ with

$$\begin{aligned}
\mathbf{A}\,\mathbf{1} &= \mathbf{A}_{00}\,\mathbf{1} + \mathbf{A}_{01}\,c + \mathbf{A}_{02}\,c \otimes c + \mathbf{A}_{03}\,c \otimes c \otimes c + \ldots\,, \\
\mathbf{A}\,c &= \mathbf{A}_{10}\,\mathbf{1} + \mathbf{A}_{11}\,c + \mathbf{A}_{12}\,c \otimes c + \mathbf{A}_{13}\,c \otimes c \otimes c + \ldots\,, \\
\mathbf{A}\,c \otimes c &= \mathbf{A}_{20}\,\mathbf{1} + \mathbf{A}_{21}\,c + \mathbf{A}_{22}\,c \otimes c + \mathbf{A}_{23}\,c \otimes c \otimes c + \ldots\,, \\
\mathbf{A}\,c \otimes c \otimes c &= \mathbf{A}_{30}\,\mathbf{1} + \mathbf{A}_{31}\,c + \mathbf{A}_{32}\,c \otimes c + \mathbf{A}_{33}\,c \otimes c \otimes c + \ldots\,, \\
&\vdots
\end{aligned}$$

is represented in the matrix form as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} & \mathbf{A}_{02} & \mathbf{A}_{03} & \cdots \\ \mathbf{A}_{10} & \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \cdots \\ \mathbf{A}_{20} & \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \cdots \\ \mathbf{A}_{30} & \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Therefore, the formula can be expressed as

$$\langle\, \varphi^n \,\rangle = \boldsymbol{f}_{n0}\,.$$

In the matrix form, the operators $\boldsymbol{h}\,\boldsymbol{m}_2$, $\boldsymbol{h}\,\boldsymbol{m}_3$ and $\boldsymbol{h}\,\mathbf{U}$ are given by

$$\boldsymbol{h}\,\boldsymbol{m}_2 = \frac{g}{m^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \boldsymbol{h}\,\boldsymbol{m}_3 = \frac{\lambda}{m^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\boldsymbol{h}\,\mathbf{U} = \frac{1}{m^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We set $g = 0$ and let us consider φ^4 theory in the Euclidean case. We first verify that $\boldsymbol{f}_{20}$ reproduces $\langle\, \varphi^2 \,\rangle$ in perturbation theory. The action S is

$$S = \frac{1}{2}\, m^2\, \varphi^2 + \frac{1}{4}\, \lambda\, \varphi^4\,,$$

and we further set $m^2 = 1$ and $\hbar = 1$.

The two-point function $\langle\, \varphi^2 \,\rangle$ in the path integral formalism is given by

$$\begin{aligned}\langle\, \varphi^2 \,\rangle = 1 - 3\lambda + 24\lambda^2 - 297\lambda^3 + 4896\lambda^4 - 100278\lambda^5 + 2450304\lambda^6 \\ - 69533397\lambda^7 + 2247492096\lambda^8 + O(\lambda^9)\,.\end{aligned}$$

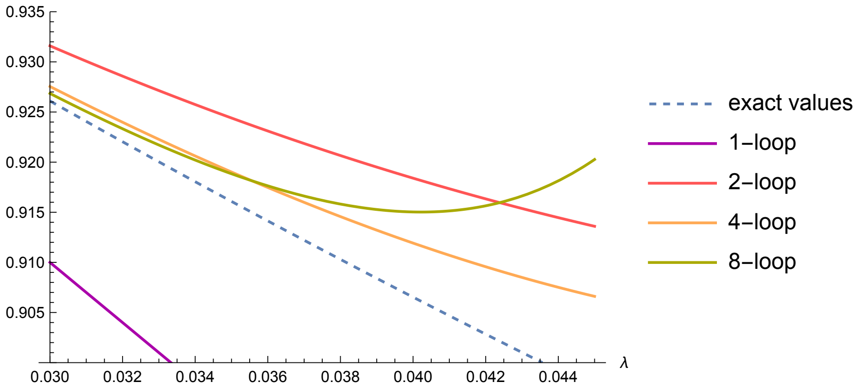
In perturbation theory, $\boldsymbol{f}$ is defined by

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}_3 - \hbar\,\boldsymbol{h}\,\mathbf{U}} = \mathbf{I} + \sum_{n=1}^{\infty} (-1)^n \left(\boldsymbol{h}\,\boldsymbol{m}_3 - \hbar\,\boldsymbol{h}\,\mathbf{U}\right)^n .$$

We calculate $\boldsymbol{f}_{20}$ to find

$$\begin{aligned}\boldsymbol{f}_{20} = 1 - 3\lambda + 24\lambda^2 - 297\lambda^3 + 4896\lambda^4 - 100278\lambda^5 + 2450304\lambda^6 \\ - 69533397\lambda^7 + 2247492096\lambda^8 + O(\lambda^9) .\end{aligned}$$

This correctly produces the perturbative expansion of $\langle\,\varphi^2\,\rangle$.


0.935
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0.034
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0.038
0.040
0.042
0.044
λ
exact values
1−loop
2−loop
4−loop
8−loop

It is possible that the inverse of the operator $\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}_3 - \hbar\,\boldsymbol{h}\,\mathbf{U}$ exists non-perturbatively for finite values of λ. When $m = 1$ and $\hbar = 1$, we have

$$\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}_3 - \hbar\,\boldsymbol{h}\,\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \lambda & 0 & 0 & \dots \\ -1 & 0 & 1 & 0 & \lambda & 0 & \dots \\ 0 & -2 & 0 & 1 & 0 & \lambda & \dots \\ 0 & 0 & -3 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & -4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}_3 - \hbar\,\boldsymbol{h}\,\mathbf{U}} = \begin{pmatrix} \boldsymbol{f}_{00} & \boldsymbol{f}_{01} & \boldsymbol{f}_{02} & \boldsymbol{f}_{03} & \boldsymbol{f}_{04} & \boldsymbol{f}_{05} & \cdots \\ \boldsymbol{f}_{10} & \boldsymbol{f}_{11} & \boldsymbol{f}_{12} & \boldsymbol{f}_{13} & \boldsymbol{f}_{14} & \boldsymbol{f}_{15} & \cdots \\ \boldsymbol{f}_{20} & \boldsymbol{f}_{21} & \boldsymbol{f}_{22} & \boldsymbol{f}_{23} & \boldsymbol{f}_{24} & \boldsymbol{f}_{25} & \cdots \\ \boldsymbol{f}_{30} & \boldsymbol{f}_{31} & \boldsymbol{f}_{32} & \boldsymbol{f}_{33} & \boldsymbol{f}_{34} & \boldsymbol{f}_{35} & \cdots \\ \boldsymbol{f}_{40} & \boldsymbol{f}_{41} & \boldsymbol{f}_{42} & \boldsymbol{f}_{43} & \boldsymbol{f}_{44} & \boldsymbol{f}_{45} & \cdots \\ \boldsymbol{f}_{50} & \boldsymbol{f}_{51} & \boldsymbol{f}_{52} & \boldsymbol{f}_{53} & \boldsymbol{f}_{54} & \boldsymbol{f}_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us evaluate the inverse of $\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}_3 - \hbar\,\boldsymbol{h}\,\mathbf{U}$ by truncation to an N by N matrix. When $N = 25$, $\boldsymbol{f}_{20}$ and $\boldsymbol{f}_{40}$ are

$$\boldsymbol{f}_{20} = \frac{1 + 140\lambda + 6660\lambda^2 + 129360\lambda^3 + 957075\lambda^4 + 1853460\lambda^5}{1 + 143\lambda + 7065\lambda^2 + 147420\lambda^3 + 1267350\lambda^4 + 3615885\lambda^5 + 1514205\lambda^6}\,,$$

$$\boldsymbol{f}_{40} = \frac{3 + 405\lambda + 18060\lambda^2 + 310275\lambda^3 + 1762425\lambda^4 + 1514205\lambda^5}{1 + 143\lambda + 7065\lambda^2 + 147420\lambda^3 + 1267350\lambda^4 + 3615885\lambda^5 + 1514205\lambda^6}\,.$$

For $\lambda = 0.04$, we find

$$\langle\, \varphi^2 \,\rangle \simeq 0.90653672\,,$$
$$\boldsymbol{f}_{20} \simeq 0.90653666$$

and

$$\langle\, \varphi^4 \,\rangle \simeq 2.3365819\,,$$
$$\boldsymbol{f}_{40} \simeq 2.3365834\,.$$

$\langle\, \varphi^2 \,\rangle$	$\lambda = 0.04$	$\lambda = 0.2$	$\lambda = 1.5$	$\lambda = 3$
exact	0.9065367244	0.7240590202	0.4066915207	0.3130156270
N	$\lambda = 0.04$	$\lambda = 0.2$	$\lambda = 1.5$	$\lambda = 3$
10	0.9059745348	0.7024793388	0.2685512367	0.1574468085
25	0.9065366639	0.7237546945	0.3751623774	0.2543859219
50	0.9065367244	0.7240552164	0.4002397294	0.2932452874
100	0.9065367244	0.7240590258	0.4072861268	0.3167705780

$\langle\, \varphi^4 \,\rangle$	$\lambda = 0.04$	$\lambda = 0.2$	$\lambda = 1.5$	$\lambda = 3$
exact	2.336581891	1.379704899	0.3955389862	0.2289947910
N	$\lambda = 0.04$	$\lambda = 0.2$	$\lambda = 1.5$	$\lambda = 3$
10	2.350636631	1.487603306	0.4876325088	0.2808510638
25	2.336583402	1.381226527	0.4165584151	0.2485380260
50	2.336581891	1.379723918	0.3998401804	0.2355849042
100	2.336581891	1.379704871	0.3951425821	0.2277431407

Let us next consider the Lorentzian case. We set $g = 0$ and consider φ^4 theory. The formula for correlation functions in the Lorentzian case is

$$\langle \varphi^n \rangle = \boldsymbol{f}_{n0} \,,$$

where

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}} \,.$$

When $m = 1$ and $\hbar = 1$, the matrix form of $\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}_3 + i\hbar\,\boldsymbol{h}\,\mathbf{U}$ is

$$\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}_3 + i\hbar\,\boldsymbol{h}\,\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \lambda & 0 & 0 & \dots \\ i & 0 & 1 & 0 & \lambda & 0 & \dots \\ 0 & 2i & 0 & 1 & 0 & \lambda & \dots \\ 0 & 0 & 3i & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 4i & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

When $N = 25$, $\boldsymbol{f}_{20}$ and $\boldsymbol{f}_{40}$ are

$$\boldsymbol{f}_{20} = \frac{-i - 140\lambda + 6660i\lambda^2 + 129360\lambda^3 - 957075i\lambda^4 - 1853460\lambda^5}{1 - 143i\lambda - 7065\lambda^2 + 147420i\lambda^3 + 1267350\lambda^4 - 3615885i\lambda^5 - 1514205\lambda^6},$$
$$\boldsymbol{f}_{40} = \frac{-3 + 405i\lambda + 18060\lambda^2 - 310275i\lambda^3 - 1762425\lambda^4 + 1514205i\lambda^5}{1 - 143i\lambda - 7065\lambda^2 + 147420i\lambda^3 + 1267350\lambda^4 - 3615885i\lambda^5 - 1514205\lambda^6}.$$

For $\lambda = 0.04$, we find

$$\langle \varphi^2 \rangle \simeq 0.1065670 - 0.969384893i\,,$$
$$\boldsymbol{f}_{20} \simeq 0.1065659 - 0.969384873i$$

and

$$\langle \varphi^4 \rangle \simeq -2.6641739 - 0.7653777i\,,$$
$$\boldsymbol{f}_{40} \simeq -2.6641477 - 0.7653782i\,.$$

$\langle \varphi^2 \rangle$	$\lambda = 0.04$	$\lambda = 0.5$
exact	$0.106567 - 0.969385i$	$0.280132 - 0.576152i$
N	$\lambda = 0.04$	$\lambda = 0.5$
10	$0.105205 - 0.969450i$	$0.444071 - 0.548360i$
25	$0.106566 - 0.969385i$	$0.274637 - 0.597967i$
50	$0.106567 - 0.969385i$	$0.277626 - 0.574320i$
100	$0.106567 - 0.969385i$	$0.279980 - 0.576085i$

$\langle \varphi^4 \rangle$	$\lambda = 0.04$	$\lambda = 0.5$
exact	$-2.66417 - 0.76538i$	$-0.560264 - 0.847696i$
N	$\lambda = 0.04$	$\lambda = 0.5$
10	$-2.63013 - 0.76374i$	$-0.888141 - 0.903280i$
25	$-2.66415 - 0.76538i$	$-0.549273 - 0.804066i$
50	$-2.66417 - 0.76538i$	$-0.555251 - 0.851360i$
100	$-2.66417 - 0.76538i$	$-0.559961 - 0.847830i$

We have presented evidence that our formula contains nonperturbative information.

Our formula, however, requires us to choose a free part of the action.

- This is not satisfactory for a nonperturbative definition of correlation functions.
- This also raises the question of background independence.

3. Perturbation theory around a nontrivial solution

Let us consider perturbation theory around a nontrivial solution. Consider the following action:

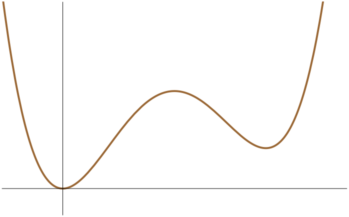
$$S = -\frac{1}{2}\, m^2\, \varphi^2 + \frac{(a+b)\, m^2}{3ab}\, \varphi^3 - \frac{m^2}{4ab}\, \varphi^4$$

with $0 < b < a$. The equation of motion is given by

$$\frac{m^2}{ab}\, \varphi\, (\varphi - b)\, (\varphi - a) = 0\,,$$

and the solutions are

$$\varphi = 0\,, b\,, a\,.$$



Let us consider perturbation theory around the nontrivial solution $\varphi = a$. In quantum field theory, we know what to do. We expand φ as

$$\varphi = a + \widetilde{\varphi}\,,$$

and the action in terms of $\widetilde{\varphi}$ is

$$S = \; - \frac{a^2\,(2b-a)\,m^2}{12b} - \frac{(a-b)\,m^2}{2b}\,\widetilde{\varphi}^{\,2} - \frac{(2a-b)\,m^2}{3ab}\,\widetilde{\varphi}^{\,3} - \frac{m^2}{4ab}\,\widetilde{\varphi}^{\,4}\,.$$

We then calculate $\langle\, \widetilde{\varphi}^{\,n}\,\rangle$, and $\langle\, \varphi^n\,\rangle$ is given by

$$\langle\, \varphi^n\,\rangle = \langle\, (\, a + \widetilde{\varphi}\,)^n\,\rangle = \sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!}\, a^{n-m}\,\langle\, \widetilde{\varphi}^{\,m}\,\rangle\,.$$

Let us describe this procedure in terms of A_∞ algebras. We first need to represent the equations of motion in the language of A_∞ algebras. The equations of motion are usually written as

$$\pi_1 \, \mathbf{M} \, \frac{1}{1-\Phi} = 0 \,,$$

where

$$\frac{1}{1-\Phi} = \sum_{n=0}^{\infty} \Phi^{\otimes n} = \mathbf{1} + \Phi + \Phi \otimes \Phi + \Phi \otimes \Phi \otimes \Phi \ldots$$

This can also be written in a form which is more convenient for us. We introduce the coderivation $\boldsymbol{\Phi}$ associated with Φ in $\mathcal{H}_1$. We then have

$$\frac{1}{1-\Phi} = e^{\boldsymbol{\Phi}} \, \mathbf{1} \,.$$

In terms of $\boldsymbol{\Phi}$, the equations of motion are written as

$$\pi_1 \, \mathbf{M} \, e^{\boldsymbol{\Phi}} \, \mathbf{1} = 0 \,.$$

Suppose that we have a nontrivial solution Φ_* to the equations of motion. We denote the corresponding coderivation by $\boldsymbol{\Phi}_*$:

$$\pi_1 \, \mathbf{M} \, e^{\boldsymbol{\Phi}_*} \, \mathbf{1} = 0 \, .$$

We expand Φ as

$$\Phi = \Phi_* + \widetilde{\Phi} \, ,$$

and the coderivation $\widetilde{\mathbf{M}}$ which describes the action in terms of $\widetilde{\Phi}$ is given by

$$\widetilde{\mathbf{M}} = e^{-\boldsymbol{\Phi}_*} \, \mathbf{M} \, e^{\boldsymbol{\Phi}_*} \, .$$

We decompose $\pi_1 \, \widetilde{\mathbf{M}}$ as

$$\pi_1 \, \widetilde{\mathbf{M}} = \widetilde{Q} \, \pi_1 + \sum_{n=2}^{\infty} \widetilde{m}_n \, \pi_n \, ,$$

and we define the coderivation $\widetilde{\mathbf{Q}}$ associated with $\widetilde{Q}$ and the coderivation $\widetilde{\boldsymbol{m}}_n$ associated with $\widetilde{m}_n$ for each n.

We define $\widetilde{\boldsymbol{m}}$ by

$$\widetilde{\boldsymbol{m}} = \sum_{n=2}^{\infty} \widetilde{\boldsymbol{m}}_n \,,$$

and we define $\widetilde{\mathbf{M}}$ by

$$\widetilde{\mathbf{M}} = \tilde{\mathbf{Q}} + \widetilde{\boldsymbol{m}} \,.$$

We then construct $\widetilde{h}$ satisfying

$$\widetilde{Q}\,\widetilde{h} + \widetilde{h}\,\widetilde{Q} = \mathbb{I}\,, \qquad \widetilde{h}^2 = 0\,,$$

and $\widetilde{\boldsymbol{h}}$ satisfying

$$\widetilde{\mathbf{Q}}\,\widetilde{\boldsymbol{h}} + \widetilde{\boldsymbol{h}}\,\widetilde{\mathbf{Q}} = \mathbf{I} - \mathbf{P}\,, \qquad \widetilde{\boldsymbol{h}}\,\mathbf{P} = 0\,, \qquad \mathbf{P}\,\widetilde{\boldsymbol{h}} = 0\,, \qquad \widetilde{\boldsymbol{h}}^2 = 0$$

with

$$\mathbf{P} = \pi_0 \,.$$

The formula for the correlation functions $\langle\, \widetilde{\Phi}^{\otimes n} \,\rangle$ is given by

$$\langle\, \widetilde{\Phi}^{\otimes n} \,\rangle = \pi_n \, \frac{1}{\mathbf{I} + \widetilde{\boldsymbol{h}}\, \widetilde{\boldsymbol{m}} + i\hbar\, \widetilde{\boldsymbol{h}}\, \mathbf{U}} \, \mathbf{1} \,.$$

The formula for the correlation functions $\langle\, \Phi^{\otimes n} \,\rangle$ for perturbation theory around the solution Φ_* can be written using $\boldsymbol{\Phi}_*$ as

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n \, e^{\boldsymbol{\Phi}_*} \frac{1}{\mathbf{I} + \widetilde{\boldsymbol{h}}\, \widetilde{\boldsymbol{m}} + i\hbar\, \widetilde{\boldsymbol{h}}\, \mathbf{U}} \, \mathbf{1} \,.$$

Let us define $\mathbf{Q}_*$, $\boldsymbol{m}_*$ and $\boldsymbol{h}_*$ by

$$\mathbf{Q}_* = e^{\boldsymbol{\Phi}_*} \, \widetilde{\mathbf{Q}} \, e^{-\boldsymbol{\Phi}_*} \,, \qquad \boldsymbol{m}_* = e^{\boldsymbol{\Phi}_*} \, \widetilde{\boldsymbol{m}} \, e^{-\boldsymbol{\Phi}_*} \,, \qquad \boldsymbol{h}_* = e^{\boldsymbol{\Phi}_*} \, \widetilde{\boldsymbol{h}} \, e^{-\boldsymbol{\Phi}_*} \,.$$

The formula for $\langle\, \Phi^{\otimes n} \,\rangle$ is then

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n \, \frac{1}{\mathbf{I} + \boldsymbol{h}_*\, \boldsymbol{m}_* + i\hbar\, \boldsymbol{h}_*\, \mathbf{U}} \, \mathbf{P} \, \mathbf{1}$$

with

$$\mathbf{P} = e^{\boldsymbol{\Phi}_*} \pi_0 \,.$$

Since

$$\widetilde{\mathbf{M}} = \widetilde{\mathbf{Q}} + \widetilde{\boldsymbol{m}} = e^{-\boldsymbol{\Phi}_*}\,\mathbf{M}\,e^{\boldsymbol{\Phi}_*}\,,$$

we find

$$\mathbf{Q}_* + \boldsymbol{m}_* = \mathbf{M}\,.$$

Namely, the sum of $\mathbf{Q}_*$ and $\boldsymbol{m}_*$ is the same as the sum of $\mathbf{Q}$ and $\boldsymbol{m}$, but $\mathbf{Q}_*$ is different from $\mathbf{Q}$:

$$\mathbf{Q}_* \neq \mathbf{Q}\,.$$

Given the action described by $\mathbf{M}$, we are making different choices for the free part. If we take $\mathbf{Q}$ to be the free part, the correlation functions are given by

$$\langle\, \Phi^{\otimes n}\,\rangle = \pi_n\, \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{1}\,.$$

If we take $\mathbf{Q}_*$ to be the free part, the correlation functions are given by

$$\langle\, \Phi^{\otimes n}\,\rangle = \pi_n\, \frac{1}{\mathbf{I} + \boldsymbol{h}_*\,\boldsymbol{m}_* + i\hbar\,\boldsymbol{h}_*\,\mathbf{U}}\,\mathbf{P}\,\mathbf{1}\,.$$

The operator $\mathbf{P}$ plays an important role. The equations of motion can be written in terms of $\mathbf{P}$ as

$$\mathbf{M}\,\mathbf{P} = 0$$

with

$$\mathbf{P} = e^{\mathbf{\Phi}}\,\pi_0\,,$$

where $\mathbf{\Phi}$ is the coderivation associated with Φ.

4. New formula

Consider the case where the action described by $\mathbf{M}$ and the free part we chose is described by $\mathbf{Q}$. Suppose that we have a solution $\Phi_*^{(0)}$ to the free theory, and we denote the associated coderivation by $\boldsymbol{\Phi}_*^{(0)}$:

$$\mathbf{Q}\,\mathbf{P}^{(0)} = 0$$

with

$$\mathbf{P}^{(0)} = e^{\boldsymbol{\Phi}_*^{(0)}}\,\pi_0\,.$$

We then use $\boldsymbol{h}$ which satisfies

$$\mathbf{Q}\,\boldsymbol{h} + \boldsymbol{h}\,\mathbf{Q} = \mathbf{I} - \mathbf{P}^{(0)}\,, \qquad \mathbf{P}^{(0)}\,\boldsymbol{h} = 0\,, \qquad \boldsymbol{h}\,\mathbf{P}^{(0)} = 0\,, \qquad \boldsymbol{h}^2 = 0$$

to describe the correlation functions as follows:

$$\langle\, \Phi^{\otimes n}\,\rangle = \pi_n\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P}^{(0)}\,\mathbf{1}\,.$$

Let us transform the formula into a form that does not involve the division of the free part and the interaction part. Since

$$\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U} = (\,\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}\,)\,(\,\mathbf{I} + i\hbar\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\boldsymbol{h}\,\mathbf{U}\,)\,,$$

we obtain

$$\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}} = \frac{1}{\mathbf{I} + i\hbar\,\mathbf{H}\,\mathbf{U}}\;\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\,,$$

where

$$\mathbf{H} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\,\boldsymbol{h}\,.$$

The formula for correlation functions is then

$$\langle\,\Phi^{\otimes n}\,\rangle = \pi_n\,\frac{1}{\mathbf{I} + i\hbar\,\mathbf{H}\,\mathbf{U}}\,\mathbf{P}\,\mathbf{1}$$

with

$$\mathbf{P} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\,e^{\boldsymbol{\Phi}_*^{(0)}}\,\pi_0\,.$$

What is the interpretation of $\mathbf{P}$ when the solution $\Phi_*^{(0)}$ does not solve the nonlinear equations of motion?

First, we can show that $\mathbf{P}$ can be written in the form

$$\mathbf{P} = e^{\mathbf{\Phi}_*} \pi_0 \,,$$

where $\mathbf{\Phi}_*$ is a coderivation associated with Φ_* in $\mathcal{H}_1$.

Second, we can show that

$$\mathbf{M}\,\mathbf{P} = 0 \,.$$

Therefore, Φ_* in $\mathcal{H}_1$ solves the nonlinear equations of motion.

For the operator $\mathbf{H}$, we can show that it satisfies

$$\mathbf{M}\,\mathbf{H} + \mathbf{H}\,\mathbf{M} = \mathbf{I} - \mathbf{P} \,, \qquad \mathbf{H}\,\mathbf{P} = 0 \,, \qquad \mathbf{P}\,\mathbf{H} = 0 \,, \qquad \mathbf{H}^2 = 0 \,.$$

New formula

Let us denote the coderivation that describes the action by $\mathbf{M}$. We choose a solution Φ_* to the equations of motion and we denote the associated coderivation by $\boldsymbol{\Phi}_*$:

$$\mathbf{M}\,\mathbf{P} = 0$$

with

$$\mathbf{P} = e^{\boldsymbol{\Phi}_*}\,\pi_0\,.$$

The formula for correlation functions is given by

$$\langle\,\Phi^{\otimes n}\,\rangle = \pi_n\,\frac{1}{\mathbf{I} + i\hbar\,\mathbf{H}\,\mathbf{U}}\,\mathbf{P}\,\mathbf{1}\,,$$

where $\mathbf{H}$ satisfies

$$\mathbf{M}\,\mathbf{H} + \mathbf{H}\,\mathbf{M} = \mathbf{I} - \mathbf{P}\,, \qquad \mathbf{H}\,\mathbf{P} = 0\,, \qquad \mathbf{P}\,\mathbf{H} = 0\,, \qquad \mathbf{H}^2 = 0\,.$$

This formula does not involve the division of the free part and the interaction part.

The operator $\mathbf{H}$ can be constructed in the following way. We define $\widetilde{\mathbf{M}}$ by

$$\widetilde{\mathbf{M}} = e^{-\mathbf{\Phi}_*}\, \mathbf{M}\, e^{\mathbf{\Phi}_*} \,.$$

We decompose $\pi_1 \, \widetilde{\mathbf{M}}$ as

$$\pi_1 \, \widetilde{\mathbf{M}} = \widetilde{Q}\, \pi_1 + \sum_{n=2}^{\infty} \widetilde{m}_n \, \pi_n \,,$$

and we define the coderivation $\widetilde{\mathbf{Q}}$ associated with $\widetilde{Q}$ and the coderivation $\widetilde{\boldsymbol{m}}_n$ associated with $\widetilde{m}_n$ for each n. We then define $\widetilde{\boldsymbol{m}}$ by

$$\widetilde{\boldsymbol{m}} = \sum_{n=2}^{\infty} \widetilde{\boldsymbol{m}}_n \,.$$

We construct $\widetilde{h}$ satisfying

$$\widetilde{Q}\,\widetilde{h} + \widetilde{h}\,\widetilde{Q} = \mathbb{I}\,, \qquad \widetilde{h}^2 = 0\,,$$

and $\widetilde{\boldsymbol{h}}$ satisfying

$$\widetilde{\mathbf{Q}}\,\widetilde{\boldsymbol{h}} + \widetilde{\boldsymbol{h}}\,\widetilde{\mathbf{Q}} = \mathbf{I} - \pi_0\,, \qquad \widetilde{\boldsymbol{h}}\,\pi_0 = 0\,, \qquad \pi_0\,\widetilde{\boldsymbol{h}} = 0\,, \qquad \widetilde{\boldsymbol{h}}^2 = 0\,.$$

The operator $\mathbf{H}$ is given by

$$\mathbf{H} = e^{\boldsymbol{\Phi}_*}\,\frac{1}{\mathbf{I} + \widetilde{\boldsymbol{h}}\,\widetilde{\boldsymbol{m}}}\,\widetilde{\boldsymbol{h}}\,e^{-\boldsymbol{\Phi}_*}$$

The formula for correlation functions is

$$\langle\,\Phi^{\otimes n}\,\rangle = \pi_n\,\frac{1}{\mathbf{I} + i\hbar\,\mathbf{H}\,\mathbf{U}}\,\mathbf{P}\,\mathbf{1} = \pi_n\,e^{\boldsymbol{\Phi}_*}\,\frac{1}{\mathbf{I} + \widetilde{\boldsymbol{h}}\,\widetilde{\boldsymbol{m}} + i\hbar\,\widetilde{\boldsymbol{h}}\,\mathbf{U}}\,\mathbf{1}\,.$$

While the construction of $\mathbf{H}$ is perturbative, this does not imply that the resulting correlation functions are perturbative, as we have demonstrated earlier.

What does the formula describe?

- It reproduces perturbation theory.
- The solution Φ_* does not have to be real.
- The Schwinger-Dyson equations are satisfied.

We claim that the formula describes correlation functions on the Lefschetz thimble associated with the solution.

Lefschetz thimble

Let us replace the real variable φ of the action S by a complex variable z. Consider a flow $z(t)$ parametrized by t in the complex z plane which satisfies the downward flow equation:

$$\frac{dz}{dt} = i\,\frac{\partial \overline{S}}{\partial \bar{z}}\,, \qquad \frac{d\bar{z}}{dt} = -\,i\,\frac{\partial S}{\partial z}\,.$$

Along the flow, the imaginary part of S increases as t increases:

$$\frac{d\,\mathrm{Im}\,S}{dt} = \frac{1}{2i}\left(\frac{dS}{dt} - \frac{d\overline{S}}{dt}\right) = \frac{1}{2i}\left(\frac{\partial S}{\partial z}\frac{dz}{dt} - \frac{\partial \overline{S}}{\partial \bar{z}}\frac{d\bar{z}}{dt}\right) = \left|\frac{\partial S}{\partial z}\right|^2 > 0\,.$$

A Lefschetz thimble associated with a solution z_* is defined by a submanifold of the z plane consisting of points that can be reached at any t by a flow that starts from z_* at $t = -\infty$.

The path integral on a Lefschetz thimble is well defined.

Let us denote the Lefschetz thimble associated with z_i by $\mathcal{J}_i$, where i labels solutions. In general, the path integral over the real variable φ should be understood as being defined by the path integral on $\mathcal{C}$ given by

$$\mathcal{C} = \sum_i n_i \, \mathcal{J}_i \,,$$

where n_i are integers and there is a procedure to determine n_i.

Previously, we considered the action

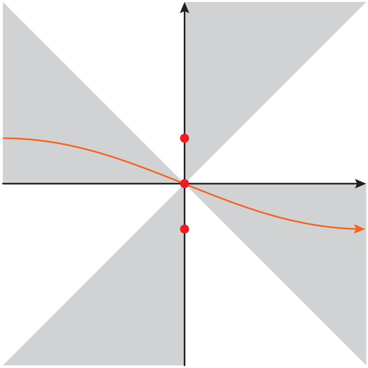
$$S = \, -\frac{1}{2} \, m^2 \, \varphi^2 - \frac{1}{4} \, \lambda \, \varphi^4$$

in the Lorentzian case.

The solutions to the equation of motion are

$$\varphi = 0\,, \pm\frac{im}{\sqrt{\lambda}}\,.$$

In this case, only the Lefschetz thimble associated with the trivial solution $\varphi = 0$ contributes.



We will present evidence that supports our claim for more nontrivial cases.

5. Evidence for the claim

Airy function

Consider the action given by

$$S = \, - \, a \, \varphi - \frac{1}{3} \, \varphi^3 \, .$$

The partition function is given by

$$Z = \int_{-\infty}^{\infty} d\varphi \, e^{\frac{i}{\hbar} S} \, .$$

When we set $\hbar = 1$, this is expressed in terms of the Airy function $\mathrm{Ai} \, (a)$ as follows:

$$Z = 2\pi \, \mathrm{Ai} \, (a) \, .$$

The correlation functions are defined by

$$\langle \varphi^n \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} d\varphi \, \varphi^n \, e^{\frac{i}{\hbar} S} \, .$$

They are given by

$$\langle \varphi^n \rangle = (i\hbar)^n \frac{1}{Z} \frac{d^n Z}{da^n} \, .$$

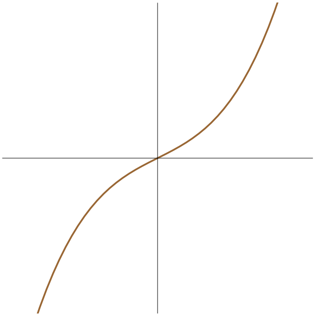
The equation of motion is given by

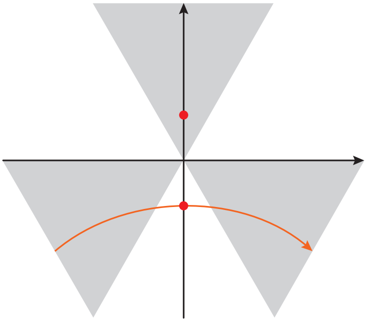
$$\varphi^2 + a = 0 \, .$$

$\mathbf{a > 0}$

When $a > 0$, the solutions to the equation of motion are

$$\varphi = \pm i\sqrt{a}\,.$$





It is known that only the Lefeschetz thimble associated with the solution $\varphi = -i\sqrt{a}$ contributes in this case. We expand φ as

$$\varphi = -i\sqrt{a} + \widetilde{\varphi}\,.$$

The action in terms of $\widetilde{\varphi}$ is

$$S = \frac{2i}{3}\,a^{3/2} + i\sqrt{a}\,\widetilde{\varphi}^{\,2} - \frac{1}{3}\,\widetilde{\varphi}^{\,3}\,.$$

The solution Φ_* is

$$\Phi_* = -i\sqrt{a}\,c\,,$$

and the associated coderivation $\boldsymbol{\Phi}_*$ is

$$\boldsymbol{\Phi}_*\,\mathbf{1} = -i\sqrt{a}\,c\,.$$

The operators $\widetilde{Q}$ and $\widetilde{m}_2$ are given by

$$\widetilde{Q}\, c = \, - 2i\sqrt{a}\, d\,,$$
$$\widetilde{m}_2\,(\, c \otimes c\,) = d\,.$$

The contracting homotopy $\widetilde{h}$ is

$$\widetilde{h}\, d = \frac{i}{2\sqrt{a}}\, c\,.$$

The formula for correlation functions are

$$\langle\, \Phi^{\otimes n}\,\rangle = \pi_n\, e^{\boldsymbol{\Phi}_*}\, \widetilde{\boldsymbol{f}}\, \mathbf{1}\,,$$

where

$$\widetilde{\boldsymbol{f}} = \frac{1}{\mathbf{I} + \widetilde{\boldsymbol{h}}\, \widetilde{\boldsymbol{m}}_2 + i\hbar\, \widetilde{\boldsymbol{h}}\, \mathbf{U}}\,.$$

The n-point functions with $n \leq 3$ are given by

$$\begin{aligned} \langle \varphi \rangle &= -i\sqrt{a} + \widetilde{\boldsymbol{f}}_{10}\,, \\ \langle \varphi^2 \rangle &= -a - 2i\sqrt{a}\,\widetilde{\boldsymbol{f}}_{10} + \widetilde{\boldsymbol{f}}_{20}\,, \\ \langle \varphi^3 \rangle &= ia\sqrt{a} - 3a\,\widetilde{\boldsymbol{f}}_{10} - 3i\sqrt{a}\,\widetilde{\boldsymbol{f}}_{20} + \widetilde{\boldsymbol{f}}_{30}\,. \end{aligned}$$

For $a = 1$ with $N = 50$, we find

$$\begin{aligned} \langle \varphi \rangle &\simeq -1.17632196714\,i\,, \\ -i\sqrt{a} + \widetilde{\boldsymbol{f}}_{10} &\simeq -1.17632196731\,i\,. \end{aligned}$$

For $a = 2$ with $N = 50$, we find

$$\begin{aligned} \langle \varphi \rangle &\simeq -1.5201633881848285\,i\,, \\ -i\sqrt{a} + \widetilde{\boldsymbol{f}}_{10} &\simeq -1.5201633881848252\,i\,. \end{aligned}$$

For $a = 0.1$ with $N = 50$, we find

$$\langle \varphi \rangle \simeq -0.7811\, i\,,$$

$$-\, i\sqrt{a} + \widetilde{\boldsymbol{f}}_{10} \simeq -0.7822\, i\,.$$

For $a = 0.1$ with $N = 100$, we find

$$\langle \varphi \rangle \simeq -0.781069\, i\,,$$

$$-\, i\sqrt{a} + \widetilde{\boldsymbol{f}}_{10} \simeq -0.781005\, i\,.$$

For $\langle \varphi^2 \rangle$, we have the following exact result:

$$\langle \varphi^2 \rangle = \, -\, a\,.$$

With $N = 50$, we found

$$-2i\sqrt{a}\, \widetilde{\boldsymbol{f}}_{10} + \widetilde{\boldsymbol{f}}_{20} = 0\,.$$

For $a = 1$ with $N = 50$, we find

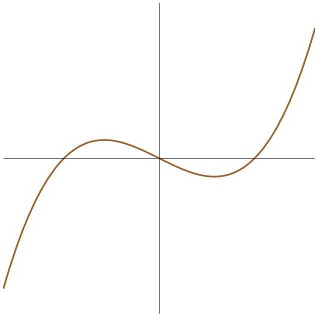
$$\langle \varphi^3 \rangle \simeq 0.17632196731\, i\,,$$

$$ia\sqrt{a} - 3a\, \widetilde{\boldsymbol{f}}_{10} - 3i\sqrt{a}\, \widetilde{\boldsymbol{f}}_{20} + \widetilde{\boldsymbol{f}}_{30} \simeq 0.17632196714\, i\,.$$

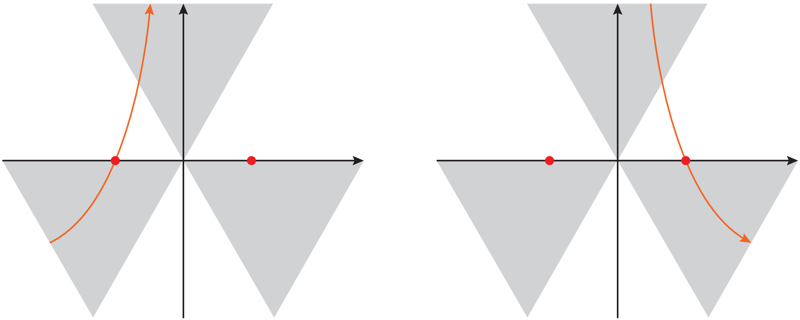
$\mathbf{a < 0}$

When $a < 0$, the solutions to the equation of motion are

$$\varphi = \pm\sqrt{-a}\,.$$



It is known that both of the two Lefeschetz thimbles associated with the solutions $\varphi = -\sqrt{-a}$ and $\varphi = \sqrt{-a}$ contribute in this case.



We denote the partition function associated with the solution $\varphi = -\sqrt{-a}$ by Z_- and the partition function associated with the solution $\varphi = \sqrt{-a}$ by Z_+. We find that

$$Z_- = \pi\,(\,\mathrm{Ai}\,(a) + i\,\mathrm{Bi}\,(a)\,)\,, \qquad Z_+ = \pi\,(\,\mathrm{Ai}\,(a) - i\,\mathrm{Bi}\,(a)\,)\,.$$

We denote the n-point function of the theory Z_- by $\langle\, \varphi^n \,\rangle_-$ and the n-point function of the theory Z_+ by $\langle\, \varphi^n \,\rangle_+$. The partition function Z of the full theory is

$$Z = Z_- + Z_+ = 2\pi \,\mathrm{Ai}\,(a)\,,$$

and the n-point function of the full theory $\langle\, \varphi^n \,\rangle$ is given by

$$\langle\, \varphi^n \,\rangle = \frac{Z_-}{Z_- + Z_+}\,\langle\, \varphi^n \,\rangle_- + \frac{Z_+}{Z_- + Z_+}\,\langle\, \varphi^n \,\rangle_+\,.$$

The n-point functions with $n \leq 3$ are given by

$$\begin{aligned} \langle\, \varphi \,\rangle_{\pm} &= \pm \sqrt{-a} + \widetilde{\boldsymbol{f}}_{10}\,, \\ \langle\, \varphi^2 \,\rangle_{\pm} &= -\, a \pm 2\sqrt{-a}\, \widetilde{\boldsymbol{f}}_{10} + \widetilde{\boldsymbol{f}}_{20}\,, \\ \langle\, \varphi^3 \,\rangle_{\pm} &= \mp\, a\sqrt{-a} - 3a\, \widetilde{\boldsymbol{f}}_{10} \pm 3\sqrt{-a}\, \widetilde{\boldsymbol{f}}_{20} + \widetilde{\boldsymbol{f}}_{30}\,. \end{aligned}$$

For the Lefschetz thimble associated with $\varphi = \,-\sqrt{-a}$, we find for $a = -1$ with $N = 50$ that

$$\begin{aligned} \langle\, \varphi \,\rangle_{-} &\simeq -1.06944263 + 0.18869689\, i\,, \\ -\sqrt{-a} + \widetilde{\boldsymbol{f}}_{10} &\simeq -1.06944243 + 0.18869651\, i \end{aligned}$$

and

$$\begin{aligned} \langle\, \varphi^3 \,\rangle_{-} &\simeq -1.06944263 - 0.81130311\, i\,, \\ a\sqrt{-a} - 3a\, \widetilde{\boldsymbol{f}}_{10} - 3\sqrt{-a}\, \widetilde{\boldsymbol{f}}_{20} + \widetilde{\boldsymbol{f}}_{30} &\simeq -1.06944243 - 0.81130349\, i\,. \end{aligned}$$

For the Lefschetz thimble associated with $\varphi = \sqrt{-a}$, we find for $a = -1$ with $N = 50$ that

$$\begin{aligned} \langle\, \varphi \,\rangle_+ &\simeq 1.06944263 + 0.18869689\, i\,, \\ \sqrt{-a} + \widetilde{\boldsymbol{f}}_{10} &\simeq 1.06944243 + 0.18869651\, i \end{aligned}$$

and

$$\begin{aligned} \langle\, \varphi^3 \,\rangle_+ &\simeq 1.06944263 - 0.81130311\, i\,, \\ -\, a\sqrt{-a} - 3a\, \widetilde{\boldsymbol{f}}_{10} + 3\sqrt{-a}\, \widetilde{\boldsymbol{f}}_{20} + \widetilde{\boldsymbol{f}}_{30} &\simeq 1.06944243 - 0.81130349\, i\,. \end{aligned}$$

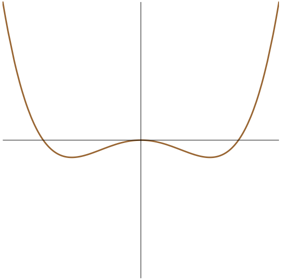
Double well

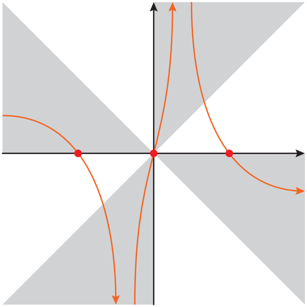
Consider the action given by

$$S = \frac{m^2}{2}\,\varphi^2 - \frac{\lambda}{4}\,\varphi^4\,.$$

We set $m^2 = 1$. The solutions to the equation of motion are

$$\varphi = 0\,, \pm\frac{1}{\sqrt{\lambda}}\,.$$





For the Lefschetz thimble associated with $\varphi = -1/\sqrt{\lambda}$, we find for $\lambda = 1$ with $N = 30$ that

$$\langle \varphi \rangle_{-} \simeq -1.204933 - 0.26012\, i\,,$$
$$-\frac{1}{\sqrt{\lambda}} + \widetilde{\boldsymbol{f}}_{10} \simeq -1.204981 - 0.26008\, i$$

and

$$\langle \varphi^2 \rangle_{-} \simeq 1.25976 + 0.43764\, i\,,$$
$$-\frac{1}{\lambda} - \frac{2}{\sqrt{\lambda}}\, \widetilde{\boldsymbol{f}}_{10} + \widetilde{\boldsymbol{f}}_{20} \simeq 1.25981 + 0.43758\, i\,.$$

For the Lefschetz thimble associated with $\varphi = 0$, we find for $\lambda = 1$ with $N = 100$ that

$$\begin{aligned} \langle\, \varphi \,\rangle_0 &= 0\,, \\ \widetilde{\boldsymbol{f}}_{10} &= 0 \end{aligned}$$

and

$$\begin{aligned} \langle\, \varphi^2 \,\rangle_0 &\simeq -0.2598 + 0.4376\, i\,, \\ \widetilde{\boldsymbol{f}}_{20} &\simeq -0.2608 + 0.4391\, i\,. \end{aligned}$$

For the Lefschetz thimble associated with $\varphi = 1/\sqrt{\lambda}$, we find for $\lambda = 1$ with $N = 30$ that

$$\langle\, \varphi \,\rangle_+ \simeq 1.204933 + 0.26012\, i\,,$$
$$\frac{1}{\sqrt{\lambda}} + \widetilde{\boldsymbol{f}}_{10} \simeq 1.204981 + 0.26008\, i$$

and

$$\langle\, \varphi^2 \,\rangle_+ \simeq 1.25976 + 0.43764\, i\,,$$
$$\frac{1}{\lambda} + \frac{2}{\sqrt{\lambda}}\, \widetilde{\boldsymbol{f}}_{10} + \widetilde{\boldsymbol{f}}_{20} \simeq 1.25981 + 0.43758\, i\,.$$

The n-point function of the full theory $\langle\, \varphi^n \,\rangle$ is given by

$$\langle\, \varphi^n \,\rangle = \frac{Z_-}{Z_- + Z_+} \langle\, \varphi^n \,\rangle_- + \frac{Z_+}{Z_- + Z_+} \langle\, \varphi^n \,\rangle_+$$

in the Airy case and by

$$\langle\, \varphi^n \,\rangle = \frac{Z_-}{Z_- + Z_0 + Z_+} \langle\, \varphi^n \,\rangle_- + \frac{Z_0}{Z_- + Z_0 + Z_+} \langle\, \varphi^n \,\rangle_0 + \frac{Z_+}{Z_- + Z_0 + Z_+} \langle\, \varphi^n \,\rangle_+$$

in the double-well case.

We need to know ratios of partition functions on different Lefschetz thimbles.

Consider a pair of actions S_i and S_f, and a solution φ_i to the equation of motion for S_i and a solution φ_f to the equation of motion of S_f. We interpolate S_i and S_f as $S(t)$ with $0 \le t \le 1$, where

$$S(0) = S_i \,, \qquad S(1) = S_f \,.$$

We also interpolate φ_i and φ_f as $\varphi(t)$ with

$$\varphi(0) = \varphi_i \,, \qquad \varphi(1) = \varphi_f \,,$$

where $\varphi(t)$ is a solution to the equation of motion for $S(t)$. The partition function Z_i of S_i for the Lefschetz thimble associated with φ_i and the partition function Z_f of S_f for the Lefschetz thimble associated with φ_f are given by

$$Z_i = \int d\varphi \, e^{\frac{i}{\hbar} S_i} \,, \qquad Z_f = \int d\varphi \, e^{\frac{i}{\hbar} S_f} \,.$$

We interpolate Z_i and Z_f as $Z(t)$:

$$Z(t) = \int d\varphi \, e^{\frac{i}{\hbar} S(t)} \,.$$

Since

$$\frac{d}{dt} \ln Z(t) = \frac{i}{\hbar} \left\langle \frac{dS(t)}{dt} \right\rangle ,$$

the ratio of the partition functions can be calculated as

$$\frac{Z_f}{Z_i} = \exp \left[\frac{i}{\hbar} \int_0^1 dt \left\langle \frac{dS(t)}{dt} \right\rangle \right] .$$

We can evaluate this using the formula for correlation functions in terms of A_∞ algebras.

6. Conclusions and discussion

For the theory described by the coderivation $\mathbf{M}$, we presented a new form of the formula for correlation functions associated with a solution Φ_* described by the coderivation $\boldsymbol{\Phi}_*$:

$$\langle \Phi^{\otimes n} \rangle = \pi_n \frac{1}{\mathbf{I} + i\hbar \, \mathbf{H} \, \mathbf{U}} \, \mathbf{P} \, \mathbf{1} \,,$$

where $\mathbf{H}$ satisfies

$$\mathbf{M} \, \mathbf{H} + \mathbf{H} \, \mathbf{M} = \mathbf{I} - \mathbf{P} \,, \qquad \mathbf{H} \, \mathbf{P} = 0 \,, \qquad \mathbf{P} \, \mathbf{H} = 0 \,, \qquad \mathbf{H}^2 = 0$$

with

$$\mathbf{P} = e^{\boldsymbol{\Phi}_*} \, \pi_0 \,.$$

This formula does not involve the division of the free part and the interaction part.

We claim that the formula describes correlation functions on the Lefschetz thimble associated with the solution, and we presented evidence that the formula contains nonperturbative information on correlation functions in the case of scalar field theories in zero dimensions.

We have not found a way to choose appropriate Lefshcetz thimbles in the language of A_∞ algebras.

We have not understood the reason why contributions from different Lefschetz thimbles are weighted by partition functions in the language of A_∞ algebras.

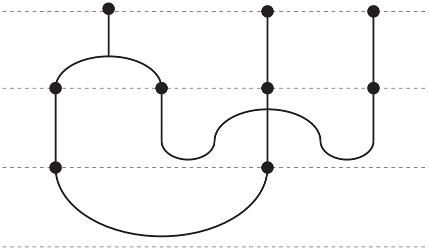
For theories in d dimensions with $d > 0$, we can decompose $\boldsymbol{f}\,\mathbf{P}$ as

$$\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P} = \prod_i \frac{1}{\mathbf{I} + \boldsymbol{h}_i\,\boldsymbol{m}_i + i\hbar\,\boldsymbol{h}_i\,\mathbf{U}}\,\mathbf{P}_i$$

with

$$\boldsymbol{h} = \sum_i \boldsymbol{h}_i\,.$$

This way we may be able to define correlation functions nonperturbatively.



We then extend our discussion to open superstring field theory, and we may be able to define string theory nonperturbatively.