Domain-Wall Skyrmions in QCD and Chiral Magnets

International Seminar-Type Online Workshop on Topological Solitons, Sep 13 /2023

Muneto Nitta

(Keio U. & Hiroshima U. WPI-SKCM^2)

Keio University
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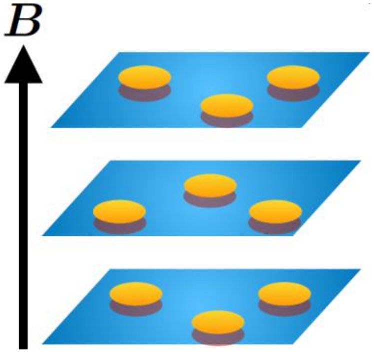


References Collaborators of the whole project

Minoru Eto(Yamagata U.), **Kentaro Nishimura**(KEK),
Zebin Qiu (Keio U.), **Yuki Amari** (Keio U.), **Calum Ross**(UCL)

1st part: DW-Skyrmions in QCD

[1] M.Eto, K.Nishimura & MN, arXiv: 2304.02940

Related papers

Quasicrystals in QCD, Z.Qiu & MN, *JHEP*, arXiv: 2304.05089
Quantum nucleation of topological solitons, M.Eto & MN, *JHEP* , arXiv: 2207.00211
Non-Abelian chiral solitons, M.Eto, K. Nishimura & MN, *JHEP* , arXiv: 2112.01381

2nd part: DW-Skyrmions in Chiral Magnets

[2] C.Ross & MN, Phys.Rev.B, arXiv: 2205.11417
[3] Y.Amari & MN, arXiv: 2307.11113

What are Skyrmions?

1961 T.H.R.Skyrme,
1983 E.Witten

Solitons in chiral Lagrangian
= Baryons(Nucleons)

3-dimensional
(a point in 3D)

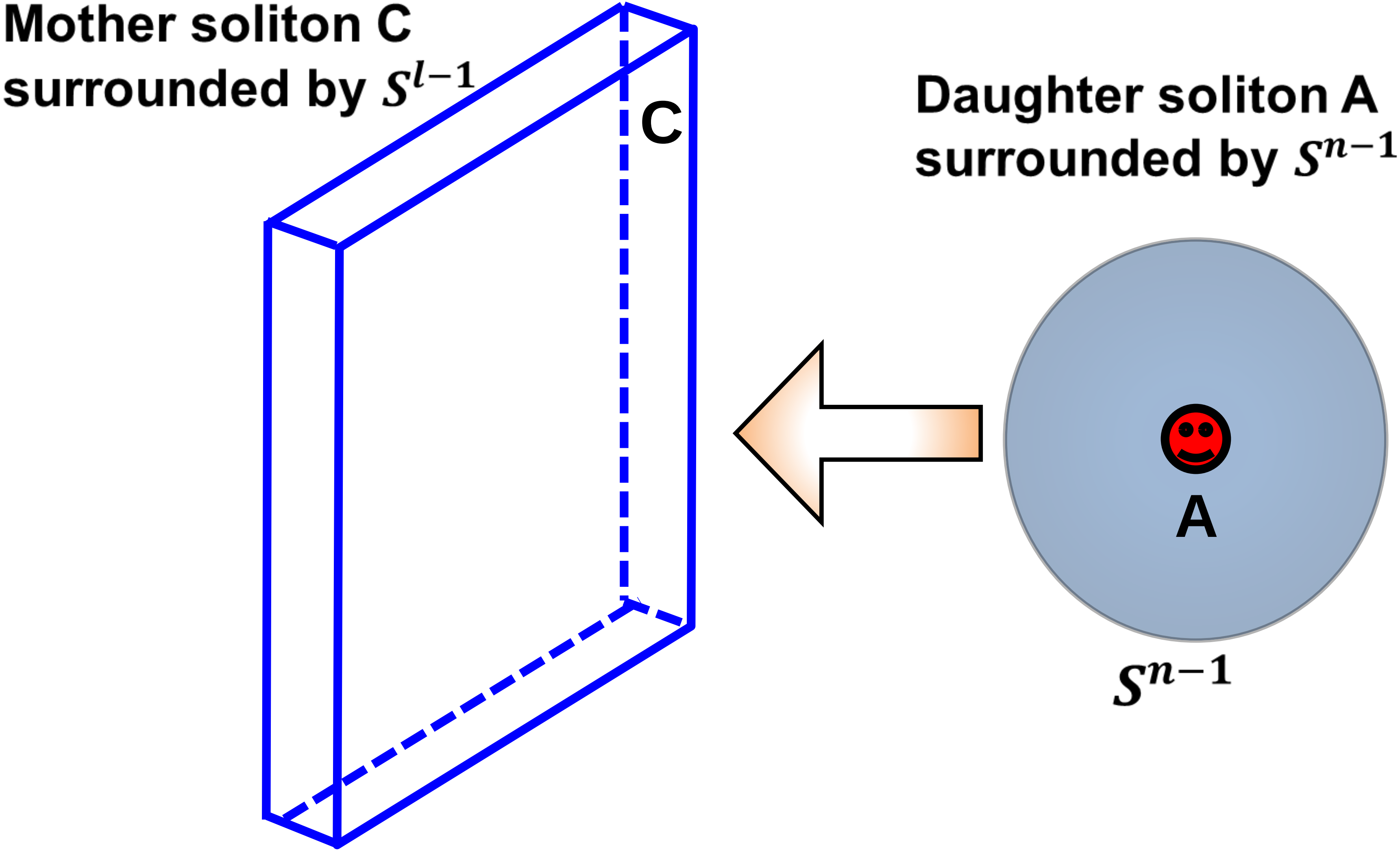
$$\pi_3\left(S^3\right) \cong \mathbb{Z}$$

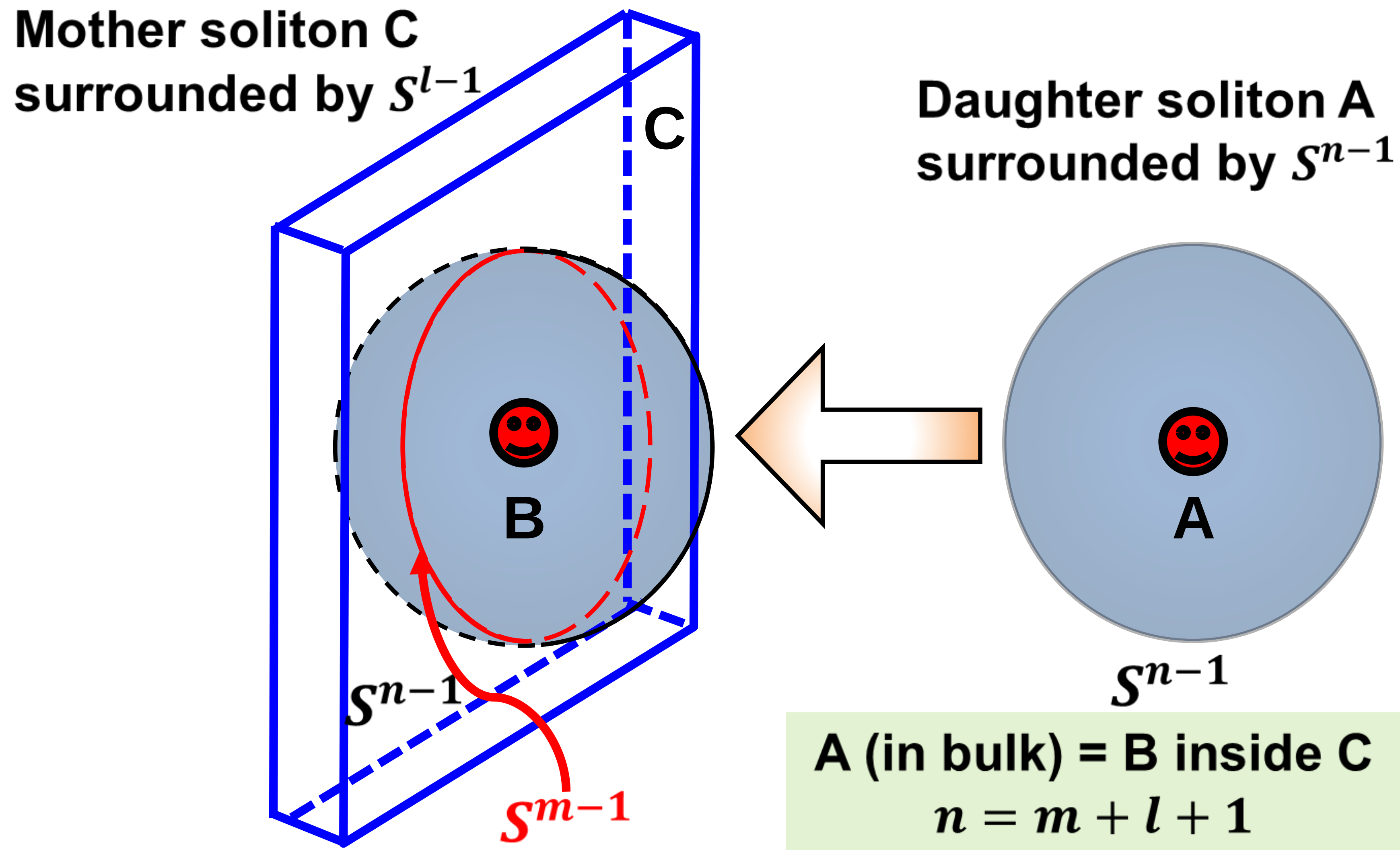
1975 A.A.Belavin & A.M.Polyakov

2D analog in magnets

2-dimensional
(a point in 2D,
a line in 3D)

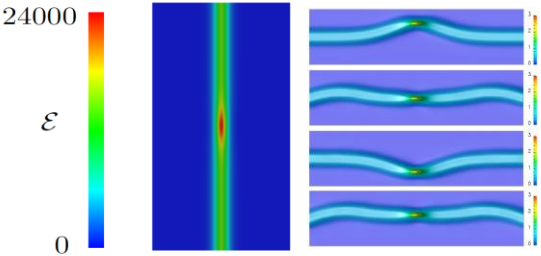
$$\pi_2\left(S^2\right) \cong \mathbb{Z}$$


Mother soliton C
surrounded by S^{l-1}
C
Daughter soliton A
surrounded by S^{n-1}
A
S^{n-1}


Mother soliton C surrounded by S^{l-1}
C
Daughter soliton A surrounded by S^{n-1}
B
A
S^{n-1}
S^{n-1}
S^{m-1}
A (in bulk) = B inside C
$n = m + l + 1$

A history of domain-wall Skyrmion

MN, Kobayashi, Gudnason, Eto, Ross ….

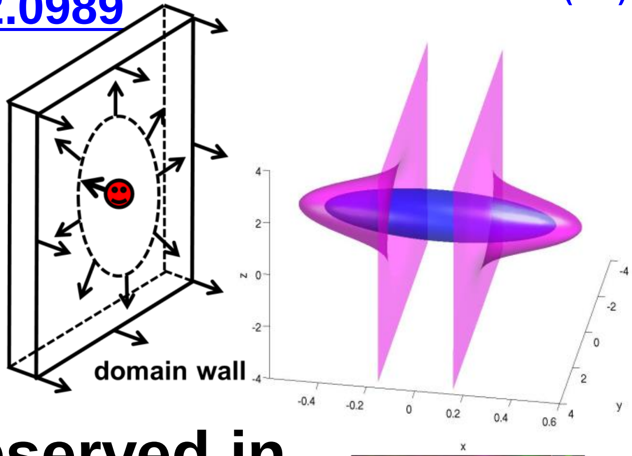


P. Jennings & P. Sutcliffe ('13)

(1) D=2+1 version

[1] MN, *PR*D86 ('12) 125004, 1207.6958
[2] M.Kobayashi & MN, *PRD*87 ('13)085003 1302.0989

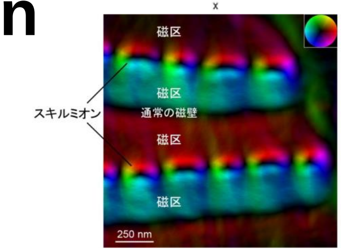
(2) D=3+1 version



[3] MN, *PRD*87 ('13) 025013 1210.2233
[4] S.B.Gudnason & MN,
*PRD*89 ('14) 085022 1403.1245
[5] M.Eto & MN, *PRD*91 ('15) 085044 1501.07038

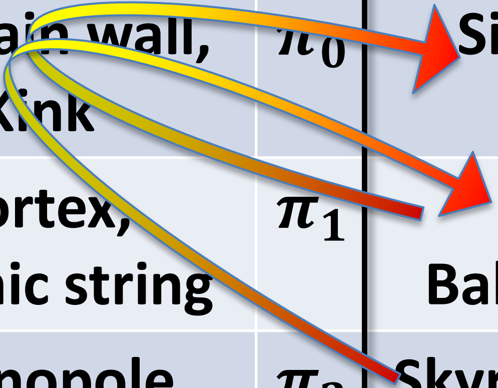
(3) In condensed matter physics, observed in chiral magnets.

T.Nagase et.al, Nature Comm. ('21)



[6] C.Ross & MN, *PRB*107 ('23) 024422 2205.11417
[7] Y.Amari & MN, 2307.11113 [hep-th]

Classification of topological solitons: 3 types



d	Defects		Textures		Gauge Structure	
1	Domain wall, Kink	π_0	Sine-Gordon soliton	π_1		
2	Vortex, Cosmic string	π_1	Lumps, Baby Skyrmion	π_2		
3	Monopole	π_2	Skyrmion, Hopfion	π_3		
4					YM instanton	π_3

$$\partial R^d \cong S^{d-1} \to G/H$$

$$\pi_{d\text{-}1}(G/H) \neq 0$$

$$R^d + \{\infty\} = S^d \to G/H$$

$$\pi_d(G/H) \neq 0$$

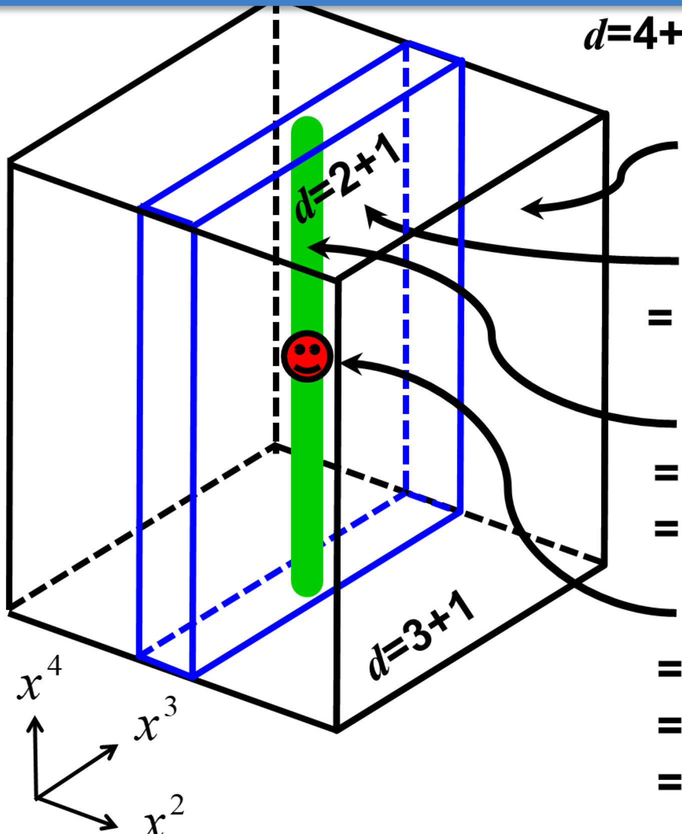
$$\partial R^d \cong S^{d-1} \to G$$

$$\pi_{d\text{-}1}(G) \neq 0$$

d: codimensions (in which solitons are particles, or on which solitons depend)

Relations among all topological solitons were obtained:

MN, *Phys.Rev.D* 105 (2022) 105006, 2202.03929



Maybe sometime, I can explain.

NA domain wall (with d=3+1 w.v.) **in bulk**

$U(N)$ NA vortex (with d=2+1 w.v.) **in bulk**
= $U(N)$ NA SG soliton in d=3+1 wall w.v.

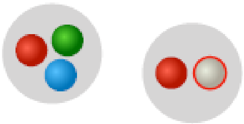
$SU(N)/U(1)^{N-1}$ monopole (with d=1+1 w.v.)
= $U(1)^{N-1}$global vortex in d=3+1 wall w.v.
= $\mathbb{C}P^{N-1}$ kink in d=2+1 vortex w.v.

$SU(N)$ YM instanton in bulk
= $SU(N)$ Skyrmion in d=3+1 wall w.v.
= $\mathbb{C}P^{N-1}$ lump in d=2+1 vortex w.v.
= $U(1)^{N-1}$ SG soliton in d=1+1 monopole w.v.

1st part

Domain-wall Skyrmions in QCD

QCD is a theory of quarks & gluons



In the vacuum, only hadrons (mesons, baryons).

➔ ***Can one calculate the nucleon mass $m_N \sim 939$ MeV ?***

(Lattice QCD, holographic QCD ?…).

➔**Low-energy QCD is described by mesons(pions, η) or Nambu-Goldstone modes. Chiral perturbation theory.**

How are baryons described? Cf: The Skyrme model.

In this talk, I show the effective nucleon mass (in a certain situation)

$$"m_N" = \frac{16\pi f_\pi^2}{3m_\pi} \sim 1.03 \text{ GeV}$$

Vacuum values

$f_\pi \sim 93\text{MeV},$

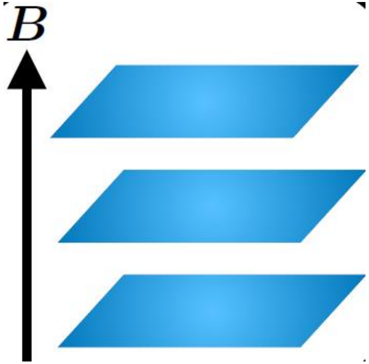
$m_\pi \sim 140 \text{ MeV}$

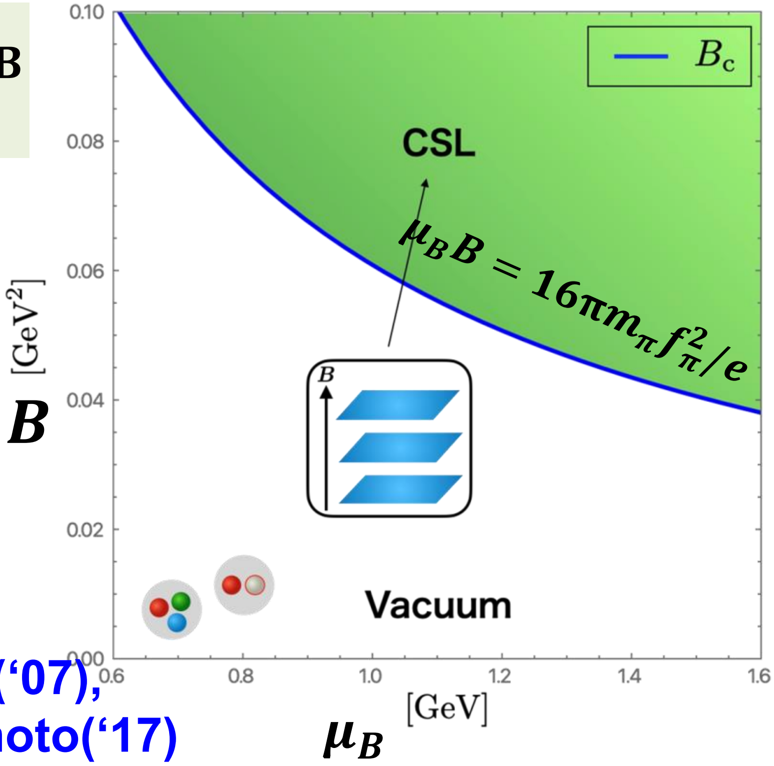
Summary of my talk

Chiral Soliton Lattice(CSL) phase

chemical pot. μ_B

magnetic field B





Son & Stephanov('07),
Brauner & Yamamoto('17)

Solitons carry baryon

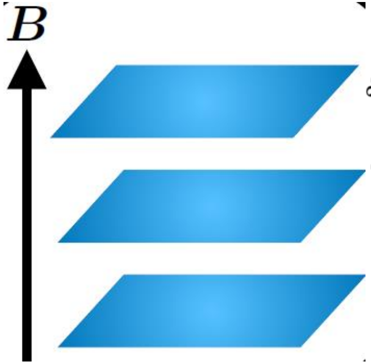
Summary of my talk

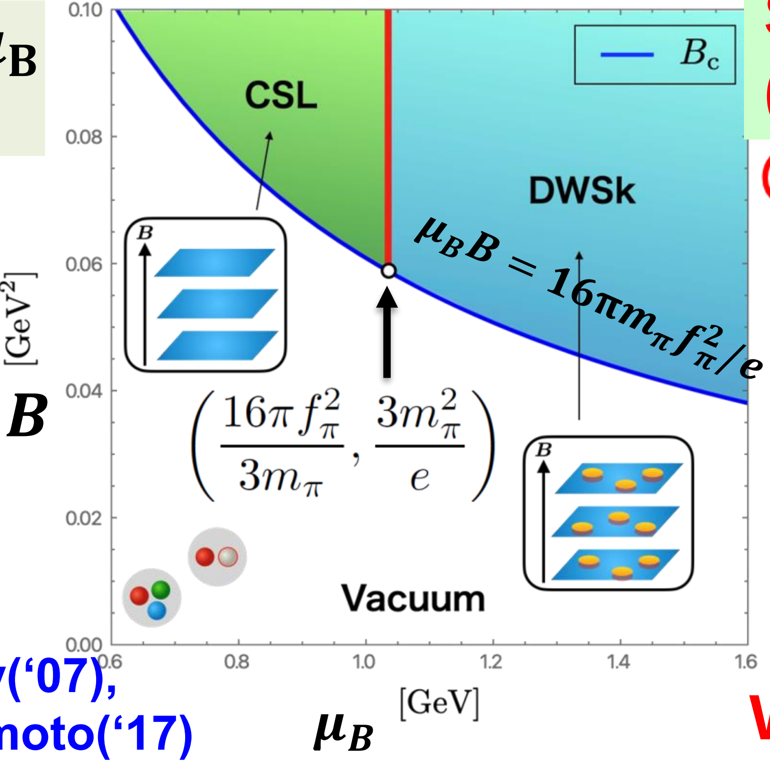
Our results: New phase in QCD

Chiral Soliton Lattice(CSL) phase

chemical pot. $\mu_{\rm B}$

magnetic field B



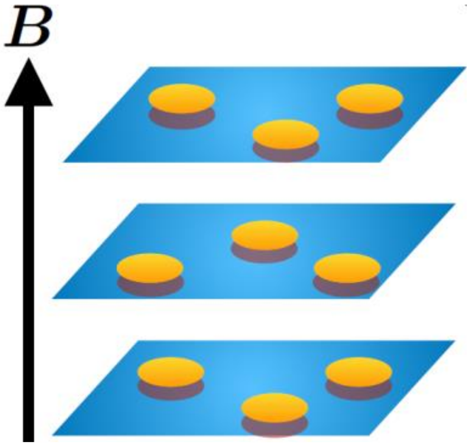


Son & Stephanov('07),
Brauner & Yamamoto('17)

Solitons carry baryon #

Domain-Wall Skyrmion Crystal (DW-SkX) phase

@ $\mu_B \geq 1.03$ GeV



Walls & Skyrmions carry baryon #

Remarkable points of our work

(1) We show this in the chiral perturbation theory @ the leading order $\mathcal{O}(p^2)$ *without higher derivative (Skyrme) term*. Thus, it is model independent.

(Skyrmions are stable without the Skyrme term.)

(2) The critical μ_B coincides with the instability of CSL via charged pion condensation (Brauner-Yamamoto '17).

Chiral sine-Gordon model in QCD [Son-Stephanov('07)]

SU(2) Nambu-Goldstone fields $\Sigma = \exp\left(\frac{i\sigma^a\pi^a}{f_\pi}\right) = \exp(i\sigma^a\chi_a)$

Chiral Lagrangian with Wess-Zumino-Witten(WZW) term

$$\mathcal{L}_{\chi\mathrm{PT}} = \frac{f_\pi^2}{4}\mathrm{Tr}\left[D_\mu\Sigma D^\mu\Sigma^\dagger + m_\pi^2(\Sigma+\Sigma^\dagger)\right] + \mathcal{L}_{\mathrm{WZW}}$$

$$\mathcal{L}_{\mathrm{WZW}} = -\left(A_\mu^{\mathrm{B}} + \frac{1}{2}A_\mu^{\mathrm{EM}}\right) j_{\mathrm{B}}^\mu$$

$$D_\mu\Sigma \equiv \partial_\mu\Sigma + \mathrm{i}eA_\mu[Q,\Sigma], \quad Q = \frac{1}{6}\mathbf{1} + \frac{1}{2}\tau_3$$

$$L_\mu = \Sigma\partial_\mu\Sigma^\dagger, \ R_\mu = \partial_\mu\Sigma^\dagger\Sigma$$

$$j_{\mathrm{B}}^\mu = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\left\{\mathrm{Tr}L_\nu L_\alpha L_\beta - \frac{3}{2}i\partial_\nu\left(A_\alpha\sigma^3(L_\beta+R_\beta)\right)\right\}$$

A constant magnetic field $\boldsymbol{B}$

A baryon chemical potential $A_\mu^{\mathrm{B}} = (\mu_{\mathrm{B}}, \vec{0})$

Ignoring charged pions $\chi_{1,2} = 0$

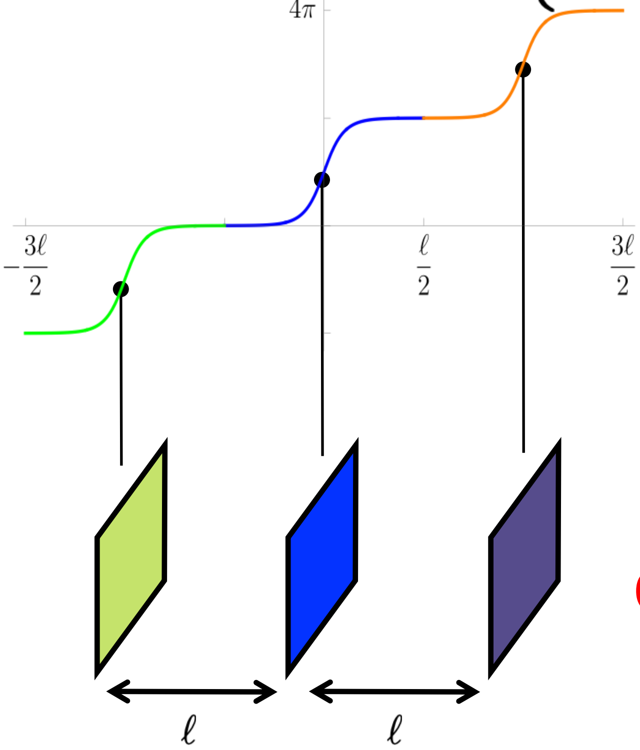
$$A_\mu^{\mathrm{B}} j_{\mathrm{B}}^\mu = \mu_{\mathrm{B}} j_{\mathrm{B}}^{\mu=0}$$

↑

Baryon# density

Previous works assume 1D structure

$$\mu_B\, j^\mu_{\rm B} = -\frac{\mu_{\rm B}}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\left\{{\rm Tr} L_\nu L_\alpha L_\beta - \underline{\frac{3}{2} i\partial_\nu\left(A_\alpha\sigma^3(L_\beta+R_\beta)\right)}\right\}$$



In 1D, this part gives

$$\sim\mu_{\rm B} B\partial_z\chi_3 \qquad (\chi_{1,2}=0)$$

magnetic field

chemical potential (finite density)

Chiral soliton lattices

Son & Stephanov('07), Brauner & Yamamoto('17)

Ignoring charged pions (chiral sine-Gordon model)
= Sine-Gordon model + topological term

$$\mathcal{L} = \frac{f_\pi^2}{2}(\partial_\mu \chi_3)^2 - f_\pi^2 m_\pi^2 (1 - \cos\chi_3) + \boxed{\frac{e\mu_{\rm B}}{4\pi^2} \boldsymbol{B} \cdot \boldsymbol{\nabla} \chi_3}$$

topological term

Cf. The same model appears in chiral magnets in which a topological term is the Dzyaloshinskii–Moriya interaction.

The condition that domain walls appear in the g.s.

$$E = 8 m_\pi f_\pi^2 \boxed{- \frac{e\mu_{\rm B} B}{2\pi}} \leq 0 \text{ !!} \quad \Leftrightarrow \quad \boxed{\mu_B B = 16\pi m_\pi f_\pi^2 / e}$$

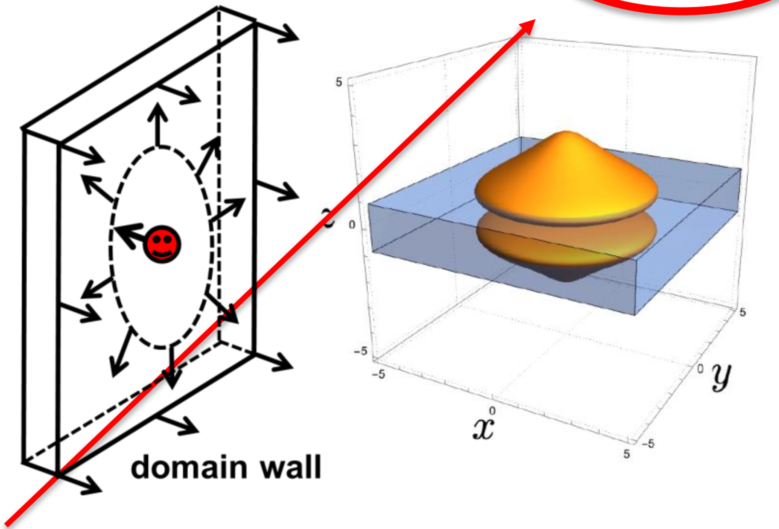
DW tension of a single soliton

Topological term

$$\chi_3 = 4 \tan^{-1} e^{m_\pi (z - Z)}$$

Previous works assume **1D** structure

$$\mu_B\, j_{\rm B}^{\mu} = -\frac{\mu_{\rm B}}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\left\{\boxed{\mathrm{Tr} L_\nu L_\alpha L_\beta} - \underline{\frac{3}{2} i\partial_\nu \left(A_\alpha \sigma^3 (L_\beta + R_\beta)\right)}\right\}$$



In 1D, this part gives

$\sim\mu_{\rm B} B \partial_z \chi_3$ $\quad(\chi_{1,2} = 0)$

magnetic field

chemical potential (finite density)

$\chi_{1,2} \neq 0$

Due to this term (≠ 0 with charged pions) in full 3D,

DW Skyrmion lattices are true ground state!!

(in certain parameter region).

§ 2 Derivation & Some more details

(1) Considering a single soliton
(2) Constructing DW world-volume effective theory
(3) Constructing lumps (baby Skyrmions)

Technical details

(1) Considering a single soliton

$$\chi_3^{\rm single} = 4\tan^{-1} e^{m_\pi (z-Z)} \qquad \Sigma_0 = {\rm e}^{{\rm i}\tau_3 \chi_3}$$

More general solution with S^2 moduli (collective coordinates)

$$\Sigma = g\Sigma_0 g^\dagger \qquad g \in SU(2)_{\rm V}$$

$$= [\mathbf{1}_2 + (u^2-1)\phi\phi^\dagger]u^{-1} \qquad u \equiv {\rm e}^{{\rm i}\chi_3^{\rm single}}$$

Non-Abelian soliton

$$\boldsymbol{\phi} \in \mathbb{C}^2, \qquad \boldsymbol{\phi}^+\boldsymbol{\phi} = 1$$

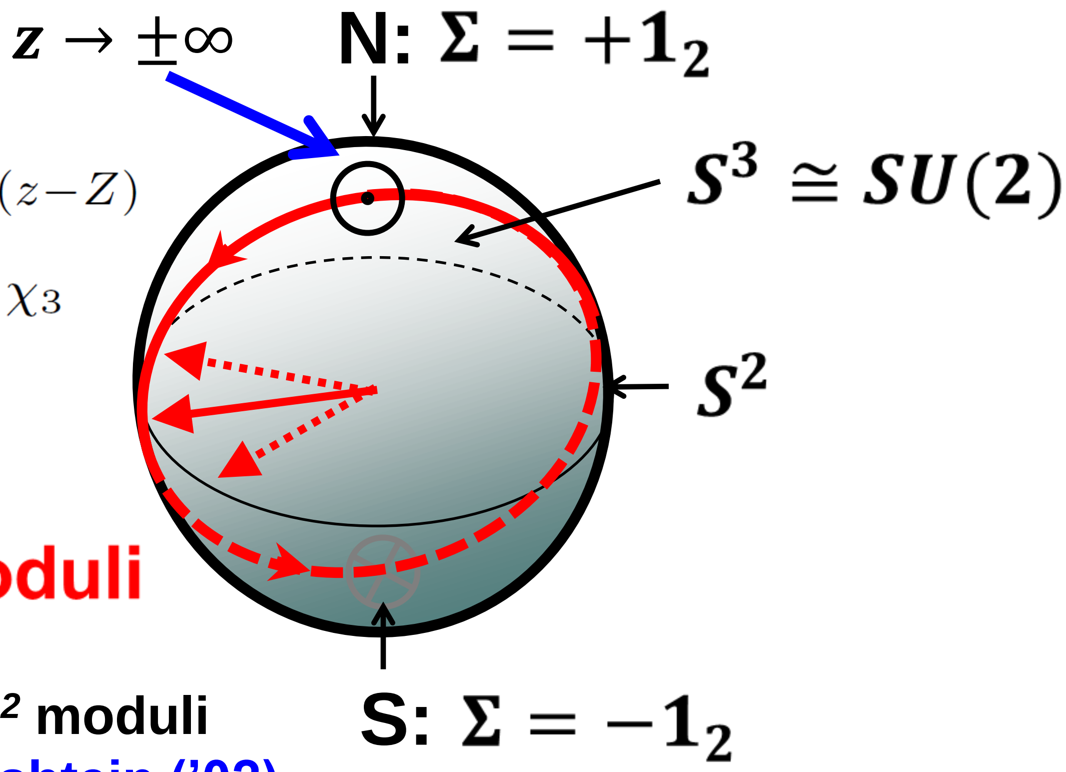
$$\boldsymbol{g\sigma_3 g^+} = 2\boldsymbol{\phi\phi}^+ - \mathbf{1}_2$$

Cf: η-solitons under rotation are also non-Abelian

Eto, Nishimura & MN, *JHEP* 08 (2022) 305, 2112.01381 [hep-ph]

$$\chi_3^{\text{single}} = 4\tan^{-1} e^{m_\pi (z-Z)}$$

$$\Sigma_0 = \mathrm{e}^{\mathrm{i}\tau_3 \chi_3}$$



Charged pions

S^2 moduli

Cf: Domain wall with S^2 moduli

Losev, Shifman & Vainshtein ('02)

Ritz, Shifman & Vainshtein ('04)

(2) Constructing DW world-volume effective theory via the moduli (Manton) approximation

(a) Promote moduli to fields on D=2+1 worldvolume

$$\phi \to \phi(x^\alpha)\,, \qquad (\alpha = 0, 1, 2)$$

Translational and orientational moduli

(b) Integrate over codimension x^3

→ **D=2+1 worldvolume effective theory**

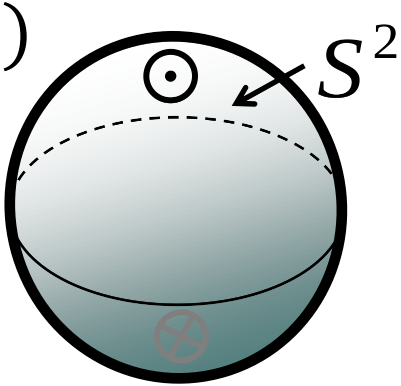
**** A well-defined & established method to construct e.g. monopole or instanton moduli space.**

A review: O(3) nonlinear sigma model

$$\mathcal{L} = \frac{1}{2}\partial^{\mu} n \partial_{\mu} n$$

$$n = (n_1, n_2, n_3)$$

$$n^2 = 1$$



Target space $S^2 = \frac{SO(3)}{SO(2)}$

$$\mathbb{C}P^1 = \frac{SU(2)}{U(1)} \cong S^2$$

$\mathbb{C}P^1$model

$$\mathcal{L} = \partial^{\mu}\phi^{+}\partial_{\mu}\phi + \phi^{+}\partial^{\mu}\phi\phi^{+}\partial_{\mu}\phi$$

$$n \equiv \phi^{\dagger}\boldsymbol{\sigma}\phi$$

$\phi \sim exp(ia)\phi,\ |\phi|^2 = 1 \quad \phi \in \mathbb{C}^2$

(2) Constructing DW world-volume effective theory via the moduli (Manton) approximation

$$\mathcal{L}_{\rm DW} = -8m_\pi f_\pi^2 + \frac{e\mu_{\rm B} B}{2\pi} + \mathcal{L}_{\rm norm} + \mathcal{L}_{\rm WZW}$$

DW tension

Topological term for DW

Gauged $\mathbb{C}P^1$ model

$$\mathcal{L}_{\rm norm} = \frac{16 f_\pi^2}{3m_\pi}\left[(\phi^\dagger D_\alpha \phi)^2 + D_\alpha \phi^\dagger D^\alpha \phi\right]$$

$$D_\alpha \phi = (\partial_\alpha + {\rm i}\tfrac{e}{2}\tau_3 A_\alpha)\phi$$

Background gauge field at $\mathcal{O}(p^2)$

$$\mathcal{L}_{\rm WZW} = \boxed{2\mu_{\rm B} q} + \frac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$$

$$n_a = \phi^\dagger \tau_a \phi$$

Topological term for 2D (baby) Skyrmions $\pi_2(S^2) \cong \mathbb{Z}$

$$q \equiv -\frac{\rm i}{2\pi}\epsilon^{ij}\partial_i \phi^\dagger \partial_j \phi = \frac{1}{8\pi}\epsilon^{ij}\boldsymbol{n}\cdot(\partial_i \boldsymbol{n} \times \partial_j \boldsymbol{n})$$

Remark: chiral perturbation theory

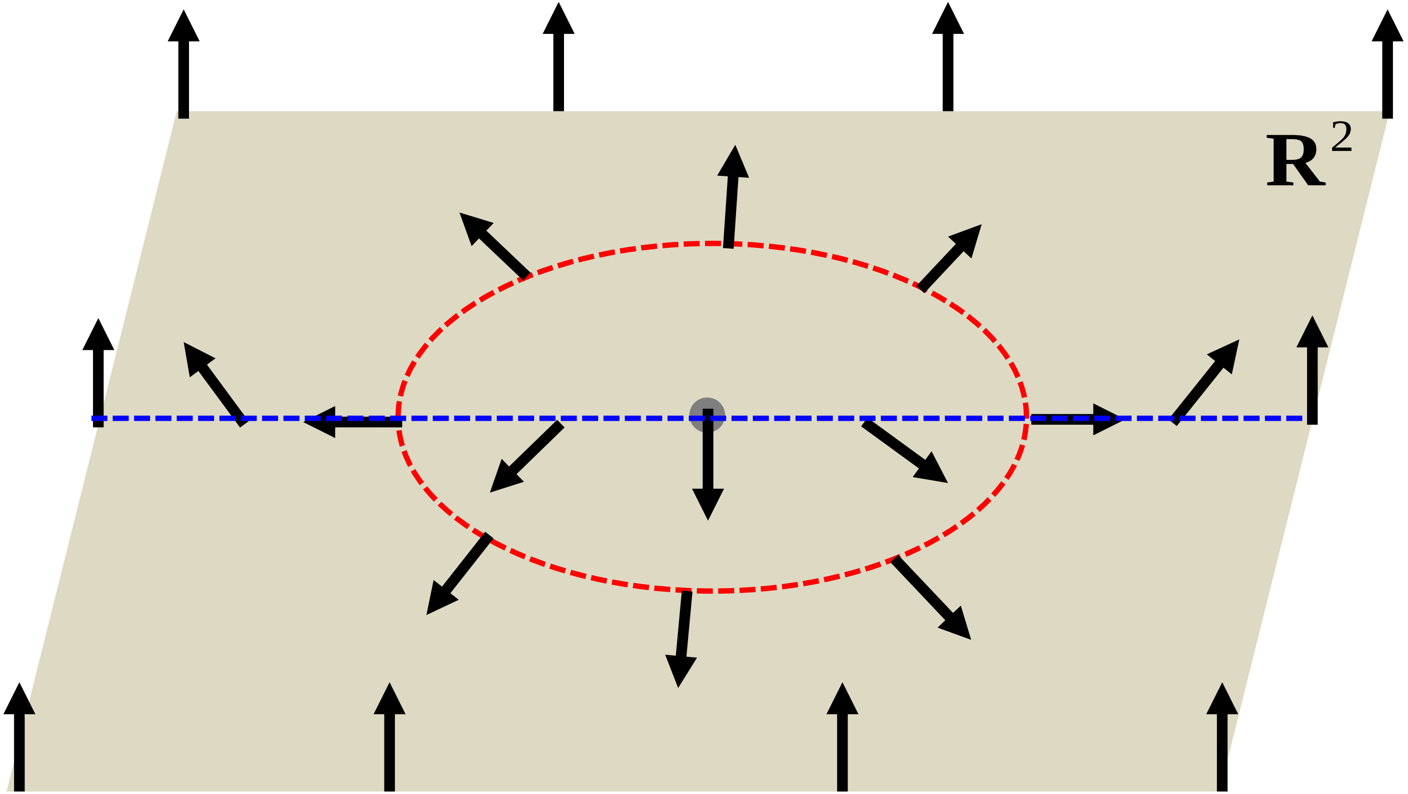
$$\partial_\mu \,, m_\pi \,, A_\mu = \mathcal{O}(p^1)\,, \qquad A_\mu^{\mathrm{B}} = \mathcal{O}(p^{-1})$$

$$\boldsymbol{F_{\mu\nu}^2 \in \mathcal{O}(p^4)}$$

Gauge field is nondynamical at the leading $\boldsymbol{\mathcal{O}(p^2)}$

2D (baby) Skyrmion (or lump)

$\pi_2(S^2) \cong \mathbb{Z}$



(3) Constructing lumps (baby Skyrmions).

$$\mathcal{H}_{\rm DW} = \frac{4f_\pi^2}{3m_\pi}(\partial_i \boldsymbol{n})^2 - 2\mu_{\rm B} q - \frac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$$

$$\partial_i \boldsymbol{n}\cdot\partial_i \boldsymbol{n} = \frac{1}{2}\underbrace{(\partial_i \boldsymbol{n} \pm \epsilon_{ij}\boldsymbol{n}\times\partial_j \boldsymbol{n})^2}_{=\,0} \pm 8\pi q$$

BPS **equation** **(the same as usual)**

Bogomol'nyi bound

$$E_{\rm DW} \geq \underbrace{\frac{32 f_\pi^2 \pi \boxed{|k|}}{3m_\pi} - 2\mu_{\rm B}\boxed{k}} + \frac{e\mu_{\rm B}B}{4\pi}\oint {\rm d}S_i\, x^i(n_3-1)$$

Can become < 0 ?

→ nontrivial constraint

$$k = \int {\rm d}^2x\, q \in \mathbb{Z}$$

lump number

BPS lumps (the same with Belavin &Polyakov)

k **lump solutions**

$$w \equiv x + \mathrm{i}y$$

$$n_3 = \frac{1-|f|^2}{1+|f|^2}, \quad f = \frac{b_{k-1}w^{k-1} + \cdots + b_0}{w^k + a_{k-1}w^{k-1} + \cdots + a_0}$$

Baryons appear pairwise

$$\boxed{N_{\mathrm{B}} = \int \mathrm{d}^3x\,\mathcal{B} = 2\int \mathrm{d}^2x\,q = 2k}$$

$\pi_2(S^2) \cong \mathbb{Z}$ **on a wall (in D=2+1)**

$\updownarrow$ **2:1**

$\pi_3(S^3) \cong \mathbb{Z}$ **in the bulk (in D=3+1)**

However, it's not the end of the story.
There are two nontrivial constraints as follows.

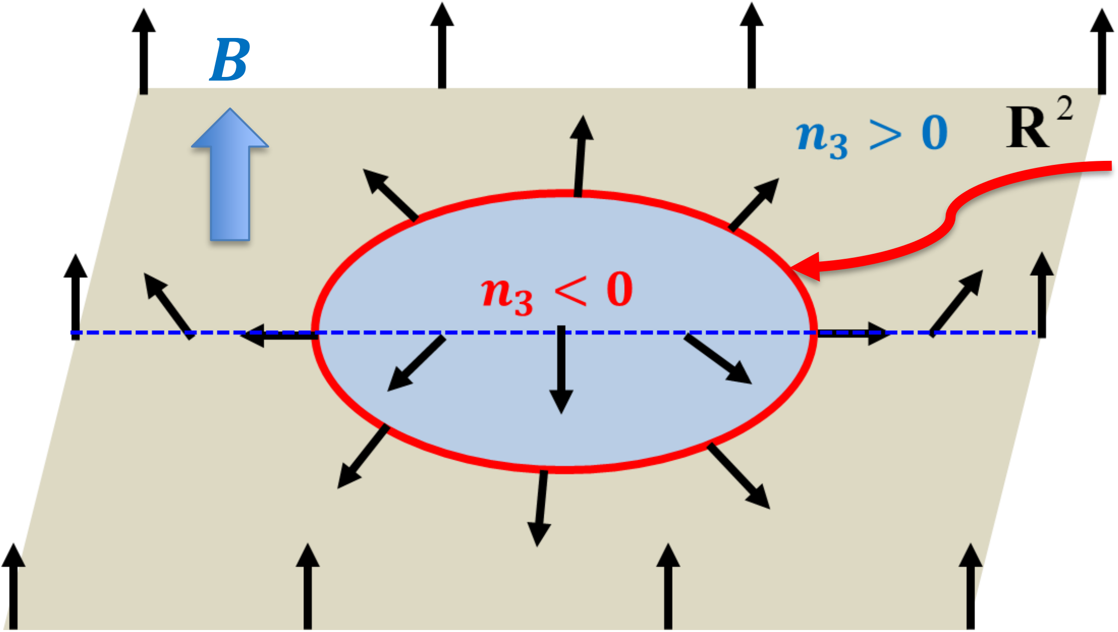
superconducting ring

$$n_1 + \mathrm{i}n_2 \to \mathrm{e}^{-\mathrm{i}\lambda}(n_1 + \mathrm{i}n_2)$$

n_3 is neutral in $\mathrm{U(1)_{EM}}$

Area (size) quantization

$$\boxed{BS_D = \int_D \mathrm{d}^2x\, B = \oint_C \mathrm{d}x^i A_i = \frac{1}{e}\oint_C \mathrm{d}x^i \partial_i \psi = \frac{2\pi k}{e}}$$



➔ stability

Charged pion ($n_1 + in_2$) condensed along this ring ($n_3 = 0$).

$$n_1 + \mathrm{i}n_2 = \mathrm{e}^{i\psi}$$

$$|D_\alpha(n_1 + \mathrm{i}n_2)|^2 = 0$$

$$\partial_\alpha \psi = eA_\alpha$$

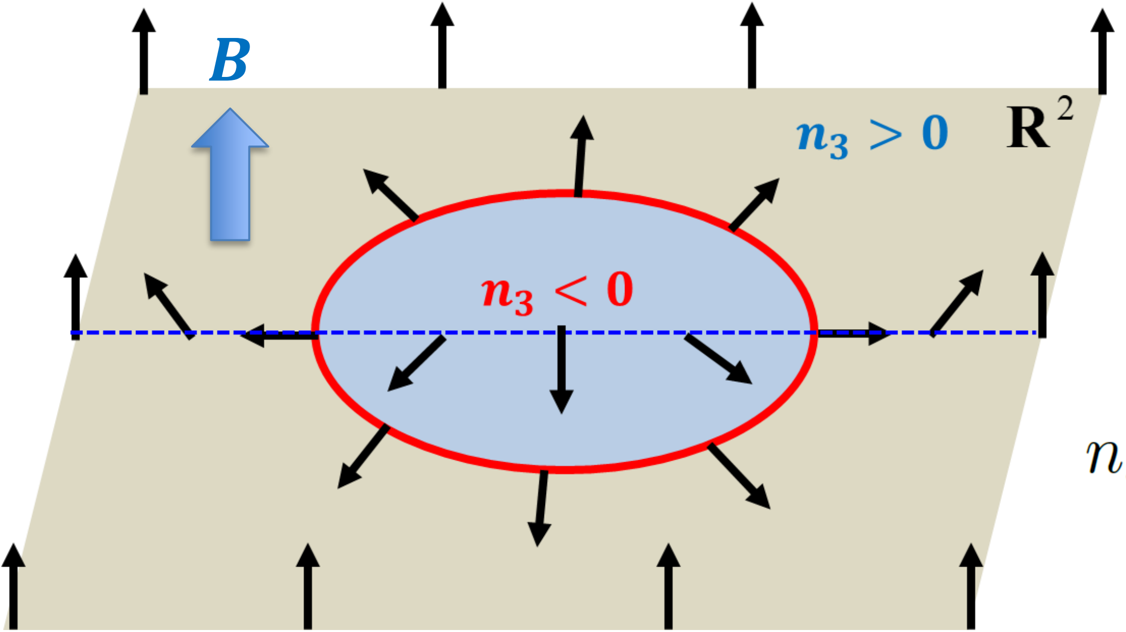
superconducting ring

$$n_1 + i n_2 \rightarrow \mathrm{e}^{-i\lambda}(n_1 + i n_2)$$

n_3 is neutral in $\mathrm{U(1)_{EM}}$

Area (size) quantization

$$BS_D = \int_D \mathrm{d}^2 x\, B = \oint_C \mathrm{d}x^i A_i = \frac{1}{e}\oint_C \mathrm{d}x^i \partial_i \psi = \frac{2\pi k}{e}$$



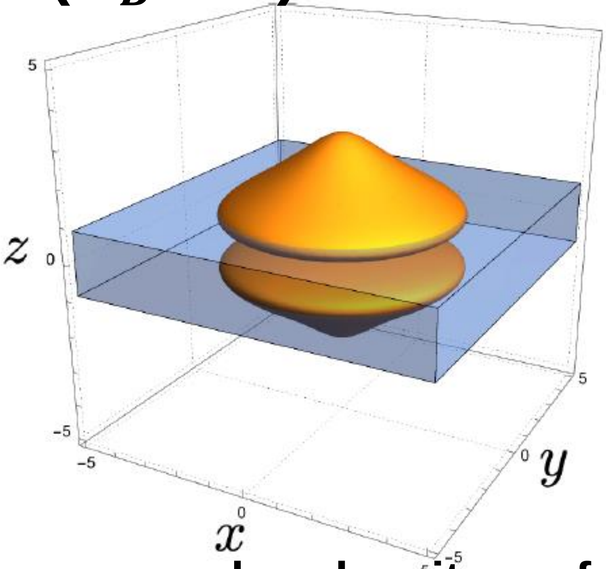
$$k = 1 \qquad f = b_0/w$$

$$n_3 = \frac{|w|^2 - |b_0|^2}{|w|^2 + |b_0|^2}$$

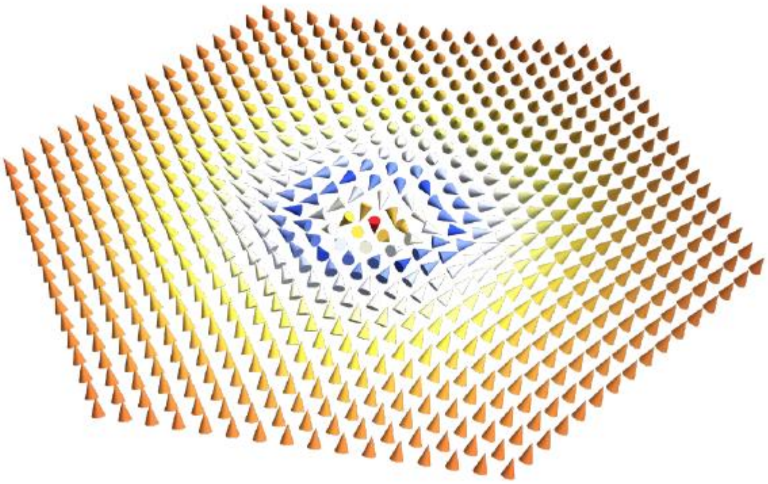
$$n_3 = 0 \ @\ |w| = |b_0|$$

$$|b_0| = \sqrt{2/eB}$$

$k = 1$ ($N_B = 2$) domain-wall Skyrmion



Iso-baryon number density surface



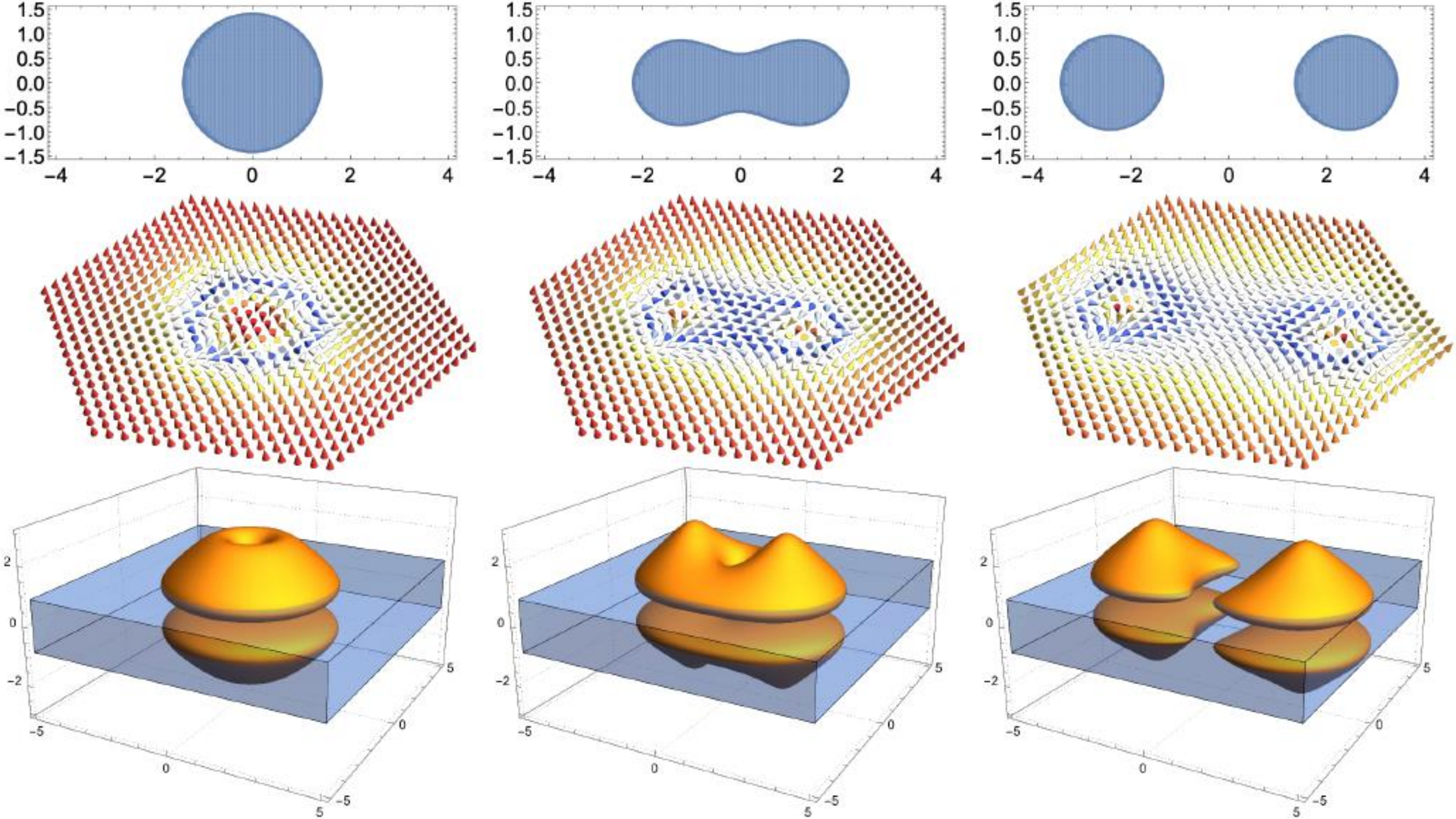
Macaron

Dorayaki

(Japanese sweets)

(kaiundo.co.jp)

$k = 2$ $(N_B = 4)$ DW Skyrmion: Area preserving deformation



DW-Skyrmion energy

Different physics between

$k = 1$ $(N_B = 2)$ & $k \geq 2$ $(N_B \geq 4)$

$$E_{\mathrm{DWSk}} = \frac{32\pi f_\pi^2}{3m_\pi}|k| - 2\mu_{\mathrm{B}}k + e\mu_{\mathrm{B}}B|b_{k-1}|^2$$

DW-Skyrmion energy

Different physics between
$k = 1$ ($N_B = 2$) & $k \geq 2$ ($N_B \geq 4$)

$$E_{\rm DWSk} = \frac{32\pi f_\pi^2}{3m_\pi}|k| - \cancel{2\mu_{\rm B}k} + e\mu_{\rm B}\cancel{B|b_{k-1}}|^2$$

$k = 1$ ($N_B = 2$) $\quad |b_0| = \sqrt{2/eB}$

A miracle cancelation between the last two terms!
Always positive energy.

DW-Skyrmion energy

Different physics between
$k = 1$ $(N_B = 2)$ & $k \geq 2$ $(N_B \geq 4)$

$$E_{\mathrm{DWSk}} = \underbrace{\frac{32\pi f_\pi^2}{3m_\pi}|k| - 2\mu_{\mathrm{B}}k}_{\textbf{Can become} < 0} + \cancel{e\mu_{\mathrm{B}}B|b_{k-1}|^2}$$

Size Quantization $|b_0| = \left(\frac{2k}{eB}\right)^{\frac{k}{2}}$

$\underline{k \geq 2\ (N_B \geq 4)}$ **Further constraint** $b_{k-1} = 0$

The condition that 2D Skyrmions appear on a wall in the ground state

$$E_{\mathrm{DWSk}} = \frac{32\pi f_\pi^2}{3m_\pi}\boxed{|k|} - 2\mu_{\mathrm{B}}\boxed{k} \leq 0$$

$$\boxed{\boldsymbol{\mu_B \geq \mu_c = \frac{16\pi f_\pi^2}{3m_\pi} \sim 1.03\ \mathrm{GeV}}}$$

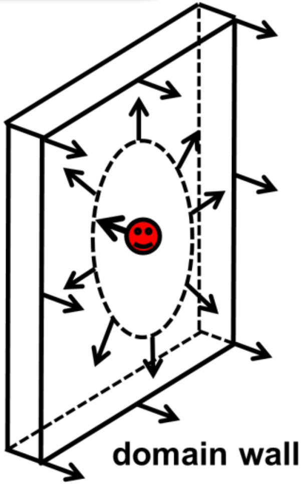
Nucleon mass in terms of pions' constants

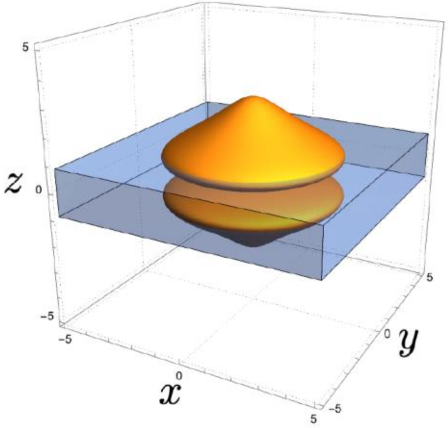
$$(\boldsymbol{\mu}_{\mathrm{B}}, \boldsymbol{B}) = \left(\frac{16\pi f_\pi^2}{3m_\pi}, \frac{3m_\pi^2}{e}\right)$$

$0.06\mathrm{GeV}^2$

$\sim 1.0 \times 10^{19}\mathrm{G}$

Future heavy-ion collider





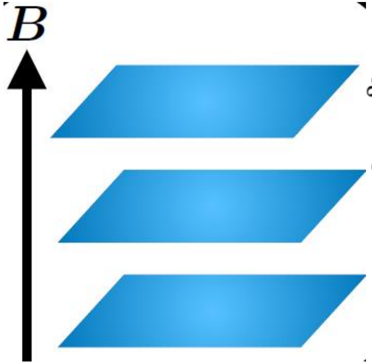
Summary of my talk

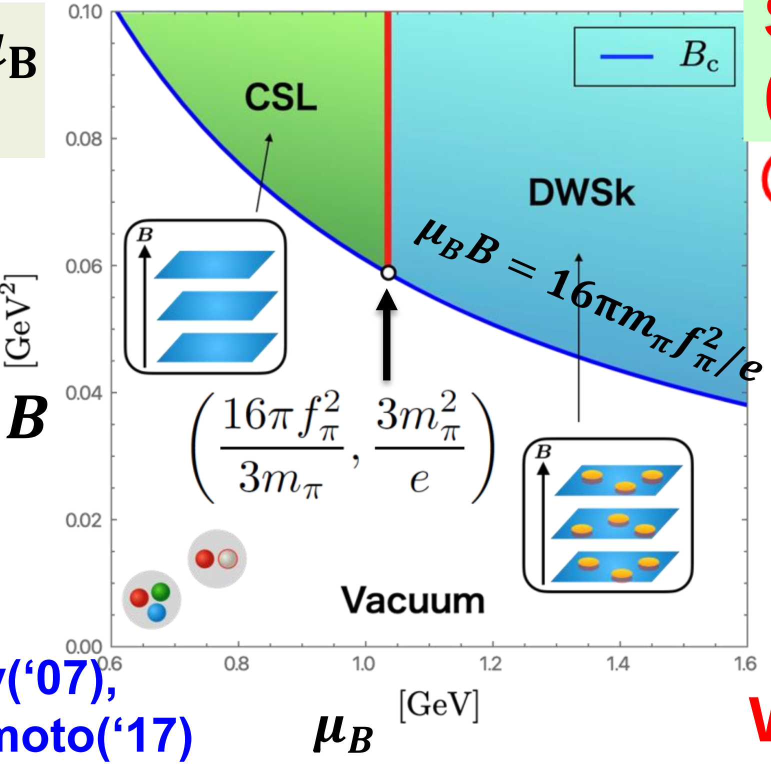
Our results: New phase in QCD

Chiral Soliton Lattice(CSL) phase

chemical pot. μ_{B}

magnetic field B



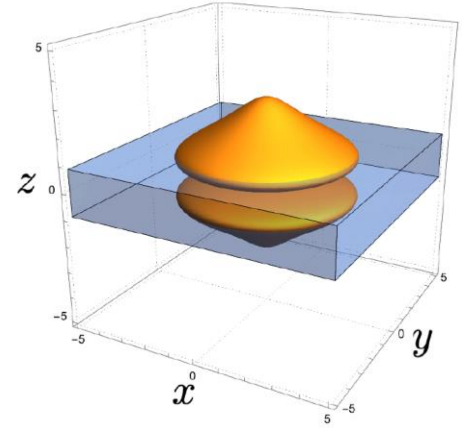


Son & Stephanov('07),
Brauner & Yamamoto('17)

Solitons carry baryon #

Domain-Wall Skyrmion Crystal (DW-SkX) phase

@ $\mu_B \geq 1.03$ GeV



Walls & Skyrmions carry baryon #

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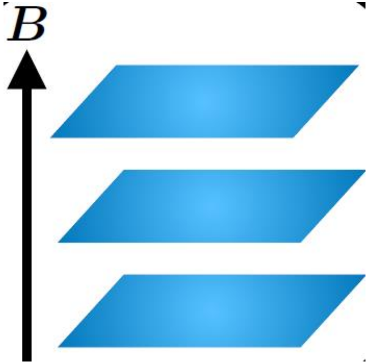

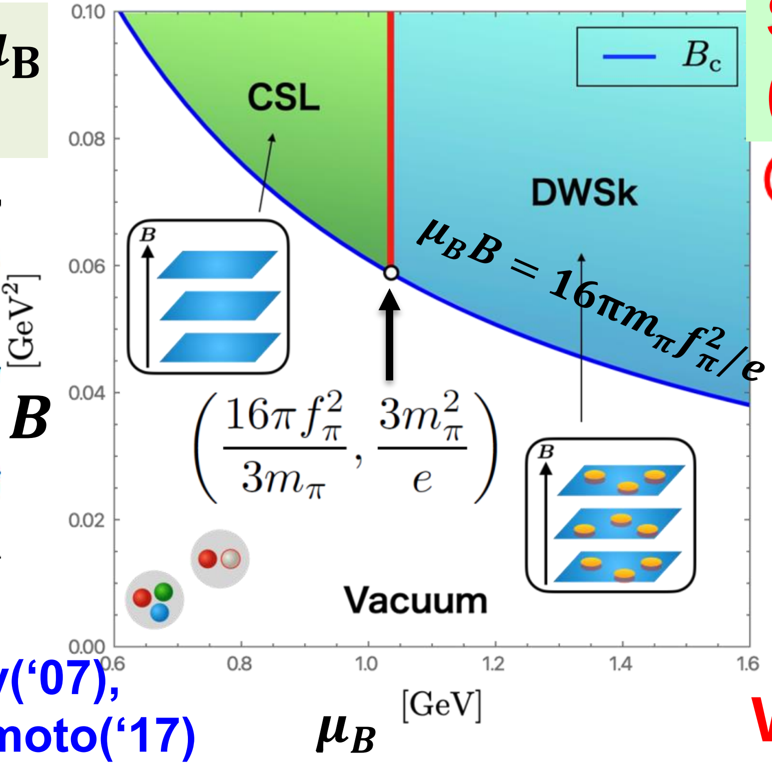

Domain-Wall Skyrmion Crystal (DW-SkX) phase

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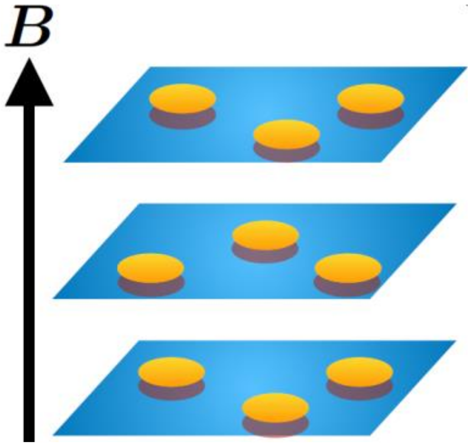

Son & Stephanov('07),
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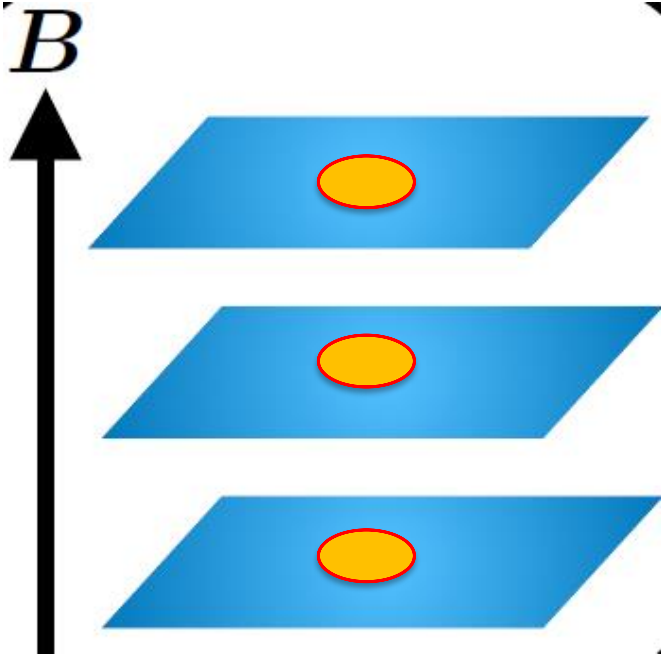
Solitons carry baryon #

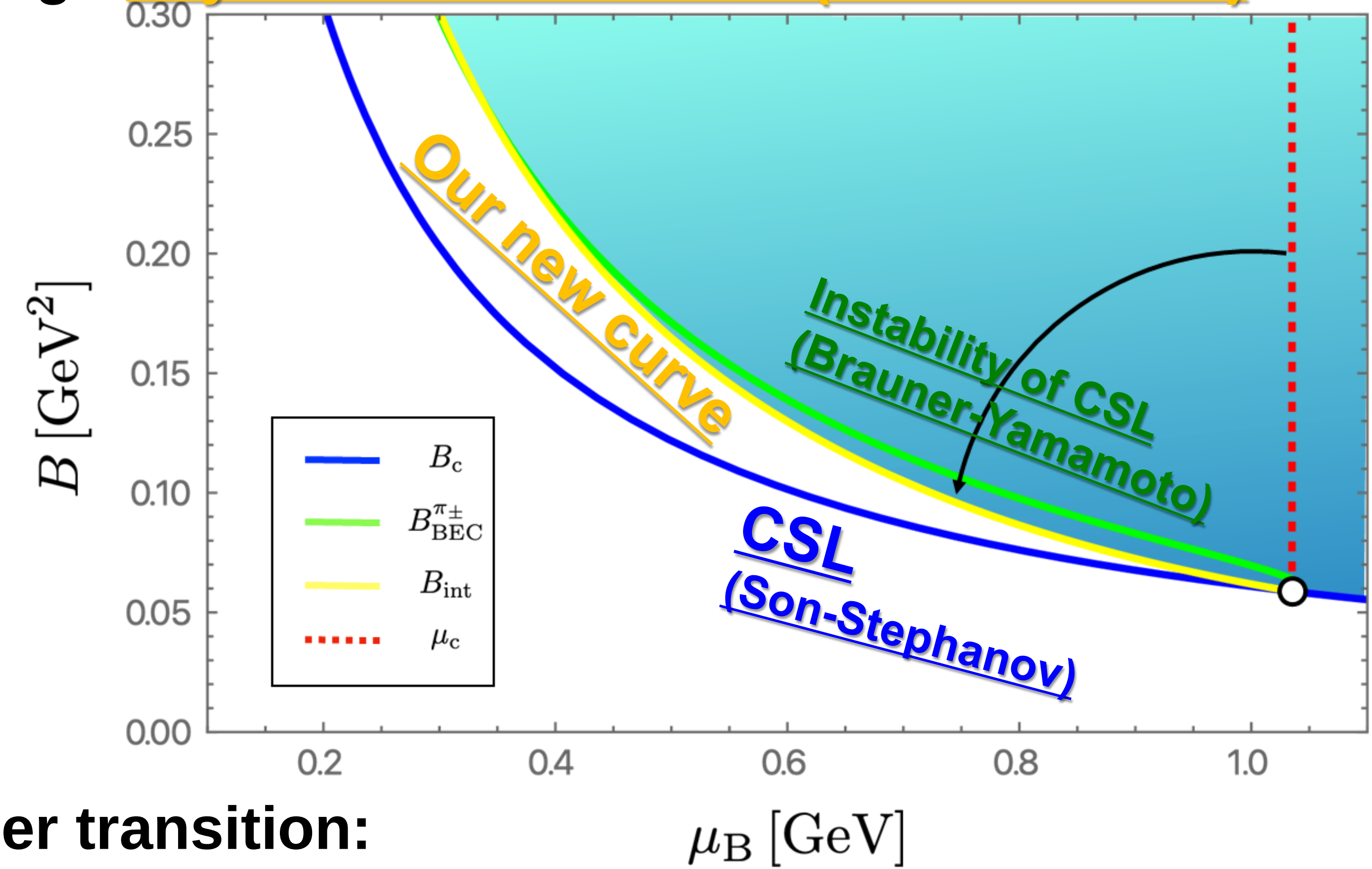
Walls & Skyrmions carry baryon #

On-going: Skyrmion-chain on CSL(multi-solitons)

Skyrmion chains



On-going: Skyrmion-chain on CSL(multi-solitons)



Our new curve

Instability of CSL (Brauner-Yamamoto)

CSL (Son-Stephanov)

1^{st} order transition:

CSL remains metastable between yellow and green curves.

(1) Quasicrystals in QCD

Z.Qiu & MN, *JHEP* 05 (2023) 170, 2304.05089 [hep-ph]

$$\phi_0 \equiv \frac{\eta}{f_\eta}, \quad \phi_3 \equiv \frac{\pi_3}{f_\pi}$$

WZW $\mathcal{L}_B = \frac{\mu}{4\pi^2}\boldsymbol{B} \cdot \left(\nabla\phi_3 + \underline{\frac{1}{3}\nabla\phi_0} \right)$ **➔both η and π modulate**

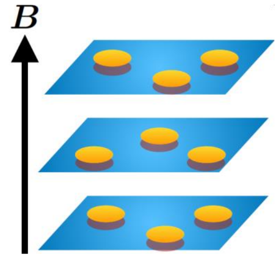
If $\alpha \equiv \frac{f_\pi^2}{f_\eta^2}$ **is**
rational ➔ lattice(crystal)
irrational ➔ quasicrystal

(2) Rotation (instead of magnetic field)

Eto, Nishimura & MN, *JHEP* 08 (2022) 305 2112.01381 [hep-ph]

WZW $\frac{\Omega\mu_\mathrm{B}^2}{2\pi^2 N_c}\partial_z\frac{\eta}{f_\eta}$ **➔ η CSL(Nishimura & Yamamoto), Non-Abelian CSL**

A lot of future directions!!



(1) Structure of SkX depending on μ_{B} & B

Interaction, triangular or square lattice

(2) CPT @ $\mathcal{O}(p^4)$ ➔ dynamical gauge field

(3) ➔ Bulk SkX (Chen, Fukushima & Qiu $B \neq 0$. Klebanov $B = 0$)

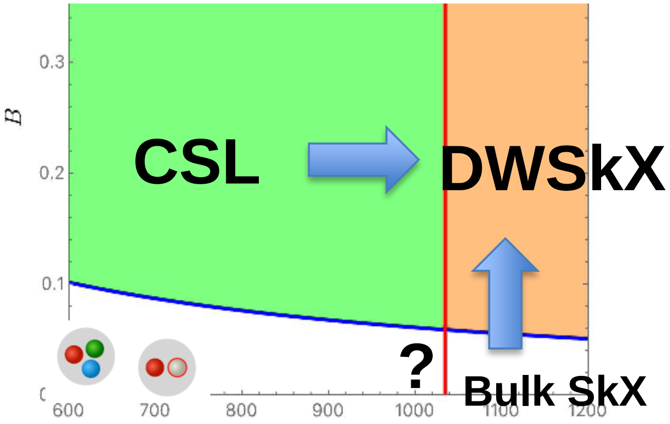
(4) Quantization ➔ proton/neutron (anyon?)

(5) $SU(3)_F$ ➔ $\mathbb{C}P^2$ model on a wall

(6) DWSkX under Rotation

(7) Nucleon mass?

(medium effect, binding energy)



Welcome to join
our collaboration !!

Chiral soliton lattice

$$\chi_3(z) = 2\mathrm{am}\left(\frac{mz}{k}, k\right) + \pi$$

Jacobi amplitude function

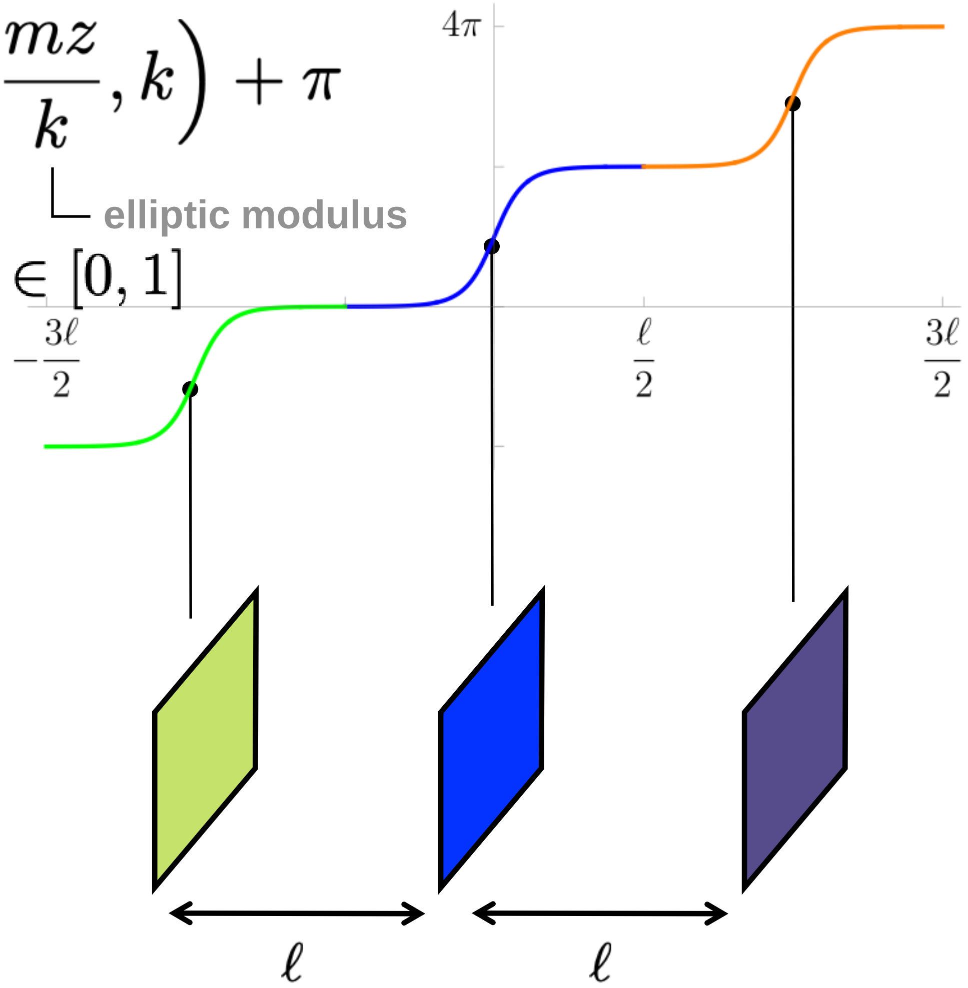
elliptic modulus

$k \in [0, 1]$

lattice spacing

$$\ell(k) = 2kK(k)/m$$

elliptic integral of 1st kind



2^{nd} part

Domain-wall Skyrmions in Chiral Magnets

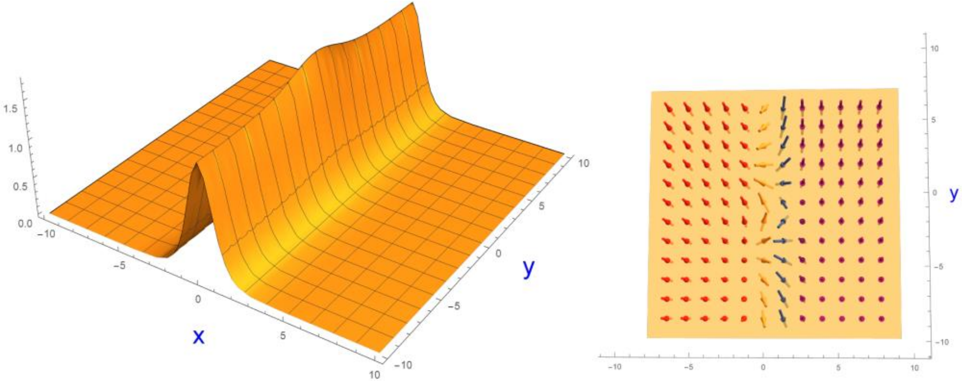
Chiral magnets

Magnetic domain wall

2D Skyrmions

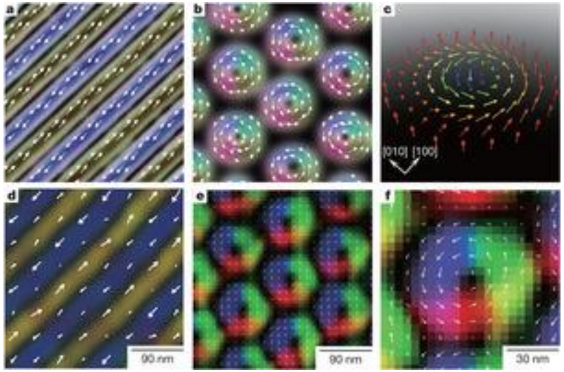
Important for nanotechnology such as
low-energy consumption magnetic memories.

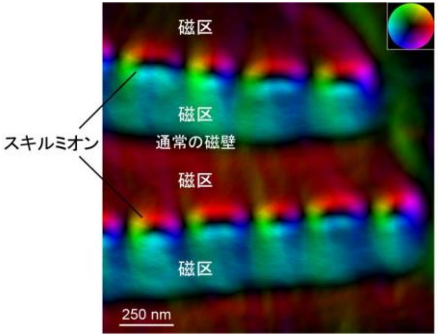
Domain-wall skyrmions



C.Ross & MN, *PRB*107 (‘23) 024422
2205.11417

a thin film of $Fe_{0.5}Co_{0.5}Si$
X. Z. Yu *et.al*
Nature 465, 901 (2010)





T.Nagase et.al,
Nature Comm. (‘21)

The model = O(3) model + the DM interaction

$$\mathcal{H} = -\frac{J}{2}\partial_k\mathbf{n}\cdot\partial_k\mathbf{n} + \underline{a^{-1}\mathbf{A}_k\cdot(\mathbf{n}\times\partial_k\mathbf{n})} + \frac{\mu^2}{2a^2}\left(1-n_z^2\right)$$

Dzyaloshinsky-Moriya (DM) interaction

SU(2) Field strength $F_{12} = \kappa^2\sigma_3$

SU(2) Gauge potential

$$\begin{aligned} A_0^a &= (0,0,0) \\ A_1^a &= -\kappa(\cos\vartheta, -\sin\vartheta, 0) \\ A_2^a &= -\kappa(\sin\vartheta, \cos\vartheta, 0) \end{aligned}$$

$$\begin{aligned} A_\mu &= A_\mu^a\sigma_a = -\kappa(0, \cos\vartheta\sigma_1 - \sin\vartheta\sigma_2, \sin\vartheta\sigma_1 + \cos\vartheta\sigma_2) \\ &= -\kappa\left(0, \begin{pmatrix} 0 & e^{i\vartheta} \\ e^{-i\vartheta} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -ie^{i\vartheta} \\ ie^{-i\vartheta} & 0 \end{pmatrix}\right) \end{aligned}$$

Spin-orbit coupling(SOC)

$\vartheta = 0$ **Dresselhaus** **Bloch** $\mathcal{H}_{\mathrm{DM}} = \kappa\mathbf{n}\cdot(\nabla\times\mathbf{n})$

$\vartheta = -\frac{\pi}{2}$ **Rashba** Néel $\mathcal{H}_{\mathrm{DM}} = \kappa(\mathbf{n}\cdot\nabla n_3 - n_3\nabla\cdot\mathbf{n})$

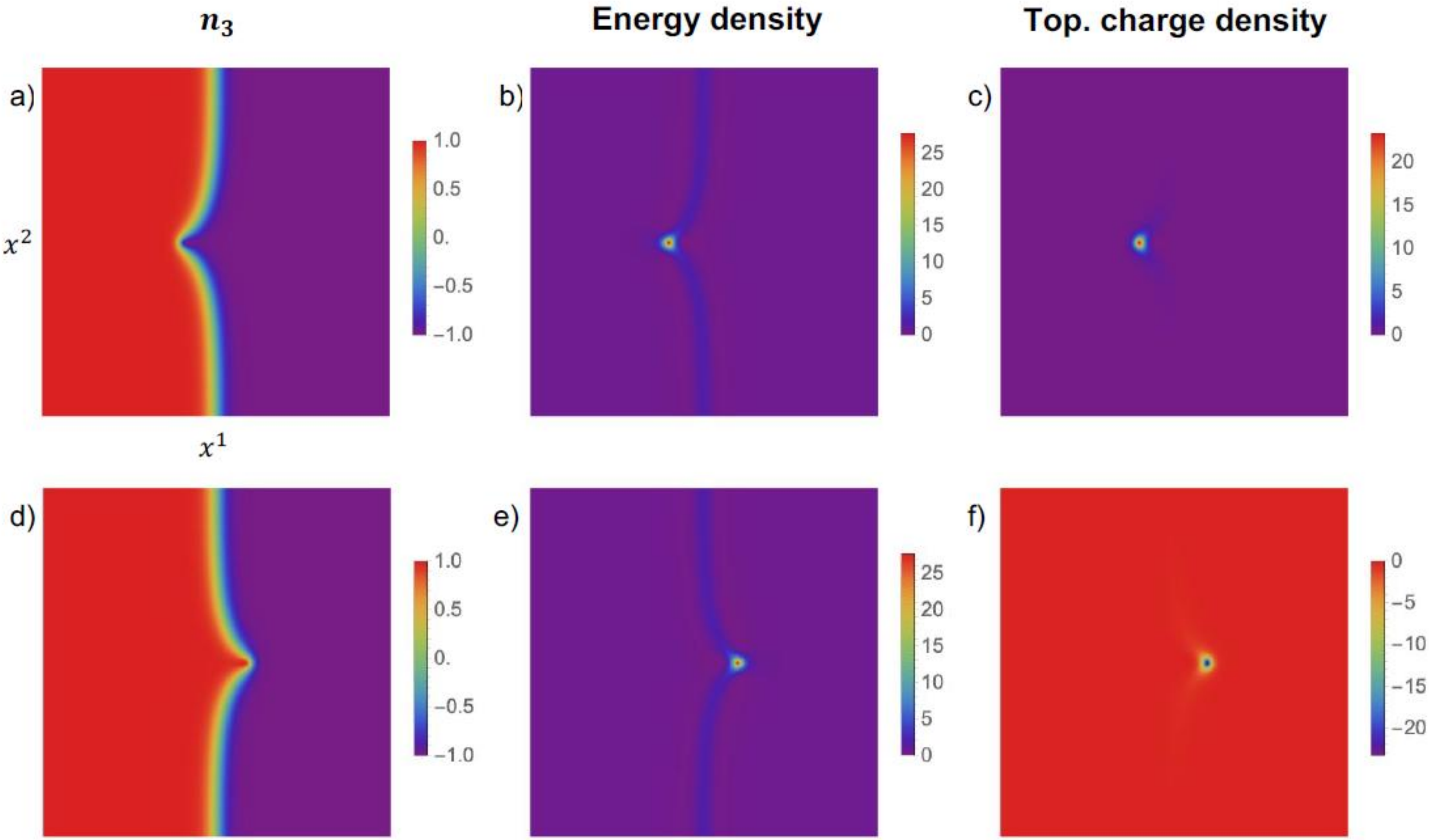
Moduli approximation:
Effective theory on a domain wall

$$\begin{aligned}\mathcal{E}_{\mathrm{eff}} &= \int dx^1 \mathcal{H} \\ &= \mu^{-1}\left\{(\partial_2 X)^2 + (\partial_2 \varphi)^2 \ - \kappa\pi\left(\mu\cos\varphi + \partial_2 X \sin\varphi\right)\right\} + \mathrm{const.}\end{aligned}$$

$$\Rightarrow \quad \begin{aligned}&\partial_2^2 \varphi - \frac{\kappa\pi}{2}\{\mu \sin\varphi - \partial_2 X \cos\varphi\} = 0 \ , \\ &\partial_2\left[\partial_2 X - \frac{\kappa\pi}{2}\sin\varphi\right] = 0 \ .\end{aligned} \quad \Rightarrow$$

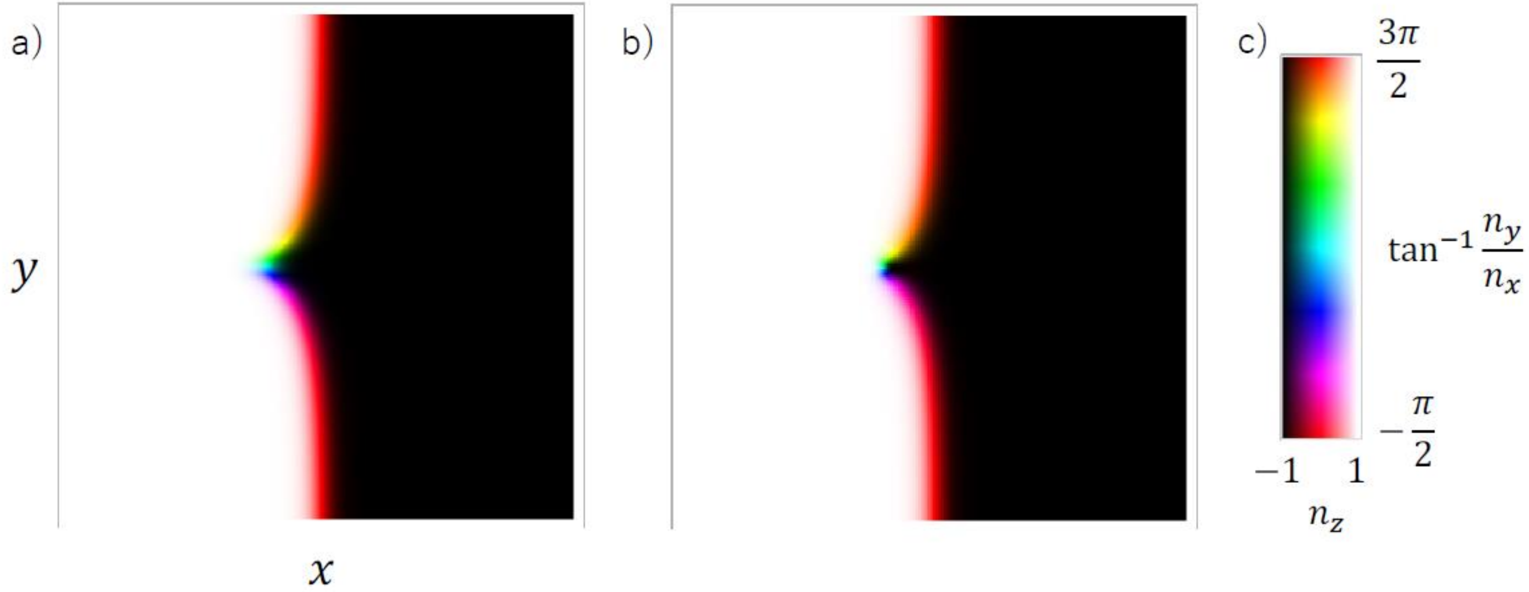
Double SG equation

Numerical solutions of DW-Skyrmion



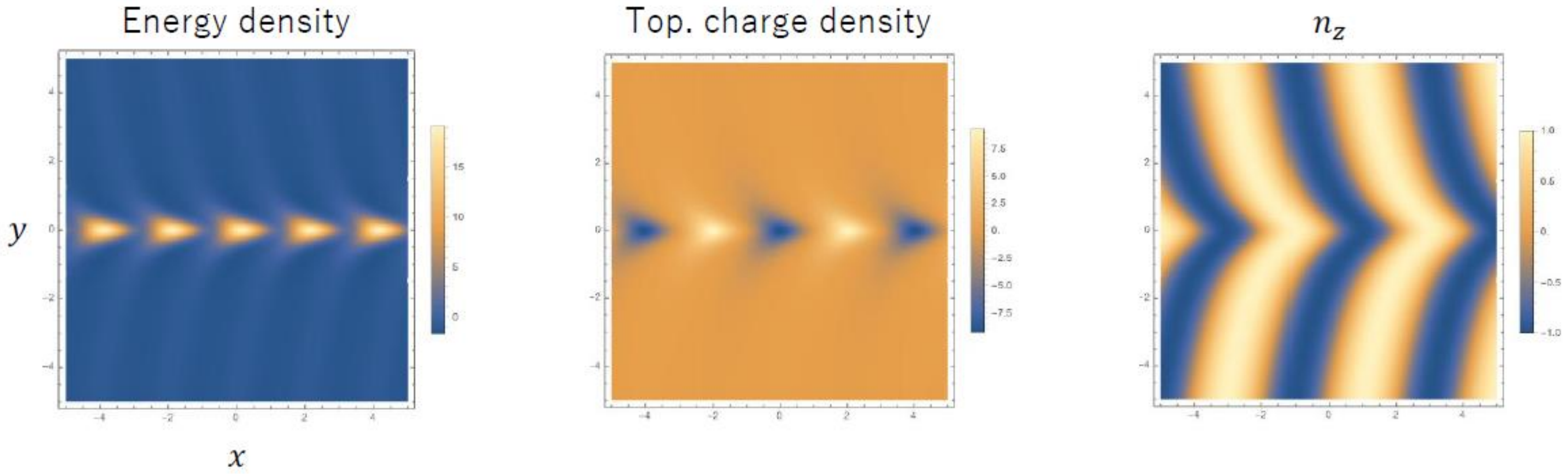
Comparison between numerics and moduli approx.

Numerical solution **Moduli approx**



➔ Application to nanotechnology such as magnetic memories.

Skyrmion chain on a chiral soliton lattice



Summary

1. Domain-wall Skyrmions are composite states of a domain wall and Skyrmions.

2. Chiral magnets

   ….. should be useful for a race-track memory

3. Chiral Lagrangian for QCD

   … appear in hadron collider, or neutron star interiors

Various scales can be commonly described in the same language.

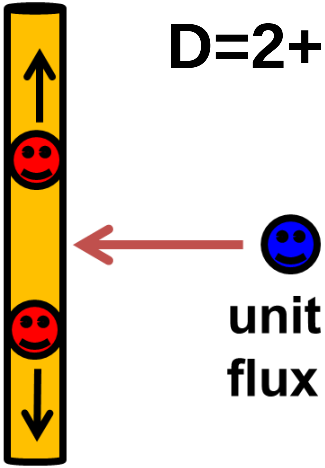
From materials to universe.

Comments on supersymmetry

(1) D=2+1 version

N =1 supersymmetry

Auzzi, Shifman & Yung ('06)



D=2+1

Two kinks

Semilocal vortex (baby skyrmion)

unit flux

(2) Baryons on domain wall in supersymmetric QCD

Armoni & Shifman ('03)