Lagrangian multiforms, the Darboux-KP system and Chern-Simons theory in infinite-dimensional space

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Outline

The talk is outlined as follows:

- A brief introduction to *Lagrangian multiform theory* - the basic idea, and an example (of a Lagrangian 2-form strucure) in 1+1 dimension;
- Present the Darboux- Kadomtsev-Petviashvili (KP) system as a generating system for the entire KP hierarchy, and its Lagrangian 3-form structure;
- Interpretation as a Chern-Simons theory in infinite-dimensional space;
- Comparison with 1+1-dimensional field theories (e.g. 4D Chern-Simons theory).

I will not go into the discrete theory, even though it has been a motivator, and the discrete systems are present in the background[1]

[1] F.W. Nijhoff, Lagrangian multiform structure of discrete and semi-discrete KP systems, in preparation, and talks given at SIDE14.2 and ICIAM2023.

Why Lagrangian multiform theory?

Key question: How to capture the property of multidimensional consistency (MDC) within a Lagrange formalism?

Multidimensional consistency: We know that many "integrable" equations, discrete and continuous possess the property of multidimenional consistency.

- continuous: commuting flows, higher symmetries & master symmetries, hierarchies;
- discrete: consistency-around-the-cube, Bäcklund transforms, higher continuous symmetries, commuting discrete flows

In all these cases we can think of the *dependent* variable a (possibly vector-valued) function of many (discrete and continuous) variables

$$u = u(n_1, n_2, \dots; x, t_1, t_2, \dots)$$

on which we can impose many equations simultaneously, and it is the *compatibility* of those equations that makes the integrability manifest.

Key problem: In a variational approach, the Euler-Lagrange (EL) equations, only produce one equation per component of the dependent variables; not an entire system of compatible equations on one and the same dependent variable!
Answer: Lagrangians of an MDC integrable theory must be differential- or difference forms in space of multi-variables!
Thus, a new variational approach to integrability was initiated by the paper:
• S. Lobb & FWN: *Lagrangian multiforms and multidimensional consistency*, J. Phys. A:Math Theor. **42** (2009) 454013

An example of a Lagrangian 2-form

Let us denote $u = u(x_1, x_2, x_3)$, $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, etc.

Consider the Lagrangians (Suris, 2012):

$$\begin{aligned}
\mathscr{L}_{12} &= \tfrac{1}{2} u_1 u_2 - \cos u \ , \\
\mathscr{L}_{13} &= \tfrac{1}{2} u_1 u_3 + \gamma \left(\tfrac{1}{2} u_{11}^2 - \tfrac{1}{8} u_1^4 \right) \ , \\
\mathscr{L}_{23} &= -\tfrac{1}{2} u_2 u_3 + \gamma \left(\tfrac{1}{2} u_1^2 \cos u + u_{11}(u_{12} - \sin u) \right) \ .
\end{aligned}$$

Then the usual Euler-Lagrange (EL) equations for these three Lagrangians yield respectively:

$$\begin{aligned}
\frac{\delta \mathscr{L}_{12}}{\delta u} &= -u_{12} + \sin u \quad \Rightarrow \quad u_{12} = \sin u \quad \text{sine-Gordon eq.} \\
\frac{\delta \mathscr{L}_{13}}{\delta u} &= \partial_1 \left(-u_3 + \tfrac{1}{2}\gamma u_1^3 + \gamma u_{111} \right) \quad \overset{u_1 = v}{\Rightarrow} \quad v_3 = \gamma (v_{111} + \tfrac{3}{2} v^2 v_1) \quad \text{mKdV eq} \\
\frac{\delta \mathscr{L}_{23}}{\delta u} &= u_{23} - \gamma \left(\tfrac{1}{2} u_1^2 \sin u + u_{11} \cos u \right) \quad \text{consistency relation} \ .
\end{aligned}$$

together with variations w.r.t. *'alien derivatives'*:

$$\frac{\delta \mathscr{L}_{23}}{\delta u_1} = \gamma \left(u_1 \cos u - u_{112} \right) = 0 \ , \quad \frac{\delta \mathscr{L}_{23}}{\delta u_{11}} = \gamma (u_{12} - \sin u) = 0 \ .$$

In fact,

$$\partial_2 \left(\text{mKdV eq} \right) \quad \Leftrightarrow \quad \partial_1 \left(\text{consist rel} \right) \quad \Leftrightarrow \quad \partial_3 \left(\text{sG eq} \right)_3$$

This suggest that the Lagrangians are components of a *Lagrangian 2-form*

$$\mathsf{L} = \mathscr{L}_{12} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathscr{L}_{23} \, \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathscr{L}_{31} \, \mathrm{d}x_3 \wedge \mathrm{d}x_1 \ ,$$

where we set $\mathscr{L}_{ji} = -\mathscr{L}_{ij}$.

Closure property and generalised EL eqs

The Lagrangian 2-form L has the following remarkable property:

$$\begin{aligned}\mathrm{d}\mathsf{L} &= (\partial_1\mathscr{L}_{23} + \partial_2\mathscr{L}_{31} + \partial_3\mathscr{L}_{12})\,\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\\ &= (\sin u - u_{12})\,\left(u_3 - \gamma u_{111} - \tfrac{1}{2}\gamma u_1^3\right)\,\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\end{aligned}$$

which has a 'double zero' when u satifies the EL equations! Thus we have the

closure property: $\boxed{\mathrm{d}\mathsf{L}|_{\mathrm{EL}} = 0}$

Action functional:

$$S[u(\boldsymbol{x});\sigma] = \int_\sigma \mathsf{L} = \int_\sigma \mathscr{L}_{12}\,\mathrm{d}x_1\wedge\mathrm{d}x_2 + \mathscr{L}_{23}\,\mathrm{d}x_2\wedge\mathrm{d}x_3 + \mathscr{L}_{31}\,\mathrm{d}x_3\wedge\mathrm{d}x_1$$

is a functional of both the field variables $u(\boldsymbol{x})$ as well as of the surface in the space of independent variables σ over which to integrate.

Multiform principle: The action S is critical w.r.t. variations $u \to u + \delta u$ of the fields, as well as w.r.t. variations $\sigma \to \sigma + \delta\sigma$ of the surfaces of integration.

Generalised EL equations: Considering a closed d-dim surface $\sigma = \partial B$ for some $d+1$-dimensional volume B, in the language of the differential bi-complex, using Stokes' theorem:

$$\int_B \mathrm{d}\mathsf{L} = \int_{\sigma=\partial B} \mathsf{L} = S \quad \Rightarrow \delta S = \int_B \delta\mathrm{d}\mathsf{L} = 0$$

for all volumes B. Hence, we have

generalised EL eqs: $\boxed{\delta S = 0 \quad \Leftrightarrow \quad \delta\mathrm{d}\mathsf{L} = 0}$

Lagrangian multi-form theory (LMFT), provides a variational approach to integrability in the sense of *multidimensional consistency* (MDC).
LMFT differs from the conventional variational approach in a number of respects:

- Lagrangians are differential- (or difference) forms (with co-dimension nonzero) in the space of independent variables;
- the action is a functional of the dependent variables (the "fields") as well as of the surfaces in the space of independent variables;
- the EL equations form a MDC (i.e. integrable) system of equations;
- the critical point of the action, i.e. solutions of a system of *generalized EL equations*, the action is independent on local variations of the surface in the space of multi-variables;
- the Lagrangians are no longer input (from tertiary considerations) but can be viewed as solutions of the system of generalized EL equations.

This new approach was initiated by the paper:
• S. Lobb & FWN: *Lagrangian multiforms and multidimensional consistency*, J. Phys. A:Math Theor. **42** (2009) 454013
Seminal work at TU Berlin (A. Bobenko & Yu. Suris and collabs.) have contributed to the development of the theory, which was there also called *theory of pluri-Lagrangian systems*.
First step to a quantum theory of LMFT, in terms of Feynman propagators, was undertaken in:
• S. King & FWN: *Quantum variational principle and quantum multiform structure: the case of quadratic Lagrangians*, Nucl. Phys. **B947** (2019) 114686

Lagrangian 3-form and KP type systems

In the case of three-dimensional equations the relevant variational structure is that of Lagrangian 3-forms. This includes the Kadomtsev-Petviashvili (KP) type systems and its generalisations. The generalised Darboux system is in this class as well.

Generalised Darboux system

The original Darboux system[2] describes conjugate nets in the theory of orthogonal curvilinear coordinates. The *generalised Darboux system* reads

$$\frac{\partial B_{qr}}{\partial \xi_p} = B_{qp}B_{pr} \ , \quad \frac{\partial B_{rq}}{\partial \xi_p} = B_{rp}B_{pq} \ ,$$
$$\frac{\partial B_{pr}}{\partial \xi_q} = B_{pq}B_{qr} \ , \quad \frac{\partial B_{rp}}{\partial \xi_q} = B_{rq}B_{qp} \ ,$$
$$\frac{\partial B_{pq}}{\partial \xi_r} = B_{pr}B_{rq} \ , \quad \frac{\partial B_{qp}}{\partial \xi_r} = B_{qr}B_{rp} \ .$$

where the B_{pq}, etc., are scalar functions (but can be readily generalised to matrices) of the independent variables ξ_p, ξ_q and ξ_r, which are continuous variables labelled by parameters p, q and r respectively (where $p \neq q \neq r \neq p$).

Remark: The integrability aspects of the Darboux system has been investigated by many authors mostly in the late 1980s and 1990s (Zakharov, Manakov, Doliwa, Santini, Konopelchenko and Bogdanov, Martinez-Alonso, etc.)

[2] G. Darboux, Mémoire sur la théorie des coordonnées curvilignes, et des systèmes orthogonaux. *Ann. Sci. de l'ÉNS*, 2eme série, tome **7** (1878) 275–348.

Multidimensional consistency of the Darboux system

A main feature of the Darboux system is the following.

Proposition: The PDE system (7) for the quantities $B_{\cdot\cdot}$ is multidimensionally consistent.

The proof is by direct computation, introducing a fourth variable ξ_s and associated lattice direction with parameter s, such that the system of independent variables is extended to include B_{ps}, B_{qs}, B_{rs} and B_{sp}, B_{sq}, B_{sr} obeying relations of the form

$$\frac{\partial B_{ps}}{\partial \xi_q} = B_{pq} B_{qs} \ , \quad \frac{\partial B_{pq}}{\partial \xi_s} = B_{ps} B_{sq} \ ,$$

etc. We then establish by direct computation from the extended system of equations comprising (7) and the PDEs w.r.t. ξ_s, the relation

$$\frac{\partial}{\partial \xi_s}\left(\frac{\partial}{\partial \xi_p} B_{qr}\right) = \frac{\partial}{\partial \xi_p}\left(\frac{\partial}{\partial \xi_s} B_{qr}\right) \ ,$$

by direct computation. Similarly all relations obtained from cross-differentiation hold by the same token.

Remark: The MDC property suggests that here is a Lagrangian multiform structure behind the Darboux system[3].

[3] FWN, *Lagrangian 3-form structure for the Darboux system and the KP hierarchy*, ArXiv:2206.14338

Lagrangian 3-form structure for the Darboux system

We now introduce the Lagrangian structure. Let us consider the following Lagrangian components

$$\begin{aligned}\mathscr{L}_{pqr} =& \tfrac{1}{2}\left(B_{rq}\partial_{\xi_p}B_{qr} - B_{qr}\partial_{\xi_p}B_{rq}\right) + \tfrac{1}{2}\left(B_{qp}\partial_{\xi_r}B_{pq} - B_{pq}\partial_{\xi_r}B_{qp}\right)\\ &+ \tfrac{1}{2}\left(B_{pr}\partial_{\xi_q}B_{rp} - B_{rp}\partial_{\xi_q}B_{pr}\right) + B_{rp}B_{pq}B_{qr} - B_{rq}B_{qp}B_{pr}\ .\end{aligned} \tag{1.1}$$

Then we have the following main statement

Theorem

The differential of the Lagrangian 3-form

$$\begin{aligned}\mathsf{L} :=& \mathscr{L}_{pqr}\,\mathrm{d}\xi_p\wedge\mathrm{d}\xi_q\wedge\mathrm{d}\xi_r + \mathscr{L}_{qrs}\,\mathrm{d}\xi_q\wedge\mathrm{d}\xi_r\wedge\mathrm{d}\xi_s +\\ &+ \mathscr{L}_{rsp}\,\mathrm{d}\xi_r\wedge\mathrm{d}\xi_s\wedge\mathrm{d}\xi_p + \mathscr{L}_{spq}\,\mathrm{d}\xi_s\wedge\mathrm{d}\xi_p\wedge\mathrm{d}\xi_q\ ,\end{aligned} \tag{1.2}$$

has a "double zero" on the solutions of the set of generalised Darboux equations (7), *i.e.* $\mathrm{d}\mathsf{L}$ *can be written as*

$$\mathrm{d}\mathsf{L} = \mathscr{A}_{pqrs}\,\mathrm{d}\xi_p\wedge\mathrm{d}\xi_q\wedge\mathrm{d}\xi_r\wedge\mathrm{d}\xi_s \tag{1.3}$$

with the coefficient $\mathscr{A}_{pqrs}$ *being a sum of products of factors which vanish on solutions of the EL equations.*

Proof.

Computing the the components of the differential dL we obtain

$$\begin{aligned}
&\partial_{\xi_s}\mathscr{L}_{pqr} - \partial_{\xi_p}\mathscr{L}_{qrs} + \partial_{\xi_q}\mathscr{L}_{rsp} - \partial_{\xi_r}\mathscr{L}_{spq} = \\
&\quad \Gamma_{s;rq}\Gamma_{p;qr} - \Gamma_{p;rq}\Gamma_{s;qr} + \Gamma_{s;qp}\Gamma_{r;pq} - \Gamma_{r;qp}\Gamma_{s;pq} \\
&\quad + \Gamma_{s;pr}\Gamma_{q;rp} - \Gamma_{q;pr}\Gamma_{s;rp} + \Gamma_{q;sr}\Gamma_{p;rs} - \Gamma_{p;sr}\Gamma_{q;rs} \\
&\quad + \Gamma_{p;sq}\Gamma_{r;qs} - \Gamma_{r;sq}\Gamma_{p;qs} + \Gamma_{q;ps}\Gamma_{r;sp} - \Gamma_{r;ps}\Gamma_{q;sp} \ ,
\end{aligned}$$

where

$$\Gamma_{p;qs} = \partial_{\xi_p} B_{qs} - B_{qp}B_{ps} \ ,$$

and similarly for the other indices. The set of generalised EL equations in this case are obtained from $\delta\mathscr{A}_{pqrs} = 0$, repeating the general arguments[4], for deriving the EL equations from the differential of the Lagrangian multiform. Thus, since all the variations δB_{pq} etc. and their first derivatives, are independent, the coefficients are precisely all the combinations $\Gamma_{r;pq}$, etc. which will have to vanish at the critical point for the action

$$\mathsf{S}[\mathbf{B}(\boldsymbol{\xi});\mathscr{V}] = \int_{\mathscr{V}} \mathsf{L} = \int_{\mathscr{W}} \mathrm{dL} \ ,$$

integrated over any arbitrary 3-dimensional closed hypersurfaces $\mathscr{V}$ in the multivariable space of all the ξ_p's, such that $\mathscr{V} = \partial\mathscr{W}$.

□

[4] Yu. Suris and M. Vermeeren, On the Lagrangian structure of integrable hierarchies, in: *Advances in Discrete Differential Geometry*, ed. A. Bobenko, (Springer Verlag, Berlin), pp. 347–378.
D. Sleigh, F.W. Nijhoff and V. Caudrelier, Variational symmetries and Lagrangian multiforms, *Lett. Math. Phys.* **110** no. 4 (2020) 805–826.

Lax multiplet

The generalised Darboux system of B-equations possesses a *Lax multiplet*[5].

Proposition: The system of B-equations arises as the compatibility conditions for the linear overdetermined system of the form

$$\frac{\partial \Phi_q}{\partial \xi_p} = B_{qp}\Phi_p \ , \quad p \neq q \ , \quad \text{or} \quad \frac{\partial \Psi_r}{\partial \xi_p} = \Psi_p B_{pr} \ . \quad \forall p \neq r \ ,$$

Remark: The Lax multiplets can be obtained from the Darboux system itself, relying on MDC, by identifying the Lax wave functions $\Phi = B_{pk}$ and $\Psi = B_{lp}$ fixing two directions in the space of independent variables, ξ_k and ξ_l, say (where k and l play the role of spectral parameters).
Furthermore, Φ and Ψ obey a linear homogeneous set of equations of the form

$$\partial_p \partial_q \Phi_r = (\partial_p \ln \Phi_q)\partial_q \Phi_r + (\partial_q \ln \Phi_p)\partial_p \Phi_r \ ,$$
$$\partial_p \partial_q \Psi_r = (\partial_p \ln \Psi_q)\partial_q \Psi_r + (\partial_q \ln \Psi_p)\partial_p \Psi_r \ .$$

A corollary to the multiform structure is a Lagrangian description of the Lax pair:
Corollary: The Lagrangian components

$$\begin{aligned}\mathscr{L}_{pq(k)} =& \tfrac{1}{2}\left(\Psi_q \partial_{\xi_p}\Phi_q - (\partial_{\xi_p}\Psi_q)\Phi_q\right) - \tfrac{1}{2}\left(\Psi_p \partial_{\xi_q}\Phi_p - (\partial_{\xi_q}\Psi_p)\Phi_p\right) \\ &+ \tfrac{1}{2}\left(B_{qp}\partial_{\xi_k}B_{pq} - B_{pq}\partial_{\xi_k}B_{qp}\right) + \Psi_p B_{pq}\Phi_q - \Psi_q B_{qp}\Phi_p \ ,\end{aligned}$$

and the corresponding Lagrangian 3-form, fixing the direction given by x_k, reduces to a 2-form:

$$\mathsf{L}_{(k)} := \mathscr{L}_{pq(k)}\,\mathrm{d}\xi_p \wedge \mathrm{d}\xi_q + \mathscr{L}_{qr(k)}\,\mathrm{d}\xi_q \wedge \mathrm{d}\xi_r + \mathscr{L}_{rp(k)}\,\mathrm{d}\xi_r \wedge \mathrm{d}\xi_p \ .$$

[5] A. Doliwa, On τ-function of conjugate nets, *J. of Nonl. Math. Phys.* **12** (2004) 244–252.

Discrete Darboux system

A discrete analogue of the Darboux system of orthogonal coordinate systems, foes back to Bogdanov & Konopelchenko, and Doliwa & Santini. The corresponding discrete analogue of the generalised Darboux system (7) reads

$$\begin{aligned} \Delta_p B_{qr} &= B_{qp} T_p B_{pr} \ , \quad \Delta_p B_{rq} = B_{rp} T_p B_{pq} \ , \\ \Delta_q B_{rp} &= B_{rq} T_q B_{qp} \ , \quad \Delta_q B_{pr} = B_{pq} T_q B_{qr} \ , \\ \Delta_r B_{pq} &= B_{pr} T_r B_{rq} \ , \quad \Delta_r B_{qp} = B_{qr} T_r B_{rp} \ , \end{aligned}$$

where the difference operator $\Delta_p = T_p - \mathrm{id}$. This system is related to other multidimensional lattice systems of matrix KP type (FWN, 1985). Now:

• *The above system of difference equations is multidimensionally consistent, and furthermore, it is consistent with the differential Darboux-KP system.*

This can be checked by direct computation.

Similarly to the continuous case we have a Lax system, and its adjoint, given by

$$\Delta_p \Phi_q = B_{qp} T_p \Phi_p \ , \quad \Delta_p \Psi_q = \Psi_p T_p B_{pq} \ ,$$

and the homogeneous linear difference system for an eigenfunctions Φ_r, Ψ_r respectively,

$$\begin{aligned} \Delta_p \Delta_q \Phi_r &= \frac{\Delta_p(T_q \Phi_q)}{T_q \Phi_q} \Delta_q \Phi_r + \frac{\Delta_q(T_p \Phi_p)}{T_p \Phi_p} \Delta_p \Phi_r \ , \\ \Delta_p \Delta_q \Psi_r &= \frac{\Delta_p \Psi_q}{T_p \Psi_q} \Delta_q(T_p \Psi_r) + \frac{\Delta_q \Psi_p}{T_q \Psi_p} \Delta_p(T_q \Psi_r) \ . \end{aligned}$$

This is the discrete analogue to the Lamé system of equations arising in the theory of conjugate nets of curvilinear coordinates[6].

[6] G.Lamé, Leçons sur les coordonnées curvilignes et leurs diverses applications,(Mallet–Bachalier, Paris, 1859).

Connection with KP system

To understand the structure of the KP system it is necesary to consider the *lattice KP system* [7] and the *KP hierarchy* [8] as part of one and the same system.
In the *direct linearisation* (DL) approach to KP, the dynamics is governed by *plane-wave factors* which take the form

$$\rho_k = \left[\prod_\nu (p_\nu - k)^{n_\nu}\right] \exp\left\{k\xi - \sum_\nu \frac{\xi_{p_\nu}}{p_\nu - k}\right\} \ ,$$

$$\sigma_{k'} = \left[\prod_\nu (p_\nu - k')^{-n_\nu}\right] \exp\left\{-k'\xi + \sum_\nu \frac{\xi_{p_\nu}}{p_\nu - k'}\right\} \ .$$

in the construction of the τ-function which obeys the Hirota equation:

$$(p-q)(T_pT_q\tau)T_r\tau + (q-r)(T_qT_r\tau)T_p\tau + (r-p)(T_rT_p\tau)T_q\tau = 0 \ ,$$

Here T_{p_ν} (p, q, r being any three of the p_ν) denotes the elementary shift in the variable n_ν associated with p_ν (which in this context) has the interpretation of a *lattice parameter* measuring the grid width in the discrete direction labelled by n_ν.
The interplay between discrete and continuous variables turns out to be an essential feature of the structure:

$$\frac{\partial\tau}{\partial\xi_p} = -\left(T_p^{-1}\frac{\mathrm{d}}{\mathrm{d}p}T_p\right)\tau := \lim_{\varepsilon\to 0}\frac{T_p^{-1}T_{p-\varepsilon}\tau - \tau}{\varepsilon\tau} \ ,$$

for any of the parameters $p_\nu = p$.

[7] Hirota R. 1981 Discrete analogue of a generalized Toda equation. *J. Phys. Soc. Japan* **50**, 3785–3791. F.W. Nijhoff, H.W. Capel, G.L. Wiersma and G.R.W. Quispel, Bäcklund transformations and three-dimensional lattice equations, *Phys. Lett.* **105A** (1984) 267–272.

[8] M. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, *RIMS Kôkyûroku* **439** (1981) 30–46.

τ-function relations

Using the identification between lattice shifts and derivatives, we can perform a limit $r \to p$ on Hirota's equation and thus obtain the following differential-difference equation for τ

$$(p-q)\left(\tau\, T_q \frac{\partial \tau}{\partial \xi_p} - (T_q\tau)\,\frac{\partial \tau}{\partial \xi_p}\right) = \tau\, T_q\tau - (T_p\tau)\, T_q T_p^{-1}\tau \ .$$

Furthermore, the τ-function also obeys the differential-difference equation

$$1 + (p-q)^2 \frac{\partial^2 \ln \tau}{\partial \xi_p\, \partial \xi_q} = \frac{(T_p T_q^{-1}\tau)\, T_q T_p^{-1}\tau}{\tau^2} \ ,$$

which is the bilinear form of the 2D Toda equation (with the discrete variable along the skew-diagonal lattice direction in the lattice generated by the T_p and T_q shifts).

Miwa variables: The KP hierarchy can be obtained by the expansions[9]

$$\begin{aligned} t_j &= \delta_{j,1}\xi + \sum_\nu \left(\frac{\xi_{p_\nu}}{p_\nu^{j+1}} + \frac{1}{j}\,\frac{n_\nu}{p_\nu^j}\right) \\ \Rightarrow \quad T_{p_\nu}\tau &= \tau\left(\{t_j + \frac{1}{j p_\nu^j}\}\right) \quad \text{and} \quad \frac{\partial \tau}{\partial \xi_{p_\nu}} = \sum_{j=1}^{\infty} \frac{1}{p_\nu^{j+1}}\,\frac{\partial \tau}{\partial t_j} \ , \end{aligned}$$

where the t_j are the usual independent time-variables in the hierarchy.

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[9] T. Miwa, On Hirota's difference equations. *Proc. Japan Acad.* (1982) **58A**, 9–12.

Identification between KP and Darboux

Consider the quantities[10]

$$S_{a,b}=\frac{T_a^{-1}T_b\tau}{\tau}\quad\Rightarrow\quad B_{pq}=\frac{\sigma_p\rho_q S_{p,q}}{q-p}=\sigma_p\rho_q\frac{T_p^{-1}T_q\tau}{(q-p)\tau}\ ,\quad (q\neq p)\ .$$

The quantities S, as a consequence of the Hirota and the differential-difference equation, obey the relations[11]

$$\begin{aligned}(p-b)T_pS_{a,b}-(p-a)S_{a,b}&=(a-b)S_{a,p}T_pS_{p,b}\ ,\\ (p-a)(p-b)\frac{\partial S_{a,b}}{\partial\xi_p}&=(a-b)\left(S_{a,p}S_{p,b}-S_{a,b}\right)\ .\end{aligned}$$

These relations are compatible for all parameters p and corresponding shifts and derivatives w.r.t. the corresponding Miwa variables ξ_p.
Furthermore, the quantity $S=S_{a,b}$ obeys the following 3-dimensional partial difference equation

$$\begin{aligned}&\frac{[(p-b)T_pT_qS-(p-a)T_qS]\ [(q-b)T_qT_rS-(q-a)T_rS]}{[(p-b)T_pT_rS-(p-a)T_rS]\ [(q-b)T_pT_qS-(q-a)T_pS]}\\ &\qquad\times\frac{[(r-b)T_pT_rS-(r-a)T_pS]}{[(r-b)T_qT_rS-(r-a)T_qS]}=1\end{aligned}$$

which is essentially the lattice Schwarzian KP equation[12].

[10] We can also identify $B_{pp}=\mathscr{C}\partial_{\xi_p}(\ln\tau)$, where $\mathscr{C}$ is some constant normalisation factor.

[11] Similar relations also appeared in:
L. Martinez-Alonso, B. Konopelchenko, The KP Hierarchy in Miwa coordinates, *Phys. Lett. A* **258** (1999) 272–278;

[12] I.Ya. Dorfman and F.W. Nijhoff, On a (2+1)-dimensional version of the Krichever-Novikov equation, *Phys. Lett. A* **157** (1991) 107–112.

Matrix Darboux system

A matrix generalisation of the Darboux system is given by

$$\partial_i G_{jk} = G_{ik} J_i G_{ji} \ , \quad i \neq j \neq k \neq i \ .$$

Here the G_{ij} are $N \times N$ matrix functions of dynamical variables $x_i = \xi_{l_i}^{J_i}, x_j = \xi_{l_j}^{J_j}, x_k = \xi_{l_k}^{J_k}, \ldots$, which are labelled by a continuous parameter $l.$ and also by constant matrices J_i, J_j, J_kwhich commute among themselves[13]
In fact, , i.e., $[J_i, J_j] = [J_j, J_k] = [J_k, J_i] = 0$, and we have denoted $\partial/\partial\xi_{l_j} =: \partial_j$, etc. for the sake of brevity.
The matrices J 'tune' a hierarchy of associated PDEs.

A Lagrangian for the matrix Darboux system reads[14]

$$\mathscr{L}_{ijk} = \tfrac{1}{2}\mathrm{tr}\left\{ G_{ij} J_i (\partial_k G_{ji}) J_j - (\partial_k G_{ij}) J_i G_{ji} J_j + \mathrm{cycl.\ (ijk)} \right\} \\ - \mathrm{tr}\left\{ G_{ij} J_i G_{ki} J_k G_{jk} J_j - G_{ji} J_j G_{kj} J_k G_{ik} J_i \right\} \ ,$$

which is a matrix generalisation of the Darboux Lagrangian.

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[13] In fact, one can also consider the non-commutative case $[J_i, J_j] = \Gamma_{ij}^k J_k$, in which case we get non-commuting flows on a loop group, for which a Lagrangian description was proposed, for (1+1)-dimensional hierarchies, recently in: V. Caudrelier, F.W. Nijhoff, D. Sleigh and M. Vermeeren, Lagrangian multiforms on Lie groups and noncommuting flows, ArXiv:2204.09663.

[14] F.W. Nijhoff and J.-M. Maillet, Algebraic Structure of Integrable Systems in $D = 2 + 1$ and Routes towards Multidimensional Integrability, in: *Nonlinear Evolutions*, Proceedings of the IVth NEEDS Conference, Ed. J.J.P. Léon, (World Scientific, Signapore, 1988); Preprint PAR-LPTHE 87-45, September 1987.

Matrix 3-form structure

The Lagrangian $\mathscr{L}_{ijk}$ can be viewed as a components of a Lagrangian 3-form:

$$\mathsf{L} = \sum_{i<j<k} \mathscr{L}_{ijk}\, \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \ ,$$

which obeys the following property: • *The Lagrangian 3-form* L *has a double zero on solutions of the set of matrix Darboux equations.*
The proof is computational, and in essence similar to the scalar case, (differing only in the matrix ordering within the trace). Computing the differential of L we get:

$$\mathrm{d}\mathsf{L} = \sum_{i,j,k,l} \mathscr{A}_{ijkl}\, \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \wedge \mathrm{d}x_l \ ,$$

with

$$\begin{aligned}\mathscr{A}_{ijkl} =& \tfrac{1}{2}\mathrm{tr}\,\{\Gamma_{l;i,j}J_i\Gamma_{k;j,i}J_j - \Gamma_{k;i,j}J_i\Gamma_{l;j,i}J_j \\ & + \Gamma_{l;k,i}J_k\Gamma_{j;i,k}J_i - \Gamma_{j;k,i}J_k\Gamma_{l;i,k}J_i \\ & \Gamma_{l;j,k}J_j\Gamma_{i;k,j}J_k - \Gamma_{i;j,k}J_j\Gamma_{l;k,j}J_k \pm \mathrm{cycl}\ \ (ijkl)\} \ ,\end{aligned}$$

where the quantities Γ are given by

$$\Gamma_{i;j.k} = \partial_i G_{jk} - G_{ik}J_iG_{ji} \ .$$

The double zero expansion implies that the *generalised EL equations* arising from $\delta\mathrm{d}\mathsf{L} = 0$(for all G_{ij} varied independently, for different indices) gives rise to the entire system of matrix Darboux equations. They yield the critical point of the action

$$S[G_{\cdot,\cdot}(\boldsymbol{x}); \mathscr{V}] = \int_{\mathscr{V}} \mathsf{L} \ ,$$

as a functional of all the matrix fields $G_{\cdot,\cdot}$ for all hypersurfaces $\mathscr{V}$ in the space of independent variables.

Higher-dimensional Chern-Simons actions

The conventional Chern-Simons theory over a Lie algebra $\mathfrak{g}$, with associated gauge group G, involves a $\mathfrak{g}$-valued gauge connection 1-form $\boldsymbol{A}$, and the associated curvature 2-form,

$$\boldsymbol{F} = \mathrm{d}\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A} \ .$$

Here we consider matrix-valued gauge fields only, where the gauge groups of interest are the general linear groups, $GL(n, \mathbb{R})$, endowed with the matrix trace Tr, and where the wedge product $\boldsymbol{A} \wedge \boldsymbol{A}$ is evaluated via the matrix product, and not via the Lie bracket.

The standard CS Lagrangians in dimensions 3 and 5 read

$$\begin{aligned} CS_3 &= \mathrm{Tr}\left(\boldsymbol{A} \wedge \mathrm{d}\boldsymbol{A} + \tfrac{2}{3}\boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A}\right) \ , \\ CS_5 &= \mathrm{Tr}\Big(\boldsymbol{A} \wedge \mathrm{d}\boldsymbol{A} \wedge \mathrm{d}\boldsymbol{A} + \tfrac{3}{2}\boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \mathrm{d}\boldsymbol{A} \\ &\qquad + \tfrac{3}{5}\boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A}\Big) \ , \end{aligned}$$

and they are defined through the property that

$$\begin{aligned} \mathrm{d}CS_3 &= \mathrm{Tr}\left(\boldsymbol{F} \wedge \boldsymbol{F}\right) \ , \\ \mathrm{d}CS_5 &= \mathrm{Tr}\left(\boldsymbol{F} \wedge \boldsymbol{F} \wedge \boldsymbol{F}\right) \ . \end{aligned}$$

Remark: *Coming from the perspective of the multiform variational principle, we recognise in the latter relations a similarity between Chern classes and the double, respectively triple zero conditions for the Euler-Lagrange equations* $\boldsymbol{F} = 0$*.*
However, the relation $\boldsymbol{F} = 0$ is too strong for integrability!

General form of (conventional) CS actions

The general higher form of the CS Lagrangians are given by the formula

$$CS_{2n+1} = (n+1)\int_0^1 \mathrm{d}\lambda\, \mathrm{Tr}\left(\boldsymbol{A} \wedge \boldsymbol{F}_\lambda^{\wedge n}\right) ,$$

$$\text{where} \quad \boldsymbol{F}_\lambda := \lambda \mathrm{d}\boldsymbol{A} + \lambda^2 \boldsymbol{A} \wedge \boldsymbol{A} ,$$

They obey[15]

$$\mathrm{d}CS_{2n+1} = \mathrm{Tr}\left(\boldsymbol{F}^{\wedge(n+1)}\right) ,$$

where the latter expressions are $2n+2$-forms, whose variational derivative (using the multiform EL equations in the language of the variational bi-complex), are given by

$$\delta \mathrm{d}CS_{2n+1} = (n+1)\mathrm{Tr}\left(\boldsymbol{F}^{\wedge(n)} \wedge \delta\boldsymbol{F}\right) ,$$

which vanishes whenever $\boldsymbol{F} = 0$. The latter would correspond to the usual variational equations in conventional CS theory, but in our setting $\boldsymbol{F} = 0$ is too stringent a condition. In fact, we will work later with a restricted set of fields, and the corresponding equations of motion are slightly weaker than the standard zero-curvature condition.

In all these conventional CS theories, we fix the dimensionality of the $(2n+1)$-dimensional manifold $\mathscr{M}_{2n+1}$ over which the Lagrangians are integrated through

$$\mathscr{A}^{CS}_{2n+1} = \int_{\mathscr{M}_{2n+1}} CS_{2n+1} .$$

[15] Note that in dimension 1, we have $CS_1 = \mathrm{Tr}(\boldsymbol{A})$, with $\mathrm{d}CS_1 = \mathrm{Tr}(\boldsymbol{F})$.

Higher Lagrangian Multiforms from CS Lagrangians

In order to make a connection between the CS theory and Lagrangian multiforms we need to specify the gauge field $\boldsymbol{A}$ as:

$$\boldsymbol{B} = \sum_{k,l\in\mathbb{Z}} B_{kl}\,\mathrm{d}\xi_k\, E_{k,l}\ .$$

Here for simplicity $B_{kl} := B_{p_k p_l}\left((\xi_i)_{i\in\mathbb{Z}}\right)$.
The E_{kl} are generators of $GL(\infty)$ obeying[16]: $E_{k,l}E_{m,n} = \delta_{l,m}E_{k,n}$ and $\mathrm{Tr}(E_{k,l}) = \delta_{k,l}$.
matrices $(M_{ab})_{a,b\in\mathbb{Z}}$, indexed by integers, with . Note that
Computing the corresponding curvature $\boldsymbol{F}_B$ associated with the gauge field $\boldsymbol{B}$, we get

$$\boldsymbol{F}_B = \sum_{j,k,l\in\mathbb{Z}} \left(\partial_{\xi_j} B_{kl} - B_{kj}B_{jl}\right)\mathrm{d}\xi_j\wedge\mathrm{d}\xi_k\, E_{k,l}\ . \tag{1.7}$$

Note that the coefficients $\partial_{\xi_j} B_{kl} - B_{kj}B_{jl}$ are exactly of the Darboux form.
Main statement: *Computing the corresponding CS action, we get exactly the Lagrangian 3-form of the Darboux-KP system:*

$$\mathsf{L}^{(3)} = CS_3(\boldsymbol{B}) = \sum_{i,j,k\in\mathbb{Z}} \mathscr{L}^{(3)}_{ijk}\,\mathrm{d}\xi_i\wedge\mathrm{d}\xi_j\wedge\mathrm{d}\xi_k\ ,$$

in which the coefficients $\mathscr{L}^{(3)}_{ijk} = \frac{2}{3!}\mathscr{L}_{p_i p_j p_k}$ of (1.1), including a prefactor for convenience.

[16] We can think of this as generating $Mat_{\mathbb{Z}}(\mathbb{C})$ with $(E_{kl})_{ab} = \delta_{k,a}\delta_{l,b}$. The sum in the definition of $\boldsymbol{B}$ being infinite, can be understood in the 'completed graded sense', for the $\mathrm{d}\xi_k\, E_{k,l}$ are linearly independent, and hence we never get infinite sums of real numbers.

Double-zero structure

Calculating with the special gauge field $\boldsymbol{B}$ the differential of the Lagrangian 3-form, we find

$$\mathrm{d}\mathsf{L}^{(3)} = \mathrm{Tr}(\boldsymbol{F}_B \wedge \boldsymbol{F}_B) = \sum_{\substack{j,k,l,m\in\mathbb{Z} \\ \text{all indices different}}} \left(\partial_{\xi_j} B_{kl} - B_{kj}B_{jl}\right)\left(\partial_{\xi_m} B_{lk} - B_{lm}B_{mk}\right) \mathrm{d}\xi_j \wedge \mathrm{d}\xi_k \wedge \mathrm{d}\xi_m \wedge \mathrm{d}\xi_l .$$

In particular, this implies that $\mathrm{Tr}(\boldsymbol{F}_B \wedge \boldsymbol{F}_B)$, has a double zero on the solutions of the generalised Darboux system in (7), which implies that the latter arises as the EL equations of the multiform action.

Remark: Note that while $\mathrm{Tr}(\boldsymbol{F}_B \wedge \boldsymbol{F}_B)$ indeed has a double zero on the solutions of (7), the form $\boldsymbol{F}_B \wedge \boldsymbol{F}_B$ does not necessarily have such a double zero when (7) holds, as this would require that the Darboux system also extends to the case that all three labelled variables are no longer distinct.

Higher CS multiform actions

Following the connection between the conventional higher CS actions and Chern classes, we can now also postulate higher multiform actions for the Darboux-KP system. Thus, using the same gauge field $\boldsymbol{B}$ in the higher CS action we obtain the Lagrangian 5-form

$$
\begin{aligned}
\mathsf{L}^{(5)} &= \mathrm{Tr}\Big(\boldsymbol{B}\wedge \mathrm{d}\boldsymbol{B}\wedge \mathrm{d}\boldsymbol{B} \\
&\qquad + \tfrac{3}{2}\boldsymbol{B}\wedge\boldsymbol{B}\wedge\boldsymbol{B}\wedge \mathrm{d}\boldsymbol{B} + \tfrac{3}{5}\boldsymbol{B}\wedge\boldsymbol{B}\wedge\boldsymbol{B}\wedge\boldsymbol{B}\wedge\boldsymbol{B}\Big)\ , \\
&= \sum_{j,k,l,m,n\in\mathbb{Z}} \mathscr{L}^{(5)}_{jklmn}\mathrm{d}\xi_j\wedge\mathrm{d}\xi_k\wedge\mathrm{d}\xi_l\wedge\mathrm{d}\xi_m\wedge\mathrm{d}\xi_n\ ,
\end{aligned}
$$

with

$$
\begin{aligned}
\mathscr{L}^{(5)}_{jklmn} = \tfrac{1}{5!} & \sum_{j',k',l',m',n'\in\{j,k,l,m,n\}} \varepsilon_{j'k'l'm'n'} \\
& \Big[B_{p_{j'},p_{l'}}(\partial_{\xi_{p_{k'}}}B_{p_{l'},p_{n'}})(\partial_{\xi_{p_{m'}}}B_{p_{n'},p_{j'}}) \\
& + \tfrac{3}{2}B_{p_{j'},p_{k'}}B_{p_{k'},p_{l'}}B_{p_{l'},p_{n'}}(\partial_{\xi_{p_{m'}}}B_{p_{n'},p_{j'}}) \\
& + \tfrac{3}{5}B_{p_{j'},p_{k'}}B_{p_{k'},p_{l'}}B_{p_{l'},p_{m'}}B_{p_{m'},p_{n'}}B_{p_{n'},p_{j'}}\Big]\ ,
\end{aligned}
$$

where ε_{jklmn} is the 5-dimensional Levi-Civita symbol.
As a consequence of the construction, the Lagrangian 5-form has the property that

$$
\mathrm{d}\mathsf{L}^{(5)} = \mathrm{Tr}\left(\boldsymbol{F}_B\wedge\boldsymbol{F}_B\wedge\boldsymbol{F}_B\right)\ .
$$

This again leads to the fact that $\mathrm{d}\mathsf{L}^{(5)}$ has a triple zero on the solutions still of the same Darboux-KP system, as it has the MDC property! (Again $\boldsymbol{F}_B\wedge\boldsymbol{F}_B\wedge\boldsymbol{F}_B$ does not necessarily have a triple zero on the solutions of the Darboux-KP system!)

Generating CS multiform Lagrangian

Similarly, all higher Lagrangian multiforms $\mathsf{L}^{(2n+1)}$ of odd degree can be constructed in the same way, leading to the formula:

$$\begin{aligned}\mathrm{d}\mathsf{L}^{(2n+1)} = \mathrm{Tr}\left(\boldsymbol{F}_B^{\wedge n}\right) = &\sum_{j_1,l_1,j_2,l_2,\ldots,j_n,l_n\in\mathbb{Z}} \\ &(\partial_{\xi_{j_1}} B_{l_n l_1} - B_{l_n j_1} B_{j_1 l_1})\,(\partial_{\xi_{j_2}} B_{l_1 l_2} - B_{l_1 j_2} B_{j_2 l_2})\ \ldots \\ &\ldots(\partial_{\xi_{j_n}} B_{l_{(n-1)} l_n} - B_{l_{(n-1)} j_n} B_{j_n l_l}) \\ &\mathrm{d}\xi_{j_1} \wedge \mathrm{d}\xi_{l_n} \wedge \mathrm{d}\xi_{j_2} \wedge \mathrm{d}\xi_{l_1} \wedge \ldots \wedge \mathrm{d}\xi_{j_n} \wedge \mathrm{d}\xi_{l_{(n-1)}},\end{aligned}$$

which has a n-fold zero on the solutions of the generalised Darboux-KP system. Thus, establishing a hierarchy of Lagrangian multiforms in increasingly higher odd dimensions, but all associated with the same generalised Darboux-KP system. Their action functionals are of the form:

$$\int_{\mathscr{V}_{2n+1}} CS_{2n+1}(\boldsymbol{B}),$$

for each $2n+1$-dimensional hypersurface $\mathscr{V}_{2n+1}$ embedded in $\mathbb{R}^{\mathbb{Z}}$. Thus, we can write a *generating Lagrangian multiform* as the formal sum, in powers of a dummy parameter $\hbar$,

$$\mathscr{S}_{\hbar}^{(\infty)}[\boldsymbol{B};\mathscr{V}_\infty] = \sum_{n=1}^{\infty} \frac{\hbar^n}{n+1} \int_{\mathscr{V}_{2n+1}} \mathsf{L}^{(2n+1)}.$$

integrated over the disjoint union $\mathscr{V}_\infty = \amalg_{n=1}^{\infty} \mathscr{V}_{2n+1}$ of submanifolds.

Comparison with 4D Chern-Simons theory

In order to depart from the confines of a topological field theory, an action for 1+1-dimensional integrable field theory was proposed by a line of work by Costello et al., going back to earlier ideas by Nekrassov[17]
The action functional

$$S[A] = \mathscr{K} \int_{\mathscr{M}=\Sigma\times\mathbb{CP}^1} \omega \wedge\ CS_3(\boldsymbol{A})$$

extends the usual CS action by integrating over a 4D manifold $\mathscr{M}$ with coordinates $(\tau, \sigma, z, \bar{z})$, where (τ, σ) are real space-time coordinates and z is a complex spectral variable. The gauge field is chosen as

$$A = A_\sigma \, d\sigma + A_\tau \, d\tau + A_{\bar{z}} \, d\bar{z} \, .$$

(with component $A_z = 0$), and where $\omega = \varphi(z)dz$ is a meromorphic 1-form.
In the classical context one obtain two sets of equations of motion as EL equations:

- bulk equations of motion $\omega \wedge F(A) = 0$,
- 'boundary equations' arising from the contours around singularities (defects) of ω in $\mathbb{C}P^1$: $d\,\omega \wedge \mathrm{Tr}(A \wedge \eta)$ (for all variations η).

The claim is that the possible 4D CS actions generate integrable 1+1- dimensional field theories, with the gauge field components A_σ and A_τ (up to a gauge) acting as Lax connections, and indeed possessing a classical r-matrix strcuture for suitable choices of ω.

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[17] N.A. Nekrassov, *Four Dimensional Holomorphic Theories*, PhD Thesis, Princeton University, 1996.
K. Costello, E. Witten and M. Yamazaki, *Gauge theory and integrability I-II*, arXiv:1709.09993; arXiv:1802.01579.

Connection with 1+1-dim. integrable field theories

A generalised $N \times N$ matrix Lax system, by 'compounding' the usual hierarchy of integrable time-flows, was derived[18] leading to:

$$\frac{\partial}{\partial \xi_p}\Phi_k = \frac{R_p}{p-k}\Phi_k \ , \quad \forall\ p,$$

where k is a spectral parameter, p and ξ_p as before, and the matrix coefficients R_p independent of k. Imposing, this for all p we get the MDC system

$$\partial_q R_p = \partial_p R_q \ , \quad p\partial_p R_q - q\partial_q R_p + [R_q, R_p] = 0 \ , \quad \forall\ p \neq q \ .$$

(where we abbreviated $\partial/\partial\xi_p = \partial_p$, There are several ways to resolve these relations:

$$R_p = J_p - \partial_p H = p(\partial_p g)g^{-1} \ ,$$

for some matrices H and g (without label), and where the J_p are commuting constant matrices. This leads to the equations:

$$\partial_p\partial_q H = \frac{[J_p - \partial_p H, J_q - \partial_q H]}{q-p} \ ,$$

$$p\partial_q\left((\partial_p g)g^{-1}\right) = q\partial_p\left((\partial_q g)g^{-1}\right) \ ,$$

the latter being generalized chiral field equations. Both systems have a Lagrangian structure:

$$\begin{aligned}\mathscr{L} = \quad & \tfrac{(p-q)}{2}\mathrm{tr}\left(\partial_p H\cdot\partial_q H\right) - \tfrac{1}{2}\mathrm{tr}\left([J_p, H]\partial_q H\right) \\ & +\tfrac{1}{2}\mathrm{tr}\left([J_q, H]\partial_p H\right) - \tfrac{1}{3}\mathrm{tr}\left([\partial_p H, \partial_q H]H\right) \ ,\end{aligned}$$

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[18] F.W. Nijhoff, "Linear integral transformations and hierarchies of integrable nonlinear evolution equations," Physica **D31** (1988) 339–388.
F.W. Nijhoff, "Integrable hierarchies, Lagrangian structures and noncommuting flows," in *Topics in soliton theory and exactly solvable nonlinear equations*, M. Ablowitz, M. Kruskal & B. Fuchssteiner (Eds), pp. 150–181, (World Scientific, 1987).

and respectively a special Wess-Zumino-Witten-Novikov type action:

$$\mathscr{L} = \mathrm{tr}\left(\partial_p g \cdot \partial_q g^{-1}\right) + \frac{q+p}{q-p}\int_0^1 \mathrm{d}t\,\mathrm{tr}\left([\partial_p g \cdot g^{-1}, \partial_q g \cdot g^{-1}]\frac{dg}{dt}\cdot g^{-1}\right) \ ,$$

(where t is a dummy variable and $g = g(t)$ in the second integrand depends on t, s.t. $g(0) = I$ (unit matrix), $g(1) = g(\xi_p, \xi_q)$. Curiously, a similar prefactor to the topological term appears in the work on 4D CS theory[19].

Remark: For neither Lagrangian (for H and g fields) a Lagrangian 2-form structure holds, but it was established for the general class of Zakharov-Mikhailov Lagrangians[20]. A particular case of this structure is given by the Lagrangian components

$$\mathscr{L}_{pq} = \mathrm{tr}\left(\Phi_q^{-1}\partial_p\Phi_q J_q - \Phi_p^{-1}\partial_q\Phi_p J_p\right) - (\mathrm{tr}\otimes\mathrm{tr})r_{pq}\left(R_p \otimes R_q\right) \ ,$$

where the classical r matrix appears in the 'potential term' of the Lagrangian[21]. Hence the Lagrangian 2-form

$$\mathsf{L} = \sum_{p,q}\mathscr{L}_{pq}\mathrm{d}\xi_p \wedge \mathrm{d}\xi_q \ ,$$

obeys the closure relation $\mathrm{d}\mathsf{L} = 0$ on solutions of the EL equations

$$\partial_p R_q = \partial_q R_p = \frac{[R_p, R_q]}{p-q} \ ,$$

as a consequence of the classical Yang-Baxter equation.

[19] F. Delduc, S. Lacroix, M. Magro, and B. Vicedo, "A unifying 2d action for integrable -models from 4d Chern-Simons theory," arXiv:1909.13824.

K. Costello and M. Yamazaki, "Gauge Theory And Integrability, III," arXiv:1908.02289

[20] D. Sleigh, FWN & V. Caudrelier, *A variational approach to Lax representations, J. Geom, Phys.* **142** *(2019) 66–79*

[21] V. Caudrelier, M. Stoppato & B. Vicedo, *Classical Yang-Baxter equation, Lagrangian multiforms and ultralocal integrable hierarchies, arXiv:2201.08286*

Discussion

Here some points:

- Lagrangian multiform structures seem to form a universal aspect of integrability as it represents the phenomenon of MDC at the variational level;
- Establishing a Lagrangian 3-form structure for the Darboux-KP system seems the most promising route to attain a quantum theory of the KP system;
- The connection with a CS theory in infinite-dimensional space may yield new insights into the connection between topological, conformal and integrable field theories;
- A new departure (within Lagrangian multiform theory) is to develop a variational description of non-commuting flows[22], which may yield a variational approacj to Lie group actions on manifolds;
- Potentially the quantum version of Lagrangian multiform theory [23] may lead to the introduction of a new quantum object, namely the sum over (hyper)surfaces of surface-dependent propagators. embedding space').

[22] V. Caudrelier, FWN, D. Sleigh and M. Vermeeren, Lagrangian multiforms on Lie groups and noncommuting flows, J. Geom. Phys. **187** (2023), ArXiv:2204.09663

[23] S. King & FWN: *Quantum variational principle and quantum multiform structure: the case of quadratic Lagrangians*, Nucl. Phys. **B947** (2019) 114686

THANK YOU FOR YOUR ATTENTION!