Inclusion of radiation in the collective coordinate method approach of the ϕ^4 model

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State of the art

The fundamental distinction between field dynamics and particle dynamics is that fields have infinitely many degrees of freedom.

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Dynamics, at low speed, is codified in a few degrees of freedom that are promoted to time-dependent variables

$$L_{eff}(X_i(t)) = \int_{\mathbb{R}} \mathcal{L}(\phi(x; X_i(t)))\, dx \,.$$

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A novel approach[1] (pRCCM) to emulate radiation is based on a tower of Derrick modes with increasing frequency and spatial extension

$$\phi(x; a, \boldsymbol{C}) = \phi_K(x-a) + \sum_{k=1}^{n} \frac{C_k}{k!} \left((x-a)^k \phi_K^{(k)}(x-a) \right),$$

allowing, for example, to reproduce qualitatively the fractal pattern in kink-antikink scattering and the decay of the shape mode (at short times).

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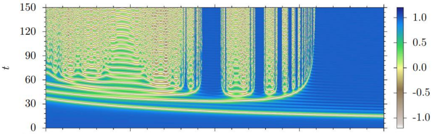

(a) Field theory

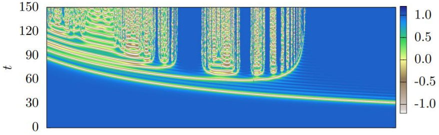

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Fig. 1: Fractal pattern in kink-antikink scattering [1]

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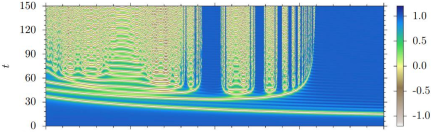

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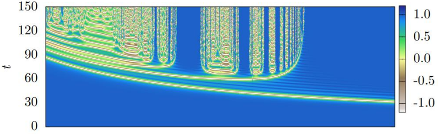

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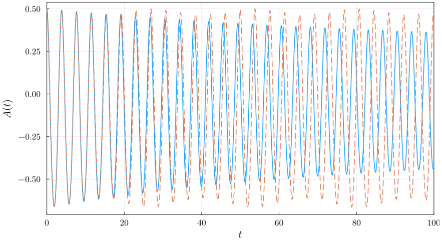

Fig. 2: Shape mode decay [2]

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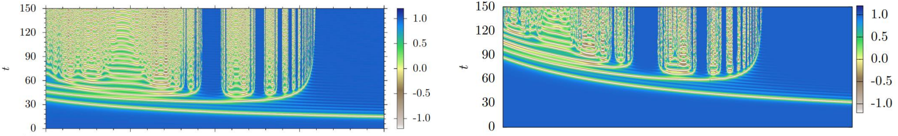

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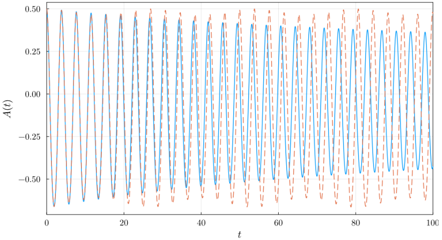

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Introduction

ϕ^4 model

The Lagrangian density reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}(\phi^2 - 1)^2 \,, \tag{2.1}$$

whose field equation looks like

$$\Box \phi + 2\phi(\phi^2 - 1) = 0 \,. \tag{2.2}$$

In addition to vacuum solutions ($\phi(x) = \pm 1$), there are non-trivial stable solutions:

$$\phi_{K(\bar{K})}(x) = \pm \tanh(x - x_0) \,. \tag{2.3}$$

The solutions with positive sign are called kinks, and the ones with negative sign are called antikinks.

Introduction

Let us consider a perturbation of the kink as follows

$$\phi(x,t) = \phi_K(x) + \eta(x,t)\,, \tag{2.4}$$

with $\eta(x,t) = \eta(x)e^{i\omega t}$.

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At linear order, the field equation looks like

$$-\eta''(x) + \left(6\phi_K(x)^2 - 2\right)\eta(x) = \omega^2\eta(x)\,. \tag{2.5}$$

The system of eigenstates and eigenvalues

$$\eta_0(x) = \frac{\sqrt{3}}{2}\,\mathrm{sech}^2 x\,, \quad \omega_0 = 0\,, \tag{2.6}$$

$$\eta_s(x) = \sqrt{\frac{3}{2}}\sinh x\,\mathrm{sech}^2 x\,, \quad \omega_s = \sqrt{3}\,, \tag{2.7}$$

$$\eta_q(x) = \frac{3\tanh^2 x - q^2 - 1 - 3iq\tanh x}{\sqrt{(q^2+1)(q^2+4)}}\,e^{iqx}\,, \quad \omega_q = \sqrt{q^2+4}\,, \tag{2.8}$$

form an orthonormal basis (Sturm-Liouville problem).

Starting point

A general field configuration close to the kink solution can be expanded as follows

$$\phi(x) = \phi_K(x) + c_0\eta_0(x) + c_s\eta_s(x) + \int_{\mathbb{R}} dq\, c_q\eta_q(x)\,. \tag{2.9}$$

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This natural assumption contains all possible degrees of freedom:

$\eta_0(x) \Rightarrow$ rigid translation.

$\eta_s(x) \Rightarrow$ change of size.

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Henceforth, we are going to assume that the evolution of the kink is governed by

$$\phi(x,t) = \phi_K(x) + c_0(t)\eta_0(x) + c_s(t)\eta_s(x) + \underbrace{\int_{\mathbb{R}} dq\, c_q(t)\eta_q(x)}_{\text{R(t,x)}} \tag{2.10}$$

for small perturbations.

Radiation from a wobbling kink

Ansatz describing a static wobbling kink

$$\phi(x,t) = \phi_K(x) + c_s(t)\eta_s(x) + \int_{\mathbb{R}} dq\, c_q(t)\eta_q(x)\,. \tag{3.1}$$

Let us assume that $c_q(t) \sim \mathcal{O}\left(c_s^2(t)\right)$.

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At second order in $c_s(t)$ the field equation looks like

$$\eta_s(x)\Big(\ddot{c}_s(t) + \omega_s^2 c_s(t)\Big) + \int_{\mathbb{R}} dq\, \eta_q(x)\Big(\ddot{c}_q(t) + \omega_q^2 c_q(t)\Big) + 6c_s^2(t)\phi_K(x)\eta_s^2(x) = 0\,. \tag{3.2}$$

Projecting onto $\eta^*_{q'}(x)$ and assuming the relations of orthogonality, we get

$$\ddot{c}_q(t) + \omega_q^2 c_q(t) - \frac{3i}{32} c_s^2(t)\sqrt{\frac{q^2+4}{q^2+1}}\,\frac{q^2(q^2-2)}{\sinh\left(\pi q/2\right)} = 0\,, \tag{3.3}$$

Radiation from a wobbling kink

We take into account that the shape mode is the only source for radiation through the initial conditions $c_q(0) = 0$ and $\dot{c}_q(0) = 0$. In addition, we will assume that $c_s(t) = A_0 \cos(\omega_s t)$.

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Then, the general solution of (3.3) takes the form

$$c_q(t) = \frac{3}{2\pi} \frac{(4\omega_s^2 - \omega_q^2) - \omega_q^2 \cos(2\omega_s t) - (4\omega_s^2 - 2\omega_q^2)\cos(\omega_q t)}{\omega_q^2(\omega_q^2 - 4\omega_s^2)} \mathcal{F}(q)\,. \qquad (3.4)$$

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Finally, the exact form of the radiation at leading order for a static wobbling kink is

$$R(t,x) = \int_{\mathbb{R}} dq\, c_q(t) \eta_q(x)\,. \tag{3.5}$$

We have been able to solve analytically this integral for all x at large t under certain approximations.

Radiation from a wobbling kink

After taking the corresponding spatial limit, we conclude that $R(t,x)$ looks asymptotically (for $x \gg 0$) like

$$R_\infty(t,x) = \frac{3\,\pi A_0^2}{2\sinh(\sqrt{2}\pi)}\sqrt{\frac{3}{8}}\cos\left(2\sqrt{3}t - 2\sqrt{2}x - \delta\right). \tag{3.6}$$

This last result is in complete agreement with Manton's work[3].

From (3.6) we can deduce the decay law for the shape mode amplitude

$$A(t) = \frac{1}{\sqrt{A_0^{-2} + 0{,}03\,t}}. \tag{3.7}$$

Our proposal seems to be adequate. Then, we will follow this working line to construct effective models. This will allow us to gather more information about the energy transfer mechanisms between the modes.

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Interaction of radiation and shape mode

Ansatz containing the shape mode and radiation

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Substituting (4.1) in (2.1) and integrating in the x-variable, we obtain, at third order,

$$\mathcal{L}_{s,q} = \frac{1}{2}\Big(\dot{c}_s^2(t) - \omega_s^2 c_s^2(t)\Big) + \pi \int_{\mathbb{R}} dq\; \Big(\dot{c}_q(t)\dot{c}_{-q}(t) - \omega_q^2 c_q(t) c_{-q}(t)\Big)$$
$$-\frac{3\pi}{16}\sqrt{\frac{3}{2}}c_s^3(t) - c_s^2(t)\int_{\mathbb{R}} dq\, f_s(q)c_q(t) + c_s(t)\int_{\mathbb{R}^2} dqdq'\, f_{sq}(q,q')c_q(t)c_{q'}(t)\,, \qquad (4.2)$$

where

$$f_s(q) = -\frac{3i\pi}{16}\sqrt{\frac{q^2+4}{q^2+1}}\frac{q^2(q^2-2)}{\sinh(\pi q/2)}\,, \qquad (4.3)$$
$$f_{sq}(q,q') = 6\int_{\mathbb{R}} dx\, \phi_K(x)\eta_s(x)\eta_q(x)\eta_{q'}(x). \qquad (4.4)$$

Interaction of radiation and shape mode

The equations of motion governing the evolution of $c_q(t)$ and $c_s(t)$ are yielded by

$$\ddot{c}_{-q}(t) + \omega_q^2\, c_{-q}(t) + \frac{1}{2} f_s(q) c_s^2(t) - \frac{1}{\pi} c_s(t) \int dq' f_{sq}(q,q') c_{q'}(t) = 0, \quad (4.5)$$

$$\ddot{c}_s(t) + \omega_s^2 c_s(t) + \frac{9\pi}{16}\sqrt{\frac{3}{2}} c_s^2(t) + 2c_s(t) \int_{\mathbb{R}} dq\, f_s(q) c_q(t)$$
$$- \int_{\mathbb{R}^2} dq dq'\, f_{sq}(q,q') c_q(t) c_{q'}(t) = 0. \quad (4.6)$$

The maximum of $f_s(q)$ takes place at $q \approx 2\sqrt{2} \equiv w \approx 2\omega_s$.

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In order to solve the system (4.5)-(4.6) numerically, we will have to choose a discretization in q. This fixes a time cut-off of order $t_c = 1/\Delta q$, beyond which our computations are no longer trustable.

Interaction of radiation and shape mode

- First experiment: Radiation emitted by a wobbling kink.

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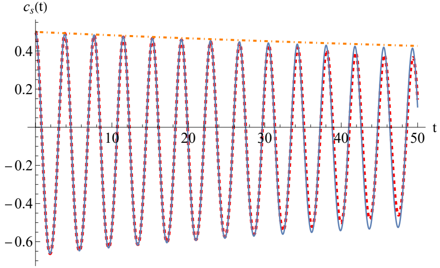
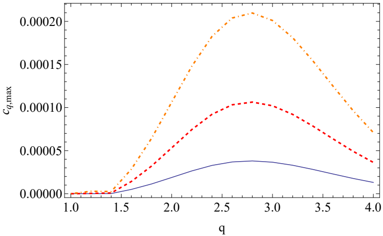
We choose the following initial conditions (I.C.)

$$c_s(0) = A_0\,, \quad c_s'(0) = 0\,, \quad c_q(0) = 0\,, \quad \text{and} \quad c_q'(0) = 0\,. \tag{4.7}$$



(a) Shape mode decay



(b) Spectrum of frequencies

Fig. 3: We have assumed $n = 20$ equidistant scattering modes in $q \in [-3, 3]$ for the decay and $n = 40$ equidistant scattering modes in $q \in [-4, 4]$ for the spectrum.

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The choice

$$c_q(t) = A_q e^{i\omega_q t}\delta(q - q_0) + A_q e^{-i\omega_q t}\delta(q + q_0) \tag{4.8}$$

describes the superposition of a kink with a combination of scattering modes of frequency $\omega_{\mathbf{q_0}}$.

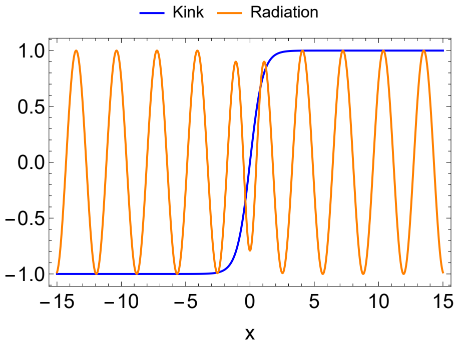

Fig. 4: Linear radiation perturbed by the kink at the origin.

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Taking (4.9) into account, the equation (4.6) reduces to

$$\ddot{c}_s(t) + \left(\omega_s^2 + f(q_0)\sin(\omega_{q_0} t)\right) c_s(t) = 0\,, \tag{4.10}$$

with

$$f(q_0) = -\frac{3\pi A_{q_0}}{4}\frac{q_0^2(q_0^2-2)}{\sinh\left(\pi q_0/2\right)}\sqrt{\frac{q_0^2+4}{q_0^2+1}}\,, \tag{4.11}$$

for small $c_s(t)$. This expression constitutes a Mathieu equation.

Instability regions $\Rightarrow \omega_s/\omega_{q_0} = k/2$.

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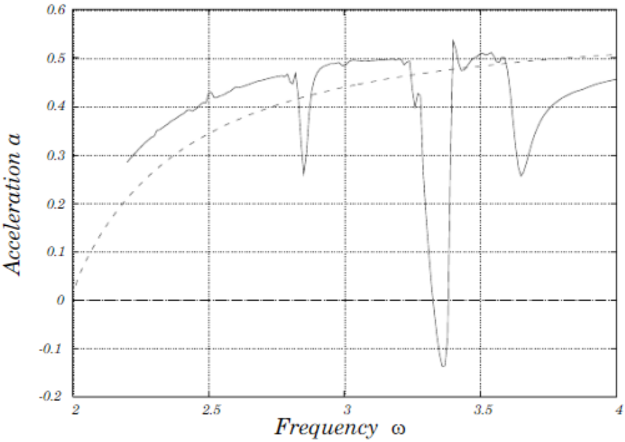

Fig. 5: Acceleration of an irradiated kink for $A_q = 0{,}16$. [4]

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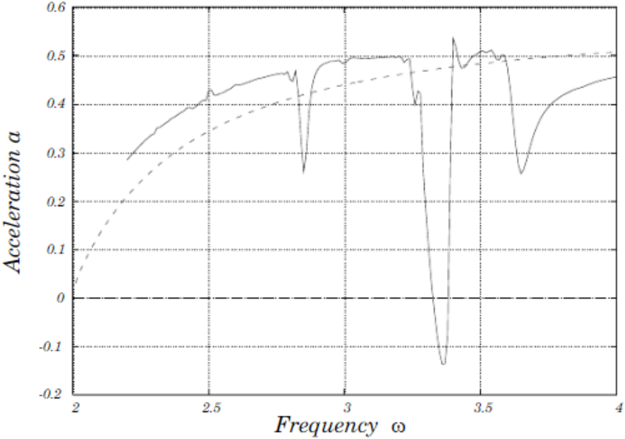

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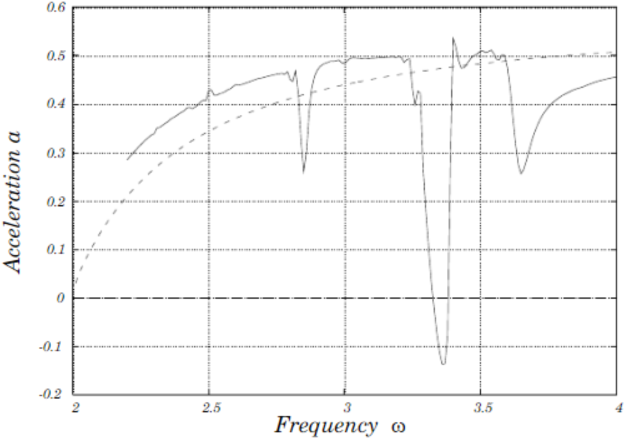

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Regarding the initial conditions

$$c_s(0) = 0\,, \quad c_s'(0) = 0\,, \tag{4.12}$$

$$c_q(0) = A_q\delta(q - q_0) + A_q\delta(q + q_0)\,, \tag{4.13}$$

$$\dot{c}_q(0) = i\omega_q A_q\delta(q - q_0) - i\omega_q A_q\delta(q + q_0)\,, \tag{4.14}$$

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$$\dot{c}_q(0) = i\omega_q A_q\delta(q - q_0) - i\omega_q A_q\delta(q + q_0)\,, \tag{4.14}$$

we are able to deduce an analytical expression for the excitation of the shape mode

$$c_s(t) = A_{q_0}^2\Omega(q_0)\left(\frac{1}{\omega_s^2} + \frac{\left(4\omega_{q_0}^2 - (\mathrm{sech}(\pi q_0) + 1)\omega_s^2\right)\cos\left(t\omega_s\right)}{\omega_s^2\left(\omega_s^2 - 4\omega_{q_0}^2\right)} + \frac{\mathrm{sech}(\pi q_0)\cos\left(2t\omega_{q_0}\right)}{\omega_s^2 - 4\omega_{q_0}^2}\right)$$

$$\Omega(q_0) = -\frac{3\sqrt{\frac{3}{2}}\pi\left(8q_0^4 + 34q_0^2 + 17\right)}{4\left(q_0^4 + 5q_0^2 + 4\right)}\,. \tag{4.15}$$

This expression is only valid for $A_{q_0} \ll 1$ (new phenomena appear[5]).

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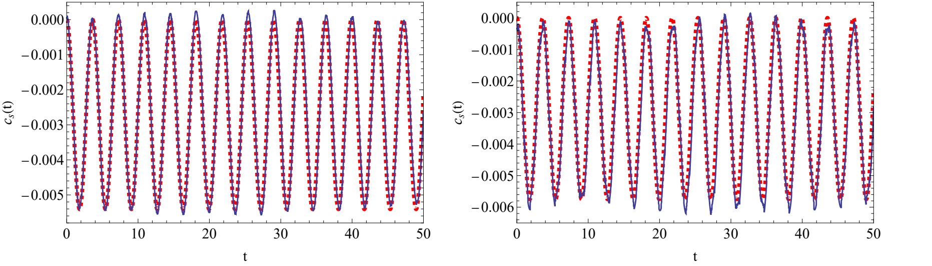
Interaction of radiation and shape mode

Comparison between the analytical approximation (dashed line) and the field theory result (solid line):



(a) $q = 2{,}0$ (b) $q = 3{,}0$

Fig. 6: We have taken into account $n = 30$ equidistant scattering modes in the interval $q \in [-3, 3]$.

Adding the translational mode

Now, we aim to generalise the previous approach allowing for translations of the kink.

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General ansatz

$$\phi(x,t) = \phi_K\big(x - a(t)\big) + c_s(t)\eta_s\big(x - a(t)\big) + \int_{\mathbb{R}} dq\, c_q(t)\eta_q\big(x - a(t)\big)\,. \quad (5.1)$$

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Once more, we have to substitute the field configuration ansatz into the Lagrangian density of the full theory (2.1) and integrate over the space.

$$\begin{aligned} \mathcal{L}_{s,q,t} &= \frac{1}{2}\Big(\dot{c}_s^2(t) - \omega_s^2 c_s^2(t)\Big) + \pi\int dq\,\Big(\dot{c}_q(t)\dot{c}_{-q}(t) - \omega_q^2 c_q(t)c_{-q}(t)\Big) + c_s^2(t)\int dq f_s(q)c_q(t) \\ &+ c_s(t)\int dq dq' f_{sq}(q,q')c_q(t)c_{q'}(t) + \frac{2}{3}\dot{a}^2(t) + \frac{\pi}{4}\sqrt{\frac{3}{2}}\dot{a}^2(t)c_s(t) + \dot{a}^2(t)\int dq f_{aa}(q)c_q(t) \\ &+ \dot{a}(t)\int dq f_{as}(q)\left(\dot{c}_s(t)c_q(t) - c_s(t)\dot{c}_q(t)\right) + \dot{a}(t)\int dq dq' f_a(q,q')\dot{c}_q(t)c_{q'}(t). \end{aligned} \quad (5.2)$$

Some terms have been neglected by assuming $|\dot{a}(t)| \ll 1$.

Adding the translational mode

The equations of motion associated to (5.2) are yielded by

$$
\begin{aligned}
\ddot{c}_{-q}(t) &+ \omega_q^2 c_{-q}(t) - \frac{1}{2\pi} c_s^2(t) f_s(q) - \frac{1}{\pi} c_s(t) \int dq' f_{sq}(q,q') c_{q'}(t) - \frac{1}{2\pi} \dot{a}^2(t) f_{aa}(q) \\
&- \frac{1}{2\pi} \ddot{a}(t) f_{as}(q) c_s(t) - \frac{1}{\pi} \dot{a}(t) f_{as}(q) \dot{c}_s(t) + \frac{1}{2\pi} \dot{a}(t) \int dq' \dot{c}_{q'}(t) \left(f_a(q,q') - f_a(q',q) \right) \\
&+ \frac{1}{2\pi} \ddot{a}(t) \int dq' f_a(q,q') c_{q'}(t) = 0, \qquad (5.3) \\
\ddot{c}_s(t) &+ \omega_s^2 c_s(t) - 2 c_s(t) \int dq f_s(q) c_q(t) - \int dq dq' f_{sq}(q,q') c_q(t) c_{q'}(t) - \frac{\pi}{4} \sqrt{\frac{3}{2}} \dot{a}^2(t) \\
&+ 2\dot{a}(t) \int dq f_{as}(q) \dot{c}_q(t) + \ddot{a}(t) \int dq f_{as}(q) c_q(t) = 0\,, \qquad (5.4) \\
\frac{4}{3} \ddot{a}(t) &+ \frac{\pi}{2} \sqrt{\frac{3}{2}} \left(\ddot{a}(t) c_s(t) + \dot{a}(t) \dot{c}_s(t) \right) + 2 \int dq f_{aa}(q) \left(\ddot{a}(t) c_q(t) + \dot{a}(t) \dot{c}_q(t) \right) \\
&+ \int dq f_{as}(q) \left(\ddot{c}_s(t) c_q(t) - c_s(t) \ddot{c}_q(t) \right) + \frac{d}{dt} \int dq dq' f_a(q,q') \dot{c}_q(t) c_{q'}(t) = 0\,. \qquad (5.5)
\end{aligned}
$$

Adding the translational mode

Let us assume that $a(t) = x_0 + v\,t$. An approximate solution of (5.4) and (5.3) is

$$c_s(t) = \frac{\pi}{4\sqrt{6}} v^2, \quad c_q(t) = -\frac{iq^2 \operatorname{csch}(\pi q/2)}{8\sqrt{(q^2+1)(q^2+4)}} v^2. \tag{5.6}$$

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If we assume the Lorentz boosted version of a kink

$$\phi(x,t) = \tanh\left(\frac{x - vt}{\sqrt{1-v^2}}\right), \tag{5.7}$$

and expand it with respect to v, at $t = 0$ we get

$$\phi(x,0) = \tanh(x) + \frac{1}{2}\left(x - x\tanh^2(x)\right) v^2 + \mathcal{O}(v^4)\,. \tag{5.8}$$

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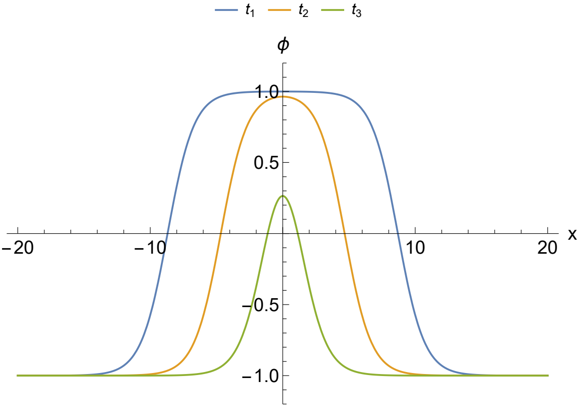
The projection of the first correction $\phi^{(1)}(x)$ onto the spectral modes gives

$$\langle \phi^{(1)}(x), \eta_s(x)\rangle = \frac{\pi}{4\sqrt{6}} v^2\,, \tag{5.9}$$

$$\langle \phi^{(1)}(x), \eta_q(x)\rangle = -\frac{i\pi q^2 \,\text{csch}(\pi q/2)}{4\sqrt{(q^2+1)(q^2+4)}} v^2\,. \tag{5.10}$$

We have proved that our general effective model describes relativistic effects.

Adding the translational mode



Connection between kink-antikink scattering and oscillon dynamics?

Effective model for the evolution of an oscillon

In order to model the profile of an oscillon, we use $\phi_o(x) = \text{sech}(x)$ [6].

First proposal

$$\Phi_{o,\ \text{rad}}(x; a, c_q) = -1 + a(t)\,\text{sech}(x/R) + \int_{\mathbb{R}} dq c_q(t) \eta_q(x/R). \qquad (6.1)$$

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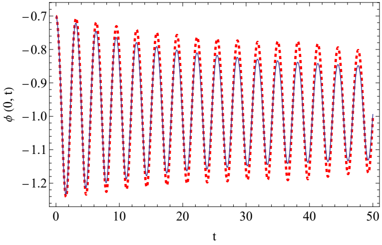
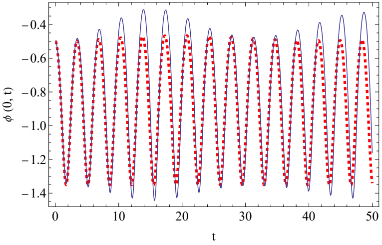


(a) $n = 20,\ a_0 = 0{,}3,\ R = 1{,}5.$



(b) $n = 10,\ a_0 = 0{,}5,\ R = 2.$

Fig. 7: Comparison between the effective model (dashed line) associated to (6.1) and field theory (solid line). The scattering modes have been taken in the interval $q \in [-5, 5]$.

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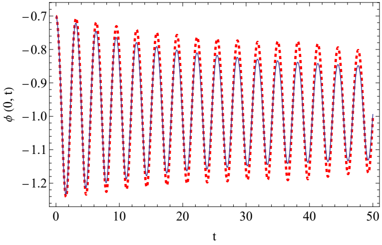

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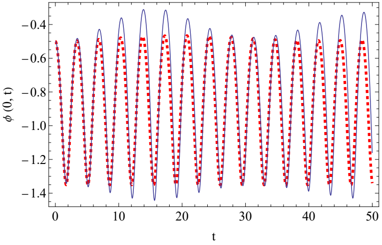

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We have modelled the oscillon through a Gaussian profile[7] for simplicity.

Moreover, we have chosen a new set of functions (that belong to $L^2(\mathbb{R})$) to describe radiation.

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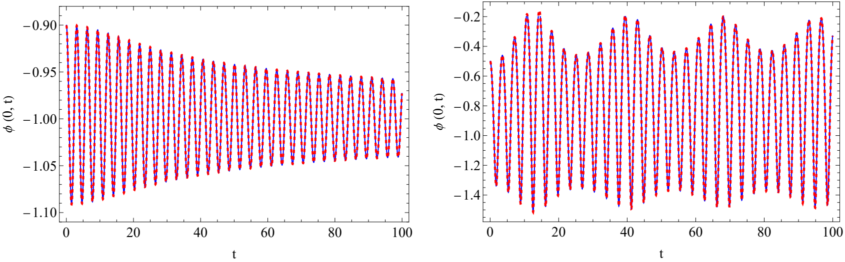

(a) $a_0 = 0{,}1,\ R = 4,\ r = 4,\ n = 7$ (b) $a_0 = 0{,}5,\ R = 4,\ r = 4,\ n = 7$

Fig. 8: $\phi(0, t)$ for different values of the initial amplitude in full numerics (solid line) and in the effective model (dashed line).

Effective model for the evolution of an oscillon

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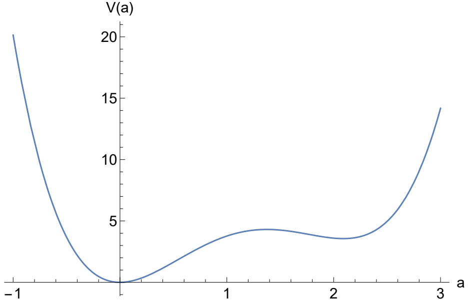
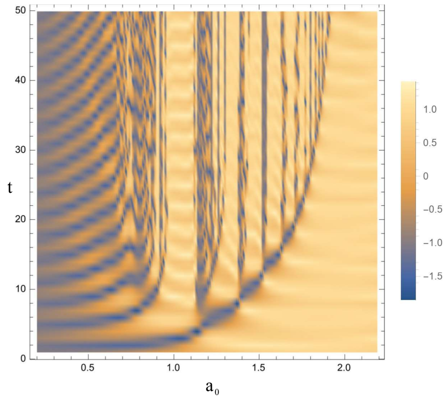
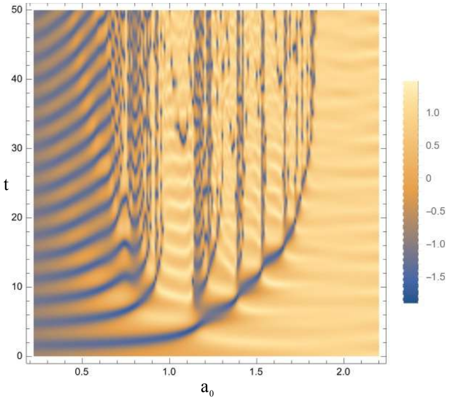


Fig. 9: Effective potential for $a(t)$ at $R = 4$.

Effective model for the evolution of an oscillon



(a) Full numerics. $R = 4$.



(b) Effective model. $R = 4$, $r = 2$ and $n = 8$.

Fig. 10: Comparison between the effective model and field theory. The color palette indicates the value of the field ϕ at the origin, $\phi(0, t)$.

Effective model for the evolution of an oscillon

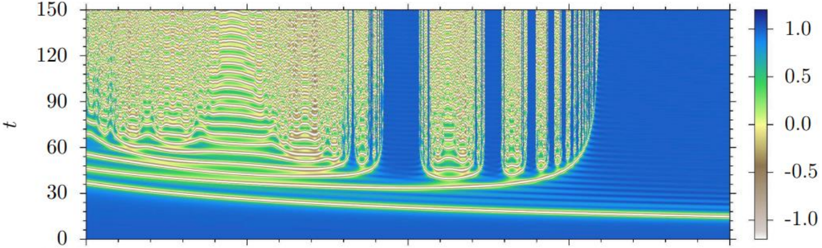


Fig. 11: Fractal pattern in kink-antikink scattering

May we describe this behaviour starting from oscillon initial data?

Summary and conclusions

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- An extremely simple effective model (system of coupled anharmonic oscillators) containing a set of functions emulating radiation is able to describe the KAK creation from initial oscillon data.

Thanks for your attention!