Wobbling kinks in a coupled two-component ϕ^4 field theory

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Outline

1.Introduction

- Among the range of topological solitons, kinks are the simplest and arise in a wide variety of scalar field theories.
- We will study one of the kinks that arise in a two-component ϕ^4 scalar field theory coupled by means of a parameter κ .
- We will unravel the behaviour of the kink solution when one of its shape modes is initially triggered. We will use a perturbative approach and then we will compare the results obtained with numerical simulations.

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- We will unravel the behaviour of the kink solution when one of its shape modes is initially triggered. We will use a perturbative approach and then we will compare the results obtained with numerical simulations.

2.Kinks in the two-component ϕ^4 model

- The model we are going to study is given by the Lagrangian density [1]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - U(\phi,\psi).$$

where

$$U(\phi,\psi) = \frac{1}{2}(\phi^2-1)^2 + \frac{1}{2}(\psi^2-1)^2 + \kappa\phi^2\psi^2 - \frac{1}{2}.$$

- The associated field equations are

$$\begin{aligned} \partial_{tt}\phi - \partial_{xx}\phi + 2\phi(\phi^2 - 1 + \kappa\psi^2) &= 0, \\ \partial_{tt}\psi - \partial_{xx}\psi + 2\psi(\psi^2 - 1 + \kappa\phi^2) &= 0. \end{aligned}$$

- For the potential $U(\phi,\psi)$ the vacua structure depends on the value of κ:

$$\mathcal{M}_{\kappa<1} = \left\{ \frac{1}{\sqrt{1+\kappa}} \begin{pmatrix} (-1)^a \\ (-1)^b \end{pmatrix}, \quad a,b=0,1 \right\},$$

$$\mathcal{M}_{\kappa>1} = \left\{ \begin{pmatrix} (-1)^a \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ (-1)^b \end{pmatrix}, \quad a,b=0,1 \right\}.$$

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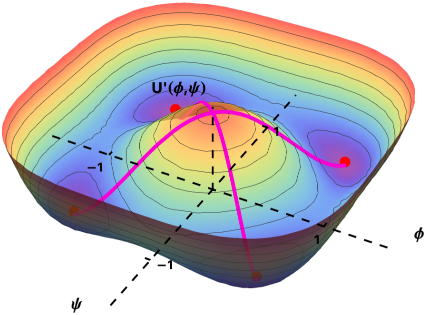
- $\underline{\kappa < \mathbf{1}}$

Kink solutions take the form:

$$K^{(a,b)}(x) = \frac{\tanh x}{\sqrt{1+\kappa}} \begin{pmatrix} (-1)^a \\ (-1)^b \end{pmatrix}, \quad a, b = 0, 1.$$

Shown over the potential:



- $\underline{\kappa > \mathbf{1}}$

Kink solutions take the form:

$$K_1^{(\pm)}(x) = \begin{pmatrix} \pm \tanh x \\ 0 \end{pmatrix}, \quad K_2^{(\pm)}(x) = \begin{pmatrix} 0 \\ \pm \tanh x \end{pmatrix}.$$

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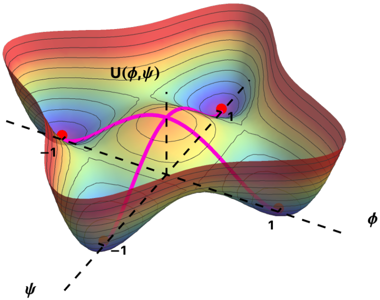

2.Stability and eigenmodes of the kink solution

- When the stability of a kink solution is studied we have to propose a solution with the structure

$$\widetilde{K}(x,t;\omega,a) = K_1^{(\pm)}(x) + a\, e^{i\omega t}\, F_\omega(x)\,.$$

where $K_1^{(\pm)}(x)$ is the kink solution and a is a small real parameter.

- Plugging this solution would lead to the problem

$$\mathcal{H}\, F_\omega(x) = \omega^2 F_\omega(x)$$

where

$$\mathcal{H} = \begin{pmatrix} -\frac{d^2}{dx^2} + 4 - 6\,\mathrm{sech}^2 x & 0 \\ 0 & -\frac{d^2}{dx^2} + 2\kappa \tanh^2 x - 2 \end{pmatrix}.$$

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2.Stability and eigenmodes of the kink solution

Longitudinal eigenmodes

Eigenfrequency (ω)	Eigenfunction
0	$(\overline{\eta}_0(x), 0)^t = (\mathrm{sech}^2 x, 0)^t$
$\overline{\omega} = \sqrt{3}$	$(\overline{\eta}_D(x), 0)^t = (\mathrm{sech}\, x\ \tanh x, 0)^t$
$\overline{\omega}^c_{\overline{q}} = \sqrt{4 + \overline{q}^2}$	$(\overline{\eta}_{\overline{q}}(x), 0)^t = \left(e^{i\overline{q}x}[-1 - \overline{q}^2 + 3\tanh^2 x - 3i\overline{q}\tanh x]\, , 0 \right)^t$

Orthogonal eigenmodes

Eigenfrequency (ω)	Eigenfunction
$\widehat{\omega}_{D,n} = \sqrt{(2n+1)\rho - n^2 - n - \frac{5}{2}}$	$(0, \widehat{\eta}_D(x))^t = (0, (\mathrm{sech}\, x)^{\rho - n - \frac{1}{2}}\ {}_2F_1(-n, 2\rho - n, \rho - n + 1/2, \frac{(1-\tanh x)}{2}))^t$
$\widehat{\omega}^c_{\widehat{q}} = \sqrt{\widehat{q}^2 + 2\kappa - 2}$	$\left(0\ ,\ \widehat{\eta}_{\widehat{q}}(x)\right)^t = \left(0,\ {}_2F_1(\frac{1}{2} - \rho, \frac{1}{2} + \rho, -i\widehat{q} + 1, \frac{(1-\tanh x)}{2})\, e^{i\widehat{q}x}\right)^t$

where $\overline{q}$ and $\widehat{q}$ are real quantities, $\rho = \sqrt{2\kappa + \frac{1}{4}}$ and n is a natural number whose maximum value is given by the relation $\kappa > \frac{n_{max}(n_{max}+1)}{2}$. For $\kappa < 3$, $\widehat{\omega}^2_{D,0} < 0$, which means that the kink is unstable in this regime.

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3.Perturbative appoach

- We are interested in studying the evolution of the system when one orthogonal mode is initially activated.
- We are going to assume a general expansion of the fields of the form [2]

$$
\begin{aligned}
\phi(x,t) &= \phi_K(x) + \overline{a}(t)\,\overline{\eta}_D(x) + \overline{\eta}(x,t), \\
\psi(x,t) &= \sum_p \widehat{a}_p(t)\,\widehat{\eta}_{D,p}(x) + \widehat{\eta}(x,t),
\end{aligned}
$$

where $\phi_K(x) = \tanh x$.
Plugging this assumption into the field equations and neglecting the smallest terms we find

$$
(\overline{a}_{tt} + \overline{\omega}^2\,\overline{a})\,\overline{\eta}_D + \overline{\eta}_{tt} - \overline{\eta}_{xx} - 2\overline{\eta} + 6\,\overline{\eta}\,\phi_K^2 + 6\overline{a}^2\,\overline{\eta}_D^2\,\phi_K + 2\kappa\,\phi_K\,(\sum_p \widehat{a}_p\widehat{\eta}_{D,p})^2 \approx 0,
$$

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3.Perturbative approach

If the previous equations are projected onto $\overline{\eta}$ and $\widehat{\eta}_{D,m}$, then:

$$(\overline{a}_{tt} + \overline{\omega}^2\,\overline{a})\,\overline{C}_D^2 + 6\,\overline{a}^2\,\overline{V} + \sum_{p,r} \widehat{a}_p\,\widehat{a}_r\,\widehat{B}_{pr} = 0,$$

$$((\widehat{a}_{tt})_m + \widehat{\omega}_m^2\,\widehat{a}_m)\,\widehat{C}_{D,m}^2 + 2\sum_{p} \overline{a}\,\widehat{a}_p\,\widehat{B}_{pm} = 0.$$

As it is assumed that we initially trigger the j-th shape mode, then

$$\widehat{a}_j(t) \approx a_0 \sin(\widehat{\omega}_j t).$$

If we now plug this formula into the first differential equation with the initial conditions

$$\overline{a}_t(0) = \overline{a}(0) = 0, \quad \widehat{a}_m(0) = \widehat{a}_m(0)_t = 0 \quad \text{with} \quad m \neq j,$$

we have that

$$\overline{a}(t) \approx \frac{a_0^2\,\widehat{B}_{jj}\left(4\widehat{\omega}_j^2 - \overline{\omega}^2 + \overline{\omega}^2\cos(2\widehat{\omega}_j t) - 4\widehat{\omega}_j^2\cos(\overline{\omega} t)\right)}{2\overline{C}_D^2\,\overline{\omega}^2(\overline{\omega}^2 - 4\widehat{\omega}_j^2)}.$$

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3.Perturbative approach

Using the previous relations in the original truncated expansion the next differential equations are found:

$$\begin{aligned} -\overline{\eta}''(x) + (6\phi_K^2 - 2 - 4\widehat{\omega}_j^2)\,\overline{\eta}(x) &= f(x), \\ -\widehat{\eta}''(x) + (2\kappa\,\phi_K^2 - 2 - \omega_\ell^2)\,\widehat{\eta}(x) &= g_\ell(x), \end{aligned}$$

where $\ell = 1, 2$ and $\omega_1 = 3\widehat{\omega}_j$, $\omega_2 = \widehat{\omega}_j + \overline{\omega}$.

These equations describe the radiation emitted on the longitudinal channel at frequency $2\widehat{\omega}_j$ and on the orthogonal one at frequencies $3\widehat{\omega}_j$ and $\widehat{\omega}_j + \overline{\omega}$.

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3.Perturbative approach

Using the **variation of parameters** method the assymptotic behaviour of the radiation terms can be computed:

$$\overline{\eta}_{2\widehat{\omega}_j} \xrightarrow{x\to\infty} \frac{i\left(\int_{-\infty}^{\infty}\overline{\eta}_{\overline{q}}(y)f(y)dy\right)}{2(\overline{q}+i)(\overline{q}+2i)}e^{-i\overline{q}x},$$

$$\widehat{\eta}_{\omega_\ell} \xrightarrow{x\to\infty} \frac{\left(\int_{-\infty}^{\infty}\widehat{\eta}_{\widehat{q}_\ell}(y)g_\ell(y)dy\right)}{2i\widehat{q}_\ell}e^{-i\widehat{q}_\ell x}.$$

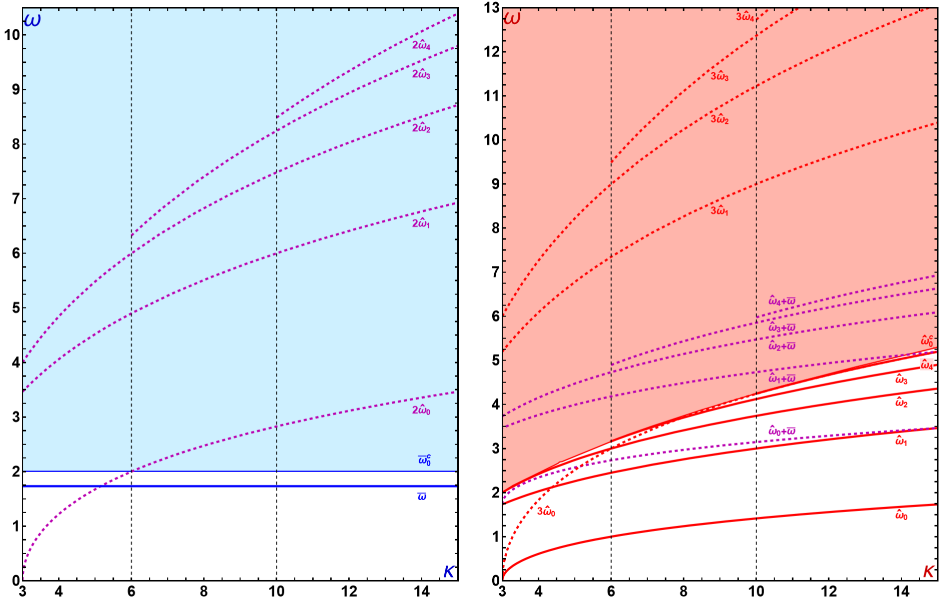
where

$$\overline{q} = 2\sqrt{\widehat{\omega}_j^2 - 1},$$

$$\widehat{q}_1 = \sqrt{9\widehat{\omega}_j^2 + 2 - 2\kappa},$$

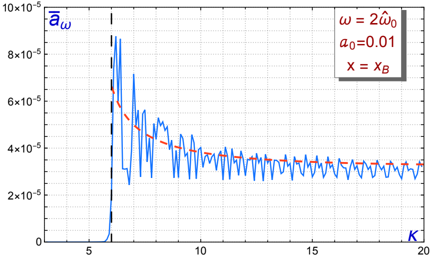
$$\widehat{q}_2 = \sqrt{(\widehat{\omega}_j + \overline{\omega})^2 + 2 - 2\kappa}.$$

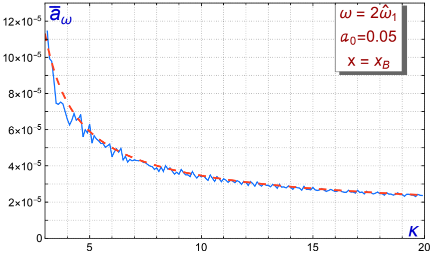
3.Perturbative approach

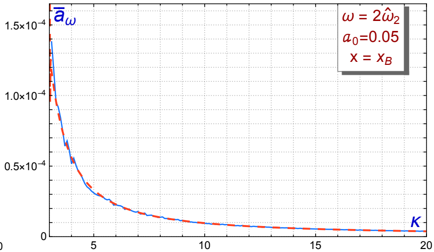


4.Numerical results: Radiation amplitudes

Longitudinal Radiation Amplitudes

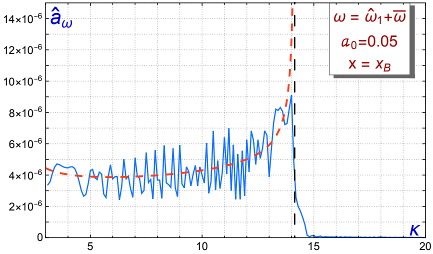


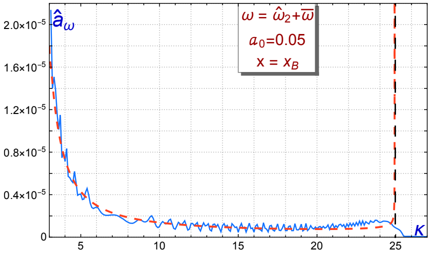




4. Numerical results: Radiation amplitudes

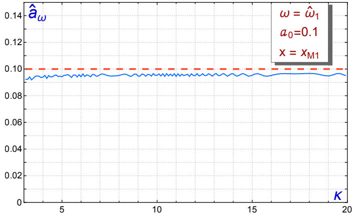
Orthogonal Radiation Amplitudes

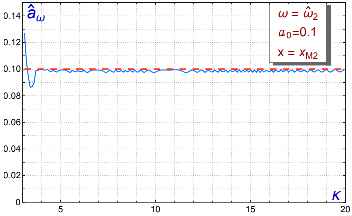




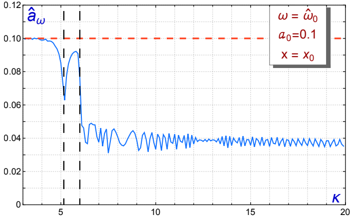
4.Numerical results: Orthogonal shape mode amplitudes

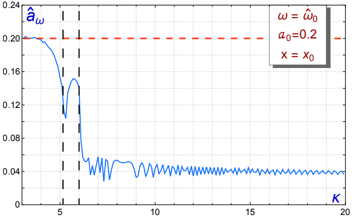
- We have assumed that the amplitude of the triggered shape mode remains constant. This assumption works fine when $\widehat{\eta}_{D,1}$ and $\widehat{\eta}_{D,2}$ are initially activated.





- But for $\widehat{\eta}_{D,0}$ a large decrease in the wobbling amplitude can be appreciated.





4.Numerical results: Orthogonal shape mode amplitudes

- A decay law for a_0 can be computed in order. First, if we trigger $\widehat{\eta}_{D,0}$ the total radiated energy flux is given by

$$\langle P \rangle = \frac{dE}{dt} = -(a_0^2\, \overline{A}'_{2\widehat{\omega}_0})^2\, (2\,\widehat{\omega}_0)\, \overline{q},$$

where $\overline{A}_{2\widehat{\omega}_0} = a_0^2\; \overline{A}'_{2\widehat{\omega}_0}$.

- On the other hand, the wobbling amplitude behaves as an harmonic oscillator on each point of the space so,

$$\mathcal{E} = \frac{1}{2}\widehat{\omega}_0^2 a_0^2 \widehat{\eta}_0^2 \to E = \int_{-\infty}^{\infty} \mathcal{E}\; dx = \frac{1}{2}\;\widehat{\omega}_0^2\;\; a_0^2\;\; \widehat{C}_{D,0}^2.$$

- This leads to the following differential equation for $a_0(t)$

$$\frac{1}{2}\,\widehat{\omega}_0^2\;\widehat{C}_{D,0}^2\;\frac{da_0^2(t)}{dt} \approx -2\widehat{\omega}_0\,\overline{A}'^2_{2\widehat{\omega}_0}\,\overline{q}\,a_0^4(t),$$

whose solution is

$$a_0(t) \approx \frac{a_0(0)}{\sqrt{1 + t\left(\dfrac{4\,\overline{q}\,a_0(0)^2\,\overline{A}'^2_{2\widehat{\omega}_0}}{\widehat{C}_{D,0}^2\,\widehat{\omega}_0}\right)}}.$$

4.Numerical results: Orthogonal shape mode amplitudes

- A decay law for a_0 can be computed in order. First, if we trigger $\widehat{\eta}_{D,0}$ the total radiated energy flux is given by

$$\langle P \rangle = \frac{dE}{dt} = -(a_0^2\,\overline{A}'_{2\widehat{\omega}_0})^2\,(2\,\widehat{\omega}_0)\,\overline{q},$$

where $\overline{A}_{2\widehat{\omega}_0} = a_0^2\ \overline{A}'_{2\widehat{\omega}_0}$.

- On the other hand, the wobbling amplitude behaves as an harmonic oscillator on each point of the space so,

$$\mathcal{E} = \frac{1}{2}\widehat{\omega}_0^2 a_0^2 \widehat{\eta}_0^2 \rightarrow E = \int_{-\infty}^{\infty} \mathcal{E}\ dx = \frac{1}{2}\ \widehat{\omega}_0^2\ a_0^2\ \widehat{C}_{D,0}^2.$$

- This leads to the following differential equation for $a_0(t)$

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5.Conclusions and future works

- Along this work, we have discussed the behaviour of an excited kink. In this process, radiation at three different frequencies has been found.
- Apart from this, the amplitudes found analytically predict the behaviour seen through numerical simulations.
- All the analytical methods used in this presentation can be extended in order to study other excited topological solitons such as Abelian-Higgs vortices.

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Thanks for your attention!!!