Graph Zeta Functions and Wilson Loops in Kazakov-Migdal Model

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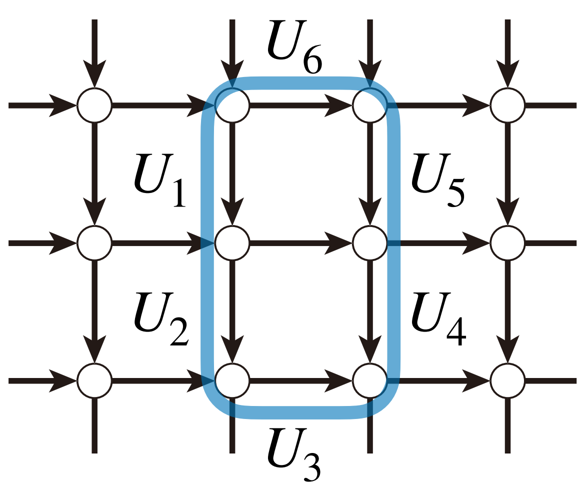
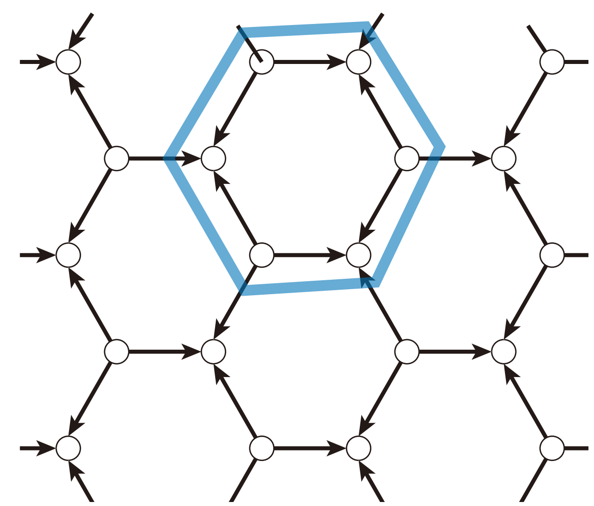
名古屋大学 多弦数理物理学セミナー (2022/10/31)

Introduction (phys)

- Cycles on the *graph* play an important role in gauge and string theory
  - Wilson loops in lattice gauge theory:

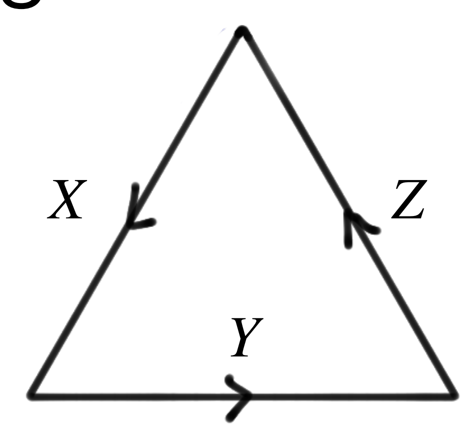


$$W_C = \mathrm{Tr} U_1 U_2 U_3 U_4^\dagger U_5^\dagger U_6^\dagger$$



$$W_C = \mathrm{Tr} U_1 U_2^\dagger U_3 U_4^\dagger U_5 U_6^\dagger$$

  - Gauge invariant operators in quiver gauge theory:



$$\mathcal{O} = \mathrm{Tr}\, XYZ$$

Introduction (math)

- Cycles on the *graph* can be counted by a kind of *zeta function*
  - Ihara introduced a Selberg zeta function of p-adic fields (1966)
  - Serre pointed out a relation to graph theory (1980)
  - Sunada gave a definition of Ihara zeta function for the regular graph and a graph theoretical proof for Ihara's theorem (1986)
  - Hashimoto gave a determinant expression by the edge matrix (1990)
  - Bass proved Ihara's theorem via the determinant expression for generic graphs (1992)
  - Bartholdi introduced two parameter extension of Ihara zeta function (1999)
- Question: Can we utilize the zeta function on the graph (Ihara zeta function) for problems on gauge or string theory?

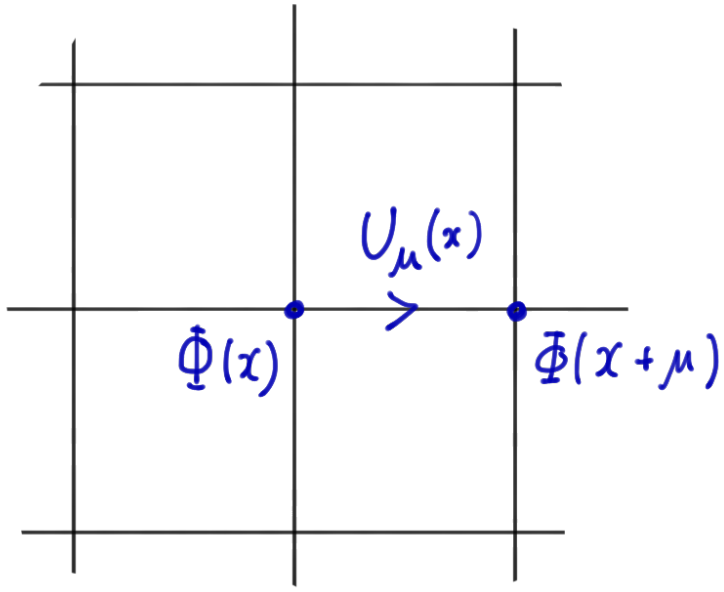
Plan of talk

1. Introduction
2. Kazakov-Migdal Model on the graph
3. Ihara Zeta Function
4. Partition Function
5. Large N Limit
6. Duality
7. Bartholdi Zeta Function and Matrix Model
8. Conclusion and Discussions

Kazakov-Migdal Model

- Kazakov-Migdal model is defined by unitary matrices $U_\mu(x)$ on links (edges) and hermite matrices $\Phi(x)$ on sites (vertices) as D-dimensional lattice gauge theory [Kazakov and Migdal (1992)]:

$$S = \sum_x N \mathrm{Tr}\left(m_0^2 \Phi(x)^2 - \sum_{\mu=1,2,\cdots D} \Phi(x) U_\mu(x) \Phi(x+\mu) U_\mu^\dagger(x) \right)$$



- After eliminating $\Phi(x)$, we get

$$\int DU D\Phi \, e^{-S[U,\Phi]} \propto \int DU \, e^{-S_{\mathrm{ind}}[U]}$$

where S_{ind} is a induced action given by

$$S_{\mathrm{ind}}[U] = \frac{1}{2} \mathrm{Tr} \log \left(\delta_{x,y} - m_0^{-2} \sum_\mu U_\mu(x) \otimes U_\mu^\dagger(x) \delta_{x+\mu,y} \right)$$

Introduction

- The induced action has the following expansion:

$$S_{\mathrm{ind}}[U] = -\frac{1}{2}\sum_{C}\frac{|\,\mathrm{Tr}\, W_C[U]\,|^2}{\ell(C) m_0^{2\ell(C)}}$$

  where

  C : lattice loops

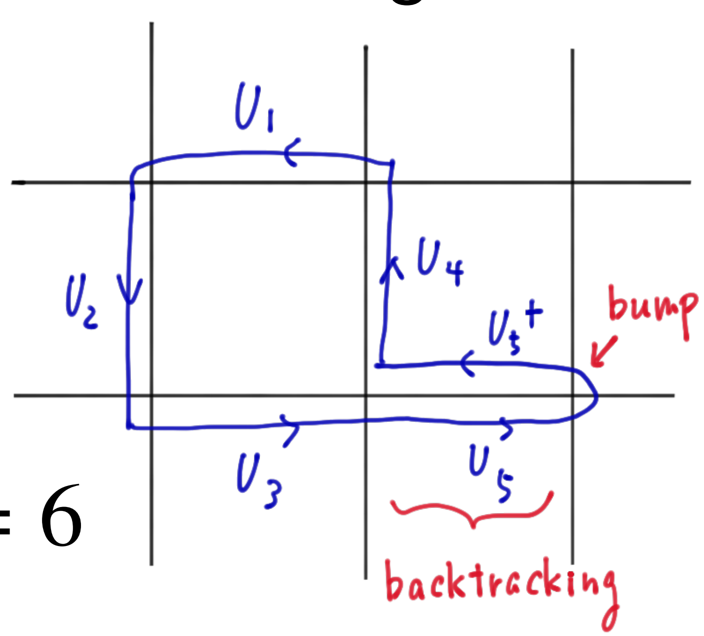
  $\ell(C)$: length of the loops

  $W_C[U]$: ordered loop product of U (Wilson loop) along C

- It had been expected that the induced action reduces to the Yang-Mills action (induced QCD) in the continuum limit, but

  - C contains "bad" (collapsed) Wilson loops

  - $\ell[C]$ does not count the net length of the loop
    e.g. $\mathrm{Tr} U_1 U_2 U_5 U_5^\dagger U_3 U_4 = \mathrm{Tr} U_1 U_2 U_3 U_4$ for $\ell(C) = 6$

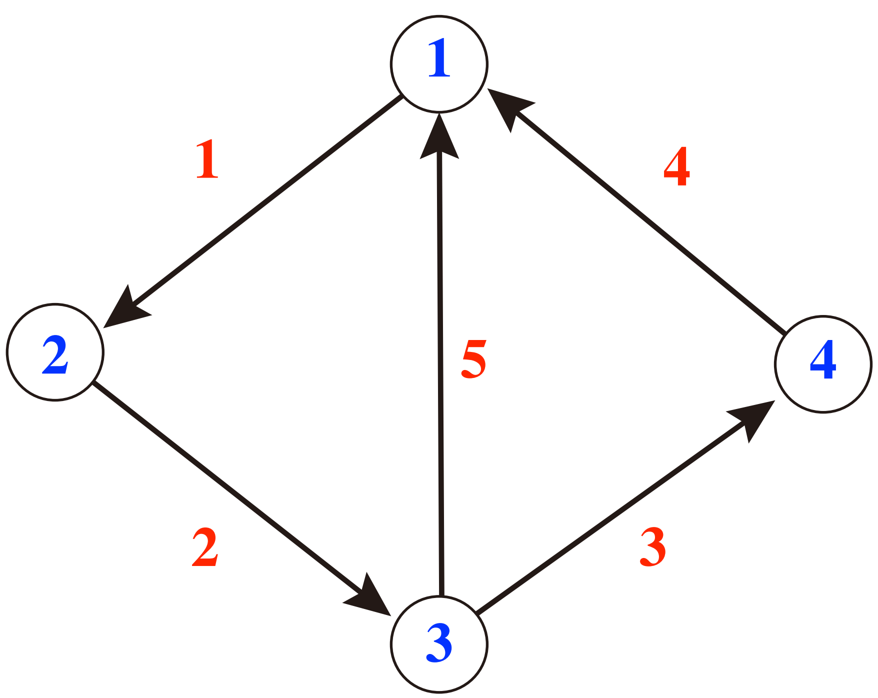


Graph Theory

- A graph G consists of the vertices V and the edges E; $G = (V, E)$
- Each edge connects two vertices between $s(e)$ and $t(e)$, where $e \in E$, $s(e), t(e) \in V$

$$s(e) \xrightarrow[e]{} t(e)$$

- Graph theory gives a mapping from the graph structure to matrices; e.g. <u>double triangle</u>



adjacency matrix:

$$A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad A_{vv'} = \left\{ \begin{array}{c} \sharp \text{ of edges} \\ \text{connecting } v \text{ and } v' \end{array} \right\}$$

incidence matrix:

$$B = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$\Rightarrow$ charge matrix of quiver gauge theory,
Dirac operator on the graph,
index theorem on the graph
[S. Matsuura and KO (2021)]

Kazakov-Migdal model on the graph

- We consider the generalized Kazakov-Migdal model defined on the graph

$$Z_{\mathrm{gKM}} = \int \prod_{v\in V} d\Phi_v \prod_{e\in E} dU_e \exp\left\{ -\beta \mathrm{Tr}\left(\frac{1}{2}\sum_{v\in V} m_v^2 \Phi_v^2 - q \sum_{e\in E} \Phi_{s(e)} U_e \Phi_{t(e)} U_e^\dagger \right) \right\}$$

where

V : a set of vertices (sites) of the graph

E : a set of edges (links) of the graph

$s(e)$: a *source* of the edge e

$t(e)$: a *target* of the edge e

- We can also integrate out the scalar field Φ_v , then get

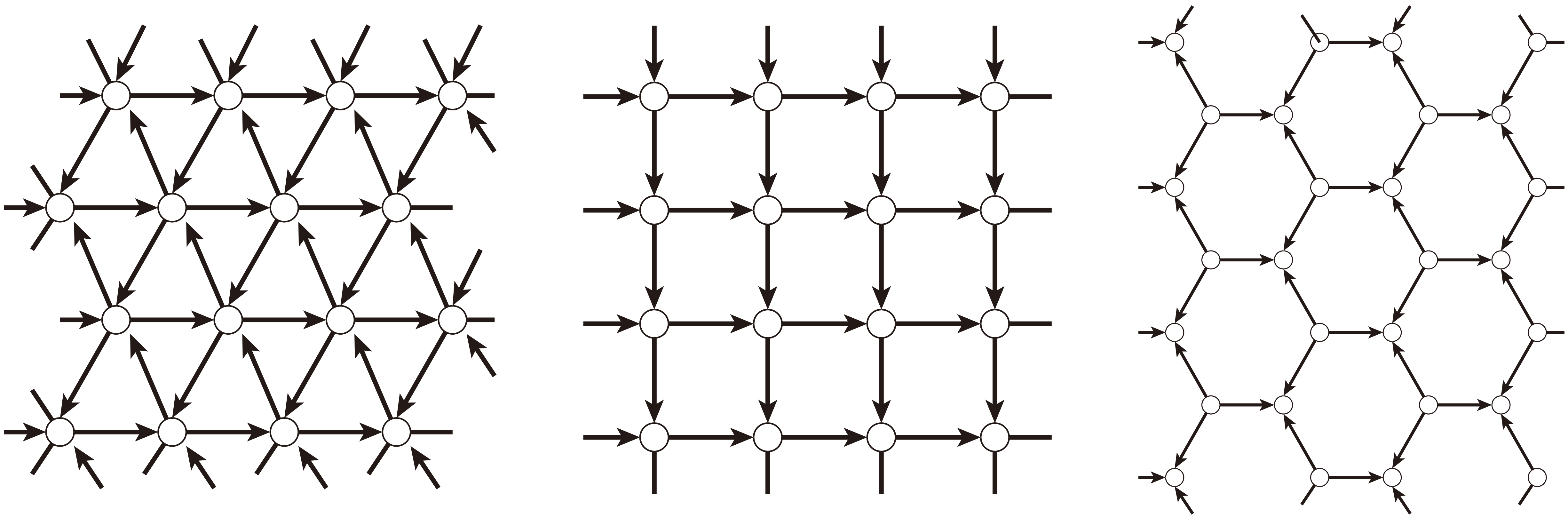
$$Z_{\mathrm{gKM}} \propto \int \prod_{e\in E} dU_e \, \exp\left\{ -\frac{1}{2} \mathrm{Tr} \log \left(m_v^2 \delta_{vv'} \otimes \mathbf{1}_{N^2} - qA[U] \right) \right\}$$

where $A[U]$ is the weighted adjacency matrix

- For the general couplings, the induced action still generates the “bad” (collapsed) Wilson loops, but we will show that they can be removed by a coupling tuning

Discretization as the graph

- At least, the flat plane can be discretized by using the graphs



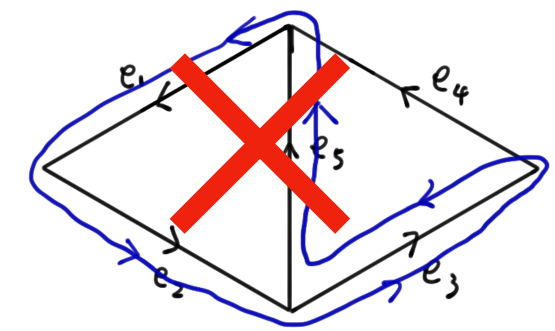
Triangle lattice

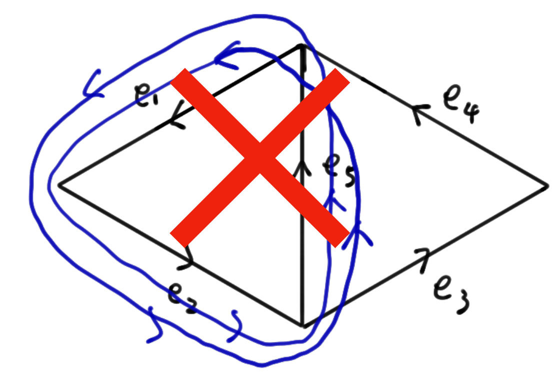
Square lattice
(Original KM model)

Hexagon lattice

- The continuum limit and emergence of the dimensionality are difficult problem

Graph Zeta Function





- The Ihara zeta function of the graph G is defined as follows [Ihara (1966)]:

$$\boxed{\zeta_G(q) \equiv \prod_{[C]:\,\text{prime cycles}} \frac{1}{1 - q^{\ell(C)}}}$$

  where the product is taken over the *prime cycles*, which are

  ✓ Neither backtracking nor tail (reduced cycle)

  ✓ Not written by a power of the reduced cycle (primitive cycle)

  ✓ Defined by a equivalence classes (a fixed cyclic ordering) $C \sim C'$ $\quad e_1e_2e_5 \sim e_2e_5e_1 \sim e_5e_1e_2$

- By definition, the Ihara zeta function counts “good” (non-collapsing) cycles only
  ⇒ A coefficient of q^k are a number of the reduced cycles of the total length of k

- Recall that the Riemann zeta function (Euler product) is defined by a infinite product of all *prime numbers*

$$\zeta(s) \equiv \prod_{p:\,\text{prime numbers}} \frac{1}{1 - p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

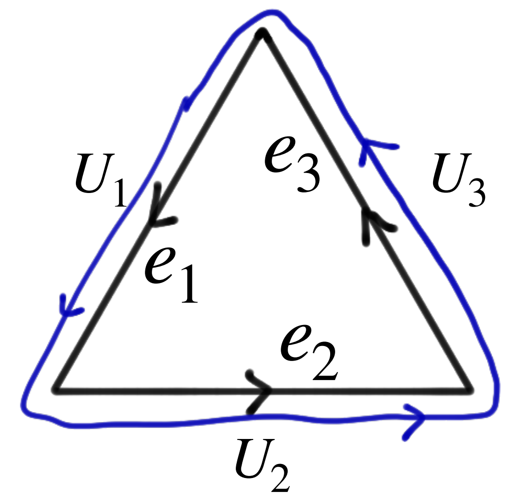
  The graph zeta function is an analogy to this (or Selberg zeta function)

Example1: triangle graph

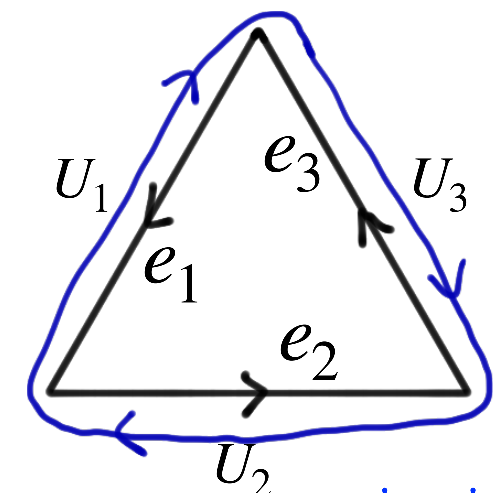
- A triangle graph is known as a cycle graph C_3 or $\hat{A}_2$ quiver diagram (Dynkin diagram)

- We have two prime cycles:

$$[C] = \{e_1e_2e_3, e_2e_3e_1, e_3e_1e_2\}$$
$$[\bar{C}] = \{\bar{e}_3\bar{e}_2\bar{e}_1, \bar{e}_1\bar{e}_3\bar{e}_2, \bar{e}_2\bar{e}_1\bar{e}_3\}$$



$W_C[U] = U_1U_2U_3$



$W_{\bar{C}}[U] = U_3^\dagger U_2^\dagger U_1^\dagger$

then

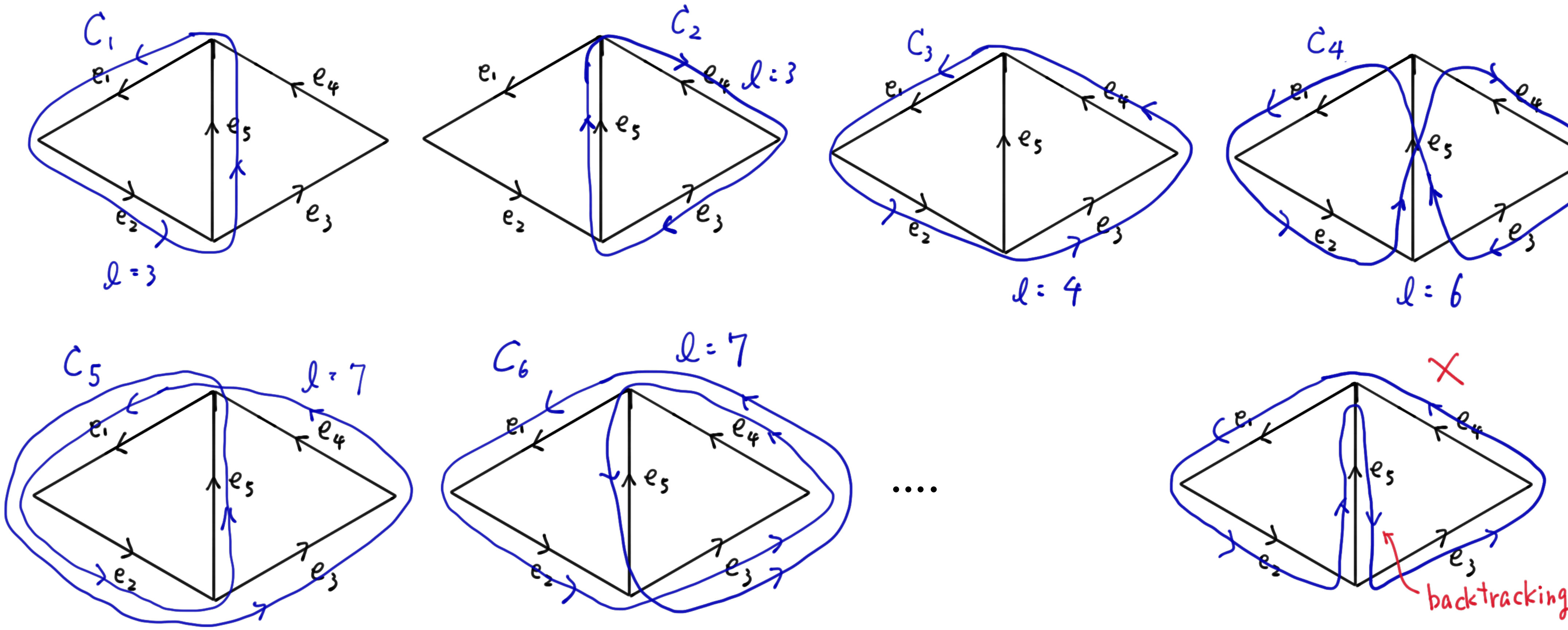
$$\zeta_{C_3}(q) = \frac{1}{(1-q^3)^2} = 1 + 2q^3 + 3q^6 + 4q^9 + 5q^{12} + \cdots$$

power of q (length)	3	6	9	12	$\cdots$
coeff	2	3	4	5	$\cdots$
cycles	C, $\bar{C}$	C^2, $C\bar{C}$, $\bar{C}^2$	C^3, $C^2\bar{C}$, $C\bar{C}^2$, $\bar{C}^3$	C^4, $C^3\bar{C}$, $C^2\bar{C}^2$ $C\bar{C}^3$, $\bar{C}^4$	$\cdots$

Easy? $\Rightarrow$ For more general graph, the prime cycles are rather complicated (infinite)

Example2: double triangle graph

- For the double triangle (DT) graph, there are infinitely many prime loops



- Then, the Euler product expression of the graph zeta function becomes the infinite product in general

$$\zeta_{\mathrm{DT}}(q) = \frac{1}{(1-q^3)^4}\frac{1}{(1-q^4)^2}\frac{1}{(1-q^6)^2}\frac{1}{(1-q^7)^4}\cdots$$

- But we have another determinant (a reciprocal of a polynomial) expression of the graph zeta function

Ihara's theorem

- For a given graph G, the Ihara zeta function is given by the following determinant formula

$$\zeta_G(q) = \frac{1}{(1-q^2)^{n_E - n_V} \det\left(I - qA + q^2(D-I)\right)}$$

where

n_V : the number of the vertices

n_E : the number of the edges

I : $n_V \times n_V$ identity matrix

D : the degree matrix ($\sharp$ of the edges attached with the vertex)

A : the adjacency matrix

$$A_{vv'} = \{\sharp \text{ of edges connecting } v \text{ and } v'\}$$

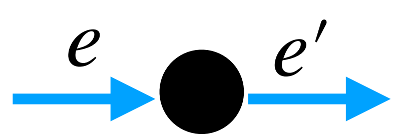
A Brief Proof

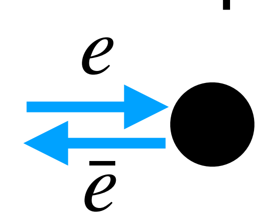
- Using an identity for the determinant between vertex and edge adjacency matrices, we can show that

$$(1-q^2)^{n_V+n_E}\det\left(I_{2n_E}-qW\right)=(1-q^2)^{2n_E}\det\left(I_{n_V}-qA+q^2(D-I_{n_V})\right)$$

where

W : the edge adjacency matrix without bumps

$W_{ee'}=1$ for 

But, $W_{e\bar{e}}=0$ for 

- Then, we obtain

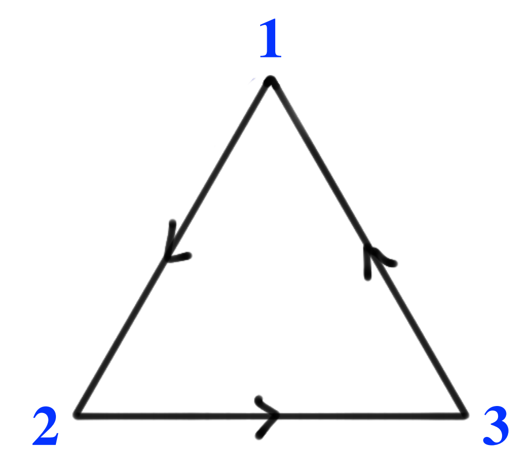
$$\zeta_G(q)=\frac{1}{\det\left(I_{2n_E}-qW\right)}=\exp\left\{\sum_{k=1}^{\infty}\frac{q^k}{k}\mathrm{Tr}W^k\right\}=\frac{1}{(1-q^2)^{n_E-n_V}\det\left(I_{n_V}-qA+q^2(D-I_{n_V})\right)}$$

Hashimoto expression

Ihara expression

Example1: triangle graph

- For the triangle graph, $n_V = n_E = 3$ and



$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

then we have

$$\zeta_{C_3}(q) = \frac{1}{\det\left(I - qA - q^2(D - I)\right)} = \frac{1}{(1 - q^3)^2}$$

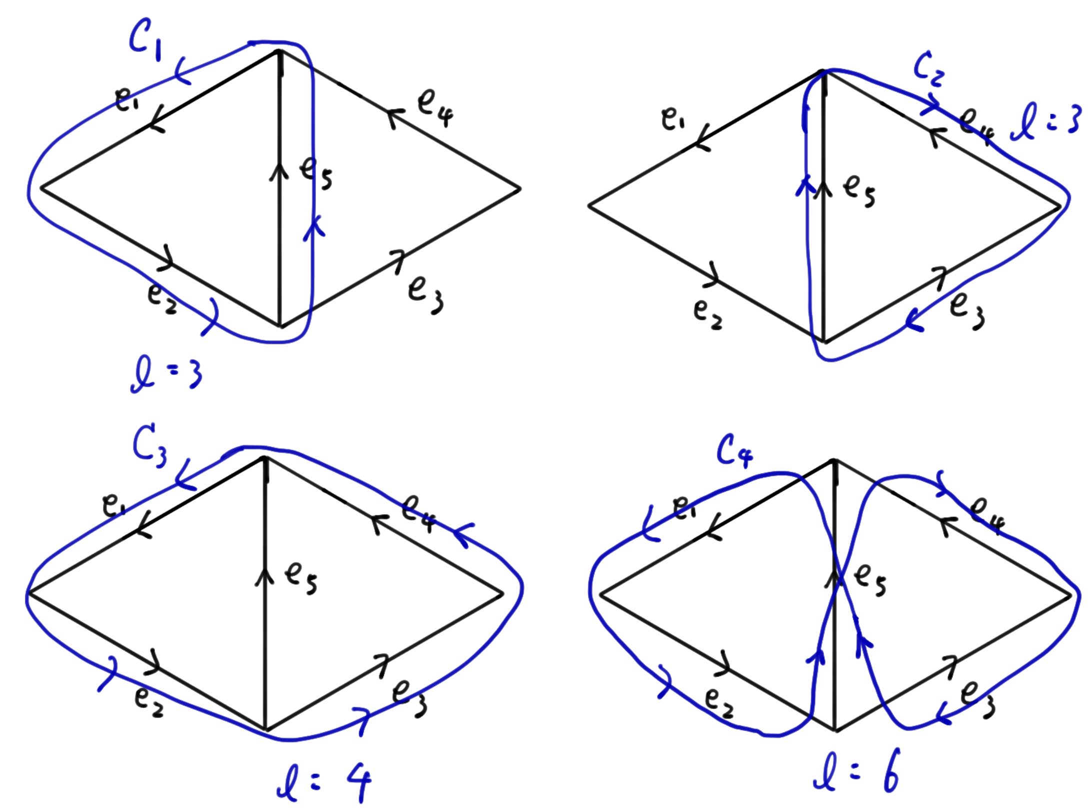
- This agrees with the previous simple observation from the graph

Example2: double triangle graph

- For the double triangle (DT) graph, $n_V = 4$, $n_E = 5$ and

$$D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\zeta_{\mathrm{DT}}(q) = \frac{1}{(1-q^2)\det\left(I - qA - q^2(D-I)\right)}$$
$$= \frac{1}{(1-q^4)(1+q^2-2q^3)(1-q^2-2q^3)}$$
$$= 1 + 4q^3 + 2q^4 + 12q^6 + 12q^7 + 3q^8 + \cdots$$



- For first few terms, the counting is as follows:

length	3	4	6	7	$\cdots$
coeff	4	2	12	12	$\cdots$
cycles	$C_1,\ \bar{C}_1,$ $C_2,\ \bar{C}_2$	$C_3,\ \bar{C}_3$	$C_1^2,\ \bar{C}_1^2,\ C_2^2,\ \bar{C}_2^2,$ $C_1\bar{C}_1,\ C_2\bar{C}_2,$ $C_1C_2,\ \bar{C}_1C_2,\ C_1\bar{C}_2,\ \bar{C}_1\bar{C}_2$ $C_4,\ \bar{C}_4$	$C_1C_3,\ \bar{C}_1C_3,\ C_2C_3,\ \bar{C}_2C_3,$ $C_1\bar{C}_3,\ \bar{C}_1\bar{C}_3,\ C_2\bar{C}_3,\ \bar{C}_2\bar{C}_3,$ $C_5,\ \bar{C}_5,\ C_6,\ \bar{C}_6$	$\cdots$

Back to the graph Kazakov-Migdal model

- Recall the partition function of the graph Kazakov-Migdal model

$$
\begin{aligned}
Z_{\mathrm{gKM}} &= \int \prod_{v\in V} d\Phi_v \prod_{e\in E} dU_e \exp\left\{-\beta \mathrm{Tr}\left(\frac{1}{2}\sum_{v\in V} m_v^2 \Phi_v^2 - q\sum_{e\in E} \Phi_{s(e)} U_e \Phi_{t(e)} U_e^\dagger\right)\right\} \\
&\propto \frac{1}{(1-q^2)^{\frac{N^2}{2}(n_E-n_V)}} \int \prod_{e\in E} dU_e \exp\left\{-\frac{1}{2}\mathrm{Tr}\log\left(m_v^2\delta_{vv'}\otimes \mathbf{1}_{N^2} - qA[U]\right)\right\} \\
&= \int \prod_{e\in E} dU_e \frac{1}{(1-q^2)^{\frac{N^2}{2}(n_E-n_V)} \det\left(m_v^2 I - qA[U]\right)^{\frac{1}{2}}}
\end{aligned}
$$

  where $A[U]$ is a unitary matrix weighted adjacency matrix

- The determinant looks like the determinant formula of the Ihara zeta function. In fact, by setting $m_v^2 = 1 + (\deg v - 1)q^2$, we get

$$
Z_{\mathrm{gKM}} \propto \int \prod_{e\in E} dU_e\, \zeta_G(q;U)^{\frac{1}{2}}
$$

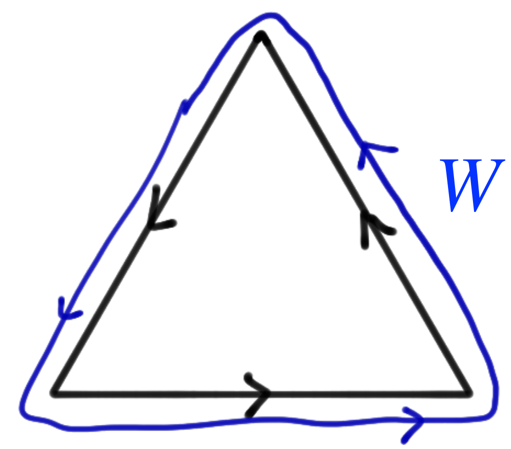
$\zeta_G(q;U)$: the unitary matrix weighted Ihara zeta function

Example1: triangle graph

- The unitary matrix weighted adjacency matrix for $U(N)$:

$$A[U] = \begin{pmatrix} 0 & U_1 \otimes U_1^\dagger & U_3^\dagger \otimes U_3 \\ U_1^\dagger \otimes U_1 & 0 & U_2 \otimes U_2^\dagger \\ U_3 \otimes U_3^\dagger & U_2^\dagger \otimes U_2 & 0 \end{pmatrix}$$

- Recalling that the Ihara zeta is a generating function of the multi-trace Wilson loops, then we obtain

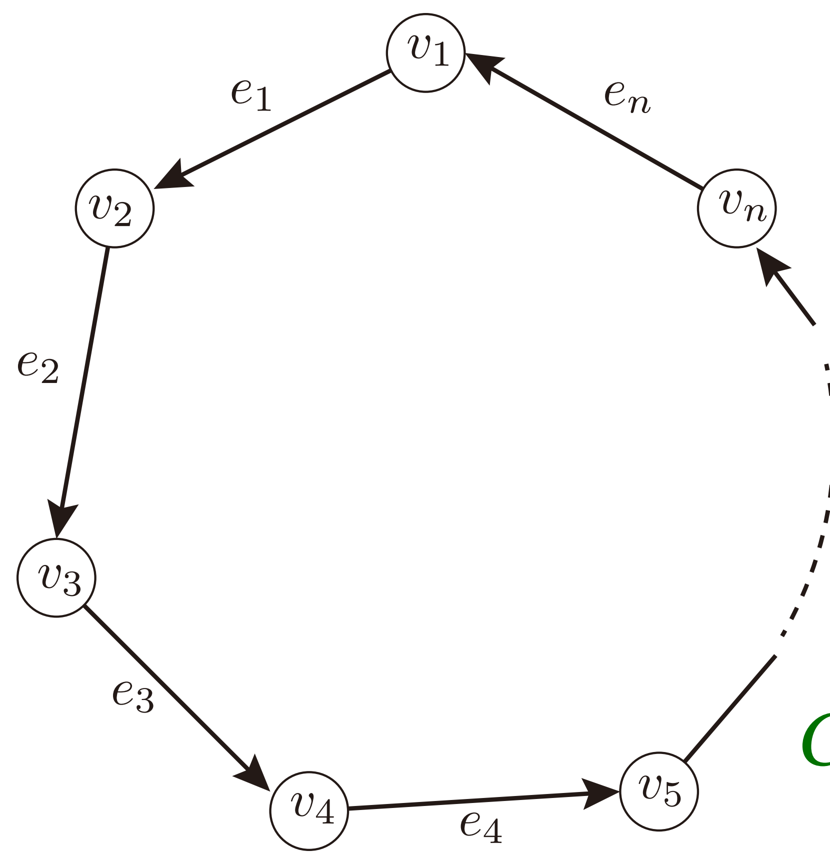


$$\begin{aligned} Z_{\mathrm{gKM}} &\propto \int dU_1 dU_2 dU_3 \frac{1}{\det\left(I - qA[U] + q^2(D-I)\right)^{\frac{1}{2}}} \\ &= \int dU_1 dU_2 dU_3 \ \exp\left\{\sum_{k=1}^{\infty} \frac{q^{3k}}{k} |\mathrm{Tr}(U_1U_2U_3)^k|^2\right\} \\ &= \int dW \left\{1 + |\mathrm{Tr}W|^2 q^3 + \frac{1}{2}\left(|\mathrm{Tr}W|^4 + |\mathrm{Tr}W^2|^2\right) q^6 + \frac{1}{6}\left(|\mathrm{Tr}W|^6 + 3|\mathrm{Tr}W|^2|\mathrm{Tr}W^2|^2 + 2|\mathrm{Tr}W^3|^2\right) q^9 + \cdots\right\} \\ &= \prod_{i=1}^{N} \frac{1}{1-q^{3i}} \end{aligned}$$

where $W \equiv U_1U_2U_3$

Cycle graph C_n

- We can generalize the previous results to cycle graphs (polygons, $\hat{A}_{n-1}$ quiver diagram)

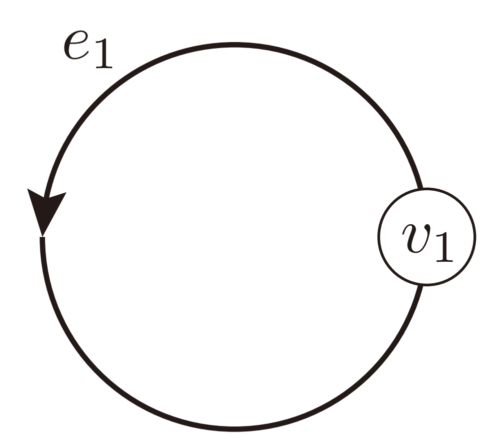


C_n graph

- After integrating over the unitary groups, we obtain

$$Z_{C_n}(q) = Z_{C_1}(q^n)$$

$$= \prod_{i=1}^{N} \frac{1}{1 - q^{ni}}$$



C_1 graph

Large N limit

- It is difficult to perform the integral over the unitary matrices in general, but the situation becomes simple in the large N limit, thanks to the decomposition (clustering) of the vev of the Wilson loops

$$
\begin{aligned}
Z_{\mathrm{gKM}} &\propto \int \prod_{e\in E} dU_e \exp\left\{ \sum_{C\in\Pi^+} \sum_{k=1}^{\infty} \frac{q^{\ell(C)k}}{k} |\mathrm{Tr} W_C[U]^k|^2 \right\} \\
&\equiv \left\langle \prod_{C\in\Pi^+} \exp\left\{ \sum_{k=1}^{\infty} \frac{q^{\ell(C)k}}{k} |\mathrm{Tr} W_C[U]^k|^2 \right\} \right\rangle \\
&\xrightarrow[N\to\infty]{} \prod_{C\in\Pi^+} \left\langle \exp\left\{ \sum_{k=1}^{\infty} \frac{q^{\ell(C)k}}{k} |\mathrm{Tr} W_C[U]^k|^2 \right\} \right\rangle \\
&= \prod_{C\in\Pi^+} \prod_{i=1}^{\infty} \frac{1}{1-q^{i\ell(C)}} = \boxed{\prod_{i=1}^{\infty} \zeta_G(q^i)^{\frac{1}{2}}}
\end{aligned}
$$

where Π^+ is a set of *chiral* prime loops (choose a one direction of the loops)

- The partition function of the graph Kazakov-Migdal model can be written by a infinite product of (square roots of) the Ihara zeta functions

Duality

- We can perform the U_e integral by using the Harish-Chandra-Itzykson-Zuber integral:

$$\int dU\, e^{t\mathrm{Tr}AUBU^\dagger} \propto \frac{\det_{i,j} e^{ta_i b_j}}{\Delta(a)\Delta(b)}$$

  where a_i, b_j are the eigenvalues and $\Delta(a), \Delta(b)$ are the Vandermonde determinants of A, B

- Then we obtain the multi matrix model for Φ_v as the partition function of the graph Kazakov-Migdal model

$$Z_{\mathrm{gKM}} \propto \int \prod_{v\in V} \prod_{i=1}^{N} d\phi_{v,i}\, e^{-\frac{1}{2}m_v^2\phi_{v,i}^2} \Delta(\phi_v)^{2-\deg v} \prod_{e\in E} \det_{i,j} e^{q\phi_{s(e),i}\phi_{t(e),j}}$$

- We can perform this integral exactly for simpler cases like the cycle graph and agrees with the results from the graph zeta function, but it is difficult to evaluate for the generic graphs

Extension to Bartholdi's zeta function

- It is known that there is a generalization of the Ihara zeta function, which contains one more parameter and counts the number of the bumps too
  $\Rightarrow$ Bartholdi's zeta function

$$\zeta'_G(q,u) \equiv \prod_{C:\text{primitive (not reduced)}} \frac{1}{1-u^{cbc(C)}q^{\ell(C)}}$$

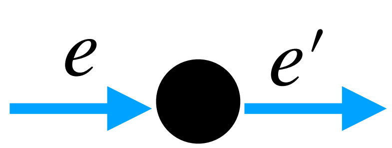
$$= \frac{1}{\left(1-(1-u)^2q^2\right)^{n_E-n_V}\det\left(I-qA+(1-u)q^2\left(D-(1-u)I\right)\right)}$$

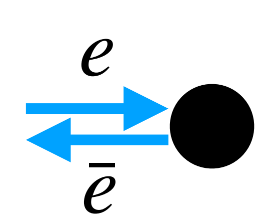
where $cbc(C)$ is cyclic bump count (# of bumps) and $\lim_{u\to 0}\zeta'_G(q,u)=\zeta_G(q)$

- A proof is similar to the Ihara zeta function

$$\zeta'_G(q,u) = \frac{1}{\det\left(I-q(W+uJ)\right)} = \frac{1}{(1-(1-u^2)q^2)^{n_E-n_V}\det\left(I-qA+(1-u)q^2(D-(1-u)I)\right)}$$

where

$W_{ee'}=1$ for 

$J_{e\bar{e}}=1$ for 

Graph Kazakov-Migdal model with bumps

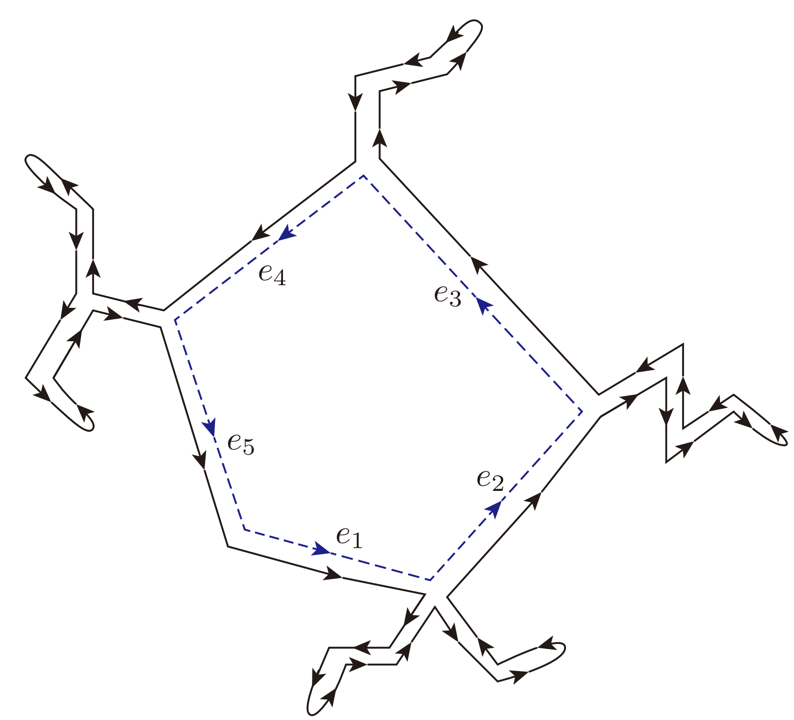
- The mass and coupling of the Kazakov-Migdal model is tuned to be

$$S_{\mathrm{gKM}} = \mathrm{Tr}\left\{ \frac{1}{2}\sum_{v\in V}\left(1-q^2(1-u)^2+q^2(1-u)\deg v\right)\Phi_v^2 - q\sum_{e\in E}\Phi_{s(e)}U_e\Phi_{t(e)}U_e^\dagger \right\}$$

- Then, the partition function becomes

$$\begin{aligned} Z_{\mathrm{gKM}} &= \int \prod_{v\in V} d\Phi_v \prod_{e\in E} dU_e e^{-\beta S_{\mathrm{gKM}}} \\ &= \left(\frac{2\pi}{\beta}\right)^{\frac{1}{2}n_V N^2} \int_{e\in E} dU_e \frac{1}{\det\left(I - qA_U + q^2(1-u)(D-(1-u)I)\right)} \\ &= \left(\frac{2\pi}{\beta}\right)^{\frac{1}{2}n_V N^2} \left(1-(1-u)^2q^2\right)^{\frac{1}{2}(n_E-n_V)N^2} \mathscr{V}_G(q,u)^{\frac{N^2}{2}} \\ &\quad \times \int_{e\in E} dU_e \prod_{C\in\Pi^+} \exp\left\{ \sum_{k=1}^{\infty} \frac{1}{k} f_C(q,u)^k \left|\mathrm{Tr}W_C(U)^k\right| \right\} \end{aligned}$$

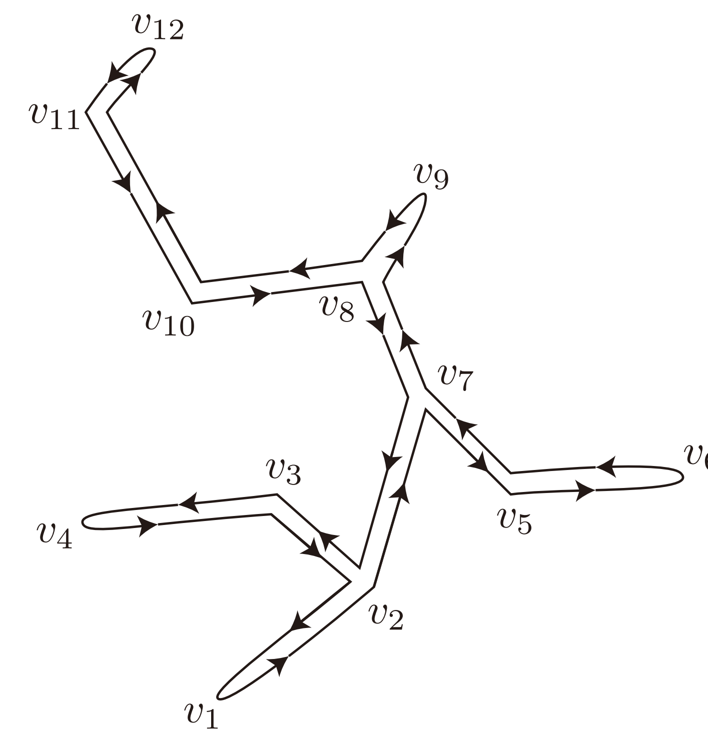
where $f_C(q,u) = \sum_{\tilde{C}:\ \text{reducible to } C} u^{cbc(\tilde{C})} q^{|\tilde{C}|}$



Contribution from the zero-area Wilson loops (collapsed cycles)

- $\mathscr{V}_G(q,u)$ is the contribution from the collapsed cycle
- For $G = C_n$,

$$\mathscr{V}_{C_n}(q,u) = \left\{ \frac{1 + (1-u^2)q^2 - \sqrt{1 - 2(1+u^2)q^2 + (1-u^2)^2 q^4}}{2q^2} \right\}^n$$



Generalized Catalan number

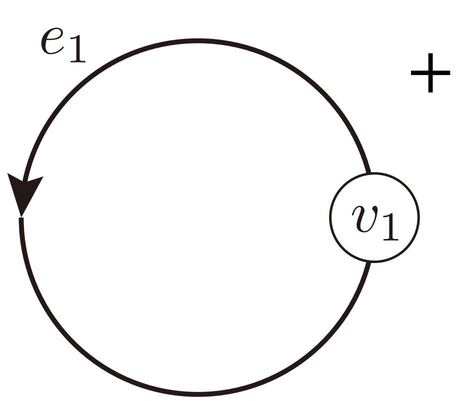
e.g.

$$\mathscr{V}_{C_1}(q,u) = 1 + u^2q^2 + (u^2+u^4)q^4 + (u^2+3u^4+u^6)q^6 + \cdots$$

$$\leftrightarrow 1 + \mathrm{Tr}UU^\dagger q^2 + (\mathrm{Tr}UUU^\dagger U^\dagger + \mathrm{Tr}UU^\dagger UU^\dagger)q^4$$

$$+(\mathrm{Tr}UUUU^\dagger U^\dagger U^\dagger + \mathrm{Tr}UU^\dagger UUU^\dagger U^\dagger + \mathrm{Tr}UUU^\dagger UU^\dagger U^\dagger + \mathrm{Tr}UUU^\dagger U^\dagger UU^\dagger$$

$$+\mathrm{Tr}UU^\dagger UU^\dagger UU^\dagger)q^6 + \cdots$$



Comparison with matrix model

- In the large N limit, $\mathscr{V}_G(q,u)$ contributes to the free energy at order N^2

$$\mathscr{F}_{\mathrm{gKM}} \sim N^2 \left(-\frac{1}{n_V} \log \mathscr{V}_G(q,u) \right) + \mathcal{O}(N)$$

- On the other hand, after eliminating U_e , we obtain the matrix model for Φ_v
- The exact semi-circle solution for this matrix model [Gross 1992] corresponds to the N^2 order contribution to the free energy
- Thus, we expect that the semi-circle solution of the matrix model at large N comes from the zero-area (collapsed) Wilson loops only $\Rightarrow$ infinite tension strings [Boulatov 1992]
- The Wilson loops with the bumps are important to understand the relation to the string theory (zigzag symmetry)

Conclusion and Discussions

- We proposed a generalization of the Kazakov-Migdal model on the graph, which reproduces the weighted Ihara zeta function

- The graph Kazakov-Migdal model generates the non-collapsing Wilson loops, which are countable

- We can perform the unitary matrix integral exactly in the large N limit and the partition function of the graph Kazakov-Migdal model is given by the infinite product of the Ihara zeta function

- We expect much more applications of the graph zeta function to the counting (index) of the gauge invariant operators (chiral rings) in quiver gauge theory