Deformations of Superstrings and Classical Yang-Baxter Equation

– Towards the Gravity/CYBE correspondence –

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Motivations

The AdS/CFT correspondence is a fascinating relation between type IIB superstring on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory.

- Integrability plays an important role in checking the conjecture.
- By considering integrable deformations, one can expect that a rich mathematical structure would appear and explain the correspondence.

Q. How can we describe deformations of gravity such as $U_q(\mathfrak{g})$?

$\Rightarrow$ we need "coordinate free" description of gravity deformation.

A. We can utilize the Classical r-matrix for characterizing deformations!

Def. of Algebra $\Leftarrow$ Classical r-matrix $\Rightarrow$ Def. of gravity

We call this the "Gravity/CYBE correspondence"

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What's the Classical r-matrix ?

For a Lie algebra $\mathfrak{g}$, the classical r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the CYBE (Classical Yang-Baxter Equation);

$$[r_{12}, r_{23}] + [r_{12}, r_{13}] + [r_{23}, r_{13}] = 0\,, \tag{1}$$

which is obtained as a classical limit of YBE;

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

by expansing $\mathcal{R}_{ij} = 1 + \hbar r_{ij} + \cdots$ and taking $\mathcal{O}(\hbar^2)$ order.

If there exists a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$, one can introduce a linear operator $R \equiv \langle r, \bullet \rangle_{\underline{2}} \in \mathrm{End}(\mathfrak{g})$.

Then, the CYBE (1) is equivalent with ($^\forall X, Y \in \mathfrak{g}$)

$$[R(X), R(Y)] - R\left([R(X), Y] + [X, R(Y)]\right) = 0\,.$$

(proof) For any $X, Y \in \mathfrak{g}$, summing up the following relations

$$
\begin{aligned}
[R(X), R(Y)] &= \langle [r_{12}, r_{13}], 1 \otimes X \otimes Y \rangle_{\underline{2},\underline{3}} \\
-R([R(X), Y]) &= \langle [r_{13}, -r_{32}], 1 \otimes X \otimes Y \rangle_{\underline{2},\underline{3}} \\
-R([X, R(Y)] &= \langle [r_{12}, r_{23}], 1 \otimes X \otimes Y \rangle_{\underline{2},\underline{3}} \ ,
\end{aligned}
$$

we have

$$
\begin{aligned}
&[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) \\
&= \langle [r_{12}, r_{13}] + [r_{13}, -r_{32}] + [r_{12}, r_{23}], 1 \otimes X \otimes Y \rangle_{\underline{2},\underline{3}} \, .
\end{aligned}
$$

If $\langle \ , \ \rangle$ is non-deg. and $r_{ij} = -r_{ji}$, the LHS reduces to the CYBE. □

The modified CYBE is a deformation of CYBE (mCYBE→CYBE, $c \to 0$);

$$
[R(X), R(Y)] - R\,([R(X), Y] + [X, R(Y)]) = -c^2 [X, Y] \, .
$$

Typical Solutions of (m)CYBE

In the $\mathfrak{sl}_2 = \langle e, h, f \rangle$ case, typical constant skew-symmetric solutions are

mCYBE

1. Drinfeld-Jimbo solution: $r = e \wedge f \equiv e \otimes f - f \otimes e$

CYBE

1. Trivial solution: $r = 0$
2. Jordanian: $r = e \wedge h$

For higher rank alg. e.g. $\mathfrak{sl}_n = \langle e_i, h_i, f_i \rangle$, there are more solutions;

4. Abelian: $r = h_i \wedge h_j \ (i \neq j)$
5. Abelian Jordanian: $r = e_i \wedge e_j \ (|i - j| > 1)$

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Yang-Baxter Sigma Models

Klimcik has introduced a systematic way of studying integrable deformations based on classical R-matrices, called Yang-Baxter sigma model. [Klimcik 2002] [Klimcik 2008]

The Lagrangian of deformed Principal Chiral Model is given by $(A = g^{-1}dg)$

$$L = \mathrm{Tr}\,(AA) \qquad \rightarrow \qquad L^{(\eta)} = \mathrm{Tr}\left(A \frac{1}{1-\eta R} A\right)$$

Deformations are characterized by the Classical R-matrix.

Recently, this approach has been generalized to $AdS_5 \times S^5$ superstrings.

- Deformations based on mCYBE [Delduc, Magro Vicedo '13]
- Deformations based on CYBE [Kawaguchi-TM Yoshida '14]

In this talk: we address deformation based on CYBE.

Green-Schwarz action of $AdS_5 \times S^5$ Superstring

The type IIB superstring action on $AdS_5 \times S^5$ was proposed by [Metsaev Tseytlin]

$$S = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \, \mathrm{Str}\left[A_\alpha d(A_\beta)\right]$$

where $\gamma^{\alpha\beta} = \mathrm{diag}(-1,1)$: ws metric and $\epsilon^{\alpha\beta}$: anti-sym. tensor

$$A_\alpha = g^{-1}\partial_\alpha g \in \mathfrak{psu}(2,2|4) \qquad \text{left-inv. 1-form}$$

$$d = P_1 + 2P_2 - P_3 \qquad P_i : \text{proj. of } \mathbb{Z}_4\text{-grading}$$

Note:

- Target space is described by supercoset $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$
- The coordinates are introduced by the parametrization of the group element $g = \mathrm{e}^{\theta_i T^i} \in PSU(2,2|4)$.
- The Lax pair is constructed $\Rightarrow$ Classically integrable [Bena Polchinski Roiban]

Yang-Baxter Deformed $AdS_5 \times S^5$ Superstring

A deformed $AdS_5 \times S^5$ superstring is proposed by [Delduc, Magro Vicedo '13]

$$S^{(\eta)} = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int d\tau d\sigma \,\mathrm{Str}\left[A_\alpha \frac{1}{1 - \eta d \circ R_g} d(A_\beta)\right]$$

where the Classical R-matrix is Drinfeld-Jimbo type, satisfying mCYBE:

$$R(X) = \mathrm{Str}[r 1 \otimes X] \qquad \text{with} \qquad r = \sum_{i<j} E_{ij} \wedge E_{ji}$$

and E_{ij} $(i, j = 1, \cdots, 8)$ are $\mathfrak{sl}(4|4)$ generators and

$$R_g(X) \equiv g^{-1} R(g X g^{-1}) g \,, \qquad d = P_1 + \frac{2}{1 - \eta^2} P_2 - P_3 \,.$$

Note:
- Trigonometric-like deformation
- The Lax pair is constructed $\Rightarrow$ Integrable deformation
- κ-symmetry is proven.

Deformations based on CYBE

Another class of deformed $AdS_5 \times S^5$ superstring is proposed by [Kawaguchi-TM Yoshida '14]

$$S^{(\eta)} = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int d\tau d\sigma \, \mathrm{Str}\left[A_\alpha \frac{1}{1 - \eta d \circ R_g} d(A_\beta)\right]$$

where the Classical R-matrix is skew-sym. and satisfying CYBE:

$$R(X) = \mathrm{Str}[r 1 \otimes X] \qquad \text{with} \qquad [[r, r]] = 0$$

and projector on supercoset is simply defined by

$$d = P_1 + 2P_2 - P_3 \, .$$

Note:
- Rational deformation (c.f. η could be normalized as 1)
- Partial deformations are possible. (e.g. deformations of S^5)
- The Lax pair is constructed $\Rightarrow$ Integrable deformation
- κ-symmetry is proven.

mCYBE v.s. CYBE

From the viewpoint of integrable models,

$$\begin{array}{ccc} \text{XXX} & \xrightarrow[\text{CYBE}]{} & \text{XXX}_{\eta} \\ {\scriptstyle \text{mCYBE}}\downarrow & & \\ \text{XXZ} & & \end{array}$$

- mCYBE gives the q-deformation : $q = e^{-\nu/g}$, $\nu = \frac{2\eta}{1+\eta^2}$ [Arutyunov Borsato Frolov]

- CYBE deformations do not change the integrability class.
  - regarded as a twisting of quasi-triangular Hopf algebra. [Reshetikhin 1990]

    $$\mathcal{R} \mapsto \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \qquad \Delta(X) \mapsto \mathcal{F}\Delta(X)\mathcal{F}^{-1}$$

  - Deformed b.g. is also obtained by duality-chain, such as TsT-trans.!
  - Classical R-matrices could classify deformations of type IIB SUGRA.

    $\Rightarrow$ "Gravity/ CYBE correspondence"

On-shell flat current

Let us introduce the deformed light-cone current $J_\pm$ by

$$J_\pm \equiv \frac{1}{1 \mp \eta R_g \circ d_\pm} A_\pm \qquad \text{with} \qquad \begin{aligned} A_\pm &= g^{-1}\partial_\pm g \\ d_\pm &= \pm P_1 + 2P_2 \mp P_3 \end{aligned} \quad .$$

Then, the eom and MC eq. of $A_\pm$ are written as

$$\mathcal{E} := \partial_- d_+(J_+) + \partial_+ d_-(J_-) + [J_-, d_+(J_+)] + [J_+, d_-(J_-)] = 0$$

$$\begin{aligned} \mathcal{Z} &:= \partial_+ A_- - \partial_- A_+ + [A_+, A_-] \\ &= \partial_+ J_- - \partial_- J_+ + [J_+, J_-] + \eta R_g(\mathcal{E}) + \eta^2 \text{CYBE} = 0 \end{aligned}$$

When the eom is satisfied $\mathcal{E} = 0$, the deformed current is also flat!

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0$$

Note: This is not true for the mCYBE.

Classical integrability

The eom $\mathcal{E} = 0$ and the MC eq. $\mathcal{Z} = 0$ are equivalent to the zero-curvature condition of the Lax connection;

$$\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = 0$$

where the Lax pair is introduced by ($\lambda \in \mathbb{C}$)[a]

$$\mathcal{L}_\pm(\lambda) \equiv J_\pm^{(0)} + \lambda J_\pm^{(1)} + \lambda^{\pm 2} J_\pm^{(2)} + \lambda^{-1} J_\pm^{(3)} \,.$$

$\Rightarrow$ The deformed sigma model is Classically integrable!

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[a] If the r-matrix satisfies the mCYBE, the Lax pair is given by [Delduc-Magro-Vicedo]

$$\mathcal{L}_\pm \equiv J_\pm^{(0)} + \lambda \sqrt{1+\eta^2} J_\pm^{(1)} + \lambda^{\pm 2} \frac{1+\eta^2}{1-\eta^2} J_\pm^{(2)} + \lambda^{-1} \sqrt{1+\eta^2} J_\pm^{(3)} \,.$$

Deformation as a gauge transformation

We have two flat currents $A_\pm$ and $J_\pm$. For the CYBE-type deformation, $J_\pm$ could be simply related to $A_\pm$ as

$$J_\pm = \tilde{g}^{-1}\partial_\pm \tilde{g} \qquad \text{with} \qquad \tilde{g} = \mathcal{F}^{-1} g\,.$$

In other words, the deformed current is a gauge transformation of $A_\pm$;

$$J_\pm = \mathcal{G} A_\pm \mathcal{G}^{-1} - \partial_\pm \mathcal{G}\mathcal{G}^{-1} \qquad \text{with} \qquad \mathcal{G} = g^{-1}\mathcal{F}^{-1} g\,.$$

Here the twist operator $\mathcal{F}$ is constructed as

$$\mathcal{F}(\sigma,\tau) = \mathrm{Pexp}\left[-\int_0^\sigma d\sigma\, J^g_\sigma\right]\mathcal{F}(0,\tau)\,.$$

by using the gauge transformed current J^g.

Note: classical analogue of the DR twist $\mathcal{R} \mapsto \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$.

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Gravity/CYBE correspondence

Classical r-matrix $\Longleftrightarrow$ Deformations of gravity solutions

$\Rightarrow$ Classical r-matrices as moduli space of deformed gravity backgrounds

A schematic picture

Deformed $AdS_5 \times S^5$ superstring action $\xrightarrow[\text{read off}]{}$ The metric and NS-NS $\mathbf{2}$-form

Classical r-matrix $\uparrow$ $\Updownarrow$

CYBE | A deformed gravity solution

Note:

- The metric and NS-NS $\mathbf{2}$-form have successfully been obtained so far.
- Further investigations are required for the RR sector.

How to compute the deformed metric ?

The bosonic part of the Lagrangian is

$$L = (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\mathrm{Tr}[A_\alpha P(J_\beta)] \qquad \text{with} \qquad J_\beta \equiv \frac{1}{1 - \eta R_g \circ P} A_\beta$$

The projected deformed current $P(J_\beta)$ is obtained by solving

$$(1 - \eta P \circ R_g) P(J_\beta) = P(A_\beta)$$

# of eqs. $= \dim G/H$

$$\begin{cases} \text{Symmetric part} \quad \gamma^{\alpha\beta}\mathrm{Tr}[A_\alpha P(J_\beta)] \Rightarrow \text{Metric} \\ \text{Skew-Sym. part} \quad \epsilon^{\alpha\beta}\mathrm{Tr}[A_\alpha P(J_\beta)] \Rightarrow \text{NS-NS } 2\text{-form} \end{cases}$$

Note:
- Could be applied for any coset G/H such as Non-symmetric
- Integrability is not necessary for the deformation

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Ex.1 Three-parameter γ-deformations of S^5

CYBE: Abelian classical r-matrix for $\mathfrak{su}(4)$ [TM-Yoshida 1404.1838]

$$r_{\rm Abe} = \mu_3 h_1 \wedge h_2 + \mu_1 h_2 \wedge h_3 + \mu_2 h_3 \wedge h_1$$

where h_1, h_2, h_3 are Cartan generators of $\mathfrak{su}(4)$.

Gravity: The resulting metric and B-field [Lunin Maldacena] [Frolov 2005]

$$ds^2 = ds^2_{\rm AdS_5} + \sum_{i=1}^{3}(d\rho_i^2 + G\rho_i^2 d\phi_i^2) + G\rho_1^2\rho_2^2\rho_3^2\left(\sum_{i=1}^{3}\hat{\gamma}_i\, d\phi_i\right)^2$$

$$B_2 = G\left(\hat{\gamma}_3\rho_1^2\rho_2^2\, d\phi_1 \wedge d\phi_2 + \hat{\gamma}_1\rho_2^2\rho_3^2\, d\phi_2 \wedge d\phi_3 + \hat{\gamma}_2\rho_3^2\rho_1^2\, d\phi_3 \wedge d\phi_1\right)$$

$$\textstyle\sum_{i=1}^{3}\rho_i^2 = 1\,, \qquad G^{-1} \equiv 1 + \hat{\gamma}_3^2\rho_1^2\rho_2^2 + \hat{\gamma}_1^2\rho_2^2\rho_3^2 + \hat{\gamma}_2^2\rho_3^2\rho_1^2$$

The parameter identification:

$$8\eta\mu_i = \hat{\gamma}_i \qquad (i = 1, 2, 3)$$

Note: Originally, obtained by TsT transformation

Ex.2 Gravity duals for Non-commutative SYM

CYBE: Abelian Jordanian classical r-matrix for $\mathfrak{su}(2,2) = \mathfrak{so}(2,4)$ [TM-Yoshida 1404.3657]

$$r_{\rm AJ} = \mu p_2 \wedge p_3 + \nu p_0 \wedge p_1 \qquad \text{where} \qquad p_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ 0 & 0 \end{pmatrix}$$

Gravity: The resulting metric and B-field [Hashimoto Itzhaki] [Maldacena Russo]

$$ds^2 = \frac{z^2}{z^4 - 4a'^2}(-dx_0^2 + dx_1^2) + \frac{z^2}{z^4 + 4a^2}(dx_2^2 + dx_3^2) + \frac{dz^2}{z^2} + d\Omega_5^2$$

$$B_2 = -\frac{a'}{z^4 - a'^2} dx_0 \wedge dx_1 + \frac{a}{z^4 + a^2} dx_2 \wedge dx_3$$

The parameter identification:

$$2\eta\mu = a^2\,, \qquad 2\eta\nu = a'^2$$

Note: Classical integrability is now proven as a byproduct!

Ex.3 New backgrounds from Jordanian r-matrix

CYBE: Jordanian classical r-matrix for $\mathfrak{su}(2,2)$ [Kawaguchi-TM Yoshida] [TM-Yoshida in preparation]

$$r_{\rm Jor} = E_{24} \wedge (\gamma E_{22} - \gamma^* E_{44}) \qquad \text{with def. param.} \qquad \gamma \in \mathbb{C}$$

Gravity: The resulting metric of AdS_5 and B-field

$$ds^2 = \frac{-2dx^+dx^- + dx_1^2 + dx_2^2 + dz^2}{2z^2} - \frac{|\gamma|^2(x_1^2 + x_2^2) + ({\rm Re}\gamma)^2 z^2}{2z^6}(dx^+)^2$$

$$B_2 = -{\rm Re}\gamma \frac{(x_1 dx_1 + x_2 dx_2 + z dz) \wedge dx^+}{2z^4} + {\rm Im}\gamma \frac{(x_2 dx_1 - x_1 dx_2) \wedge dx^+}{2z^4}$$

Note:
- Both metric and B-field are Real though $r_{\rm Jor}$ depending on $\gamma \in \mathbb{C}$
- When $\gamma \in \mathbb{R}$, the IIB SUGRA components are obtained in [Kawaguchi-TM Yoshida]
- For $\gamma \in i\mathbb{R}$, the b.g. is obtained by a TsT-transformation. [TM-Yoshida in preparation]

Ex.4 Generalization to Non-integrable background

Q. Can we generalize YB σ models to $AdS_5 \times T^{1,1}$ background? [Klebanov Witten]

($T^{1,1}$ is $U(1)$-fiber over $S^2 \times S^2$, known to be non-integrable) [Basu Pando Zayas]

$$ds^2_{T^{1,1}} = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)$$
$$+ \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 .$$

A. YES! Provided $T^{1,1}$ could be expressed as coset. [Crichigno-TM Yoshida] c.f. [Candelas de la Ossa]

- Coset construction of $T^{1,1}$; (K_3, L_3, M are Cartan gen. of numerator)

$$T^{1,1} = \frac{SU(2) \times SU(2) \times U(1)_R}{U(1)_1 \times U(1)_2}, \qquad \begin{aligned} T_1 &= K_3 + L_3 \\ T_2 &= K_3 - L_3 + 4M \end{aligned}$$

- Metric is given by $ds^2_{T^{1,1}} = \mathrm{STr}[AP(A)]$ with $A = g^{-1}dg$.

Good!, but why SuperTrace...?

What is the Origin of Supercoset?

$T^{1,1}$ is the bosonic part of supercoset with $\mathcal{N}=1$ superconformal alg. [Klebanov Witten]

$$AdS_5 \times T^{1,1} \overset{\text{bosonic}}{\subset} \frac{PSU(2,2|1) \times SU(2) \times SU(2)}{SO(1,4) \times U(1) \times U(1)}$$

Note: Need to distinguish $SU(2)\times SU(2)$: flavor sym. and $U(1)_R$: R-sym.

- In a matrix form, full symmetry of $T^{1,1}$ is embedded as

$$\left(\begin{array}{cc|c} SU(2) & 0 & 0 \\ 0 & SU(2) & 0 \\ \hline 0 & 0 & U(1)_R \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} SU(2,2) & 0 & 0 & \overline{F}^A \\ 0 & SU(2) & 0 & 0 \\ 0 & 0 & SU(2) & 0 \\ \hline F_A & 0 & 0 & U(1)_R \end{array}\right)$$

- Fund.rep. is given by $(4|1)\times(4|1)$ supermatrices rather than 5×5 mat.

$$K_i = -\frac{i}{2}\left(\begin{array}{cc|c} \sigma_i & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad L_i = -\frac{i}{2}\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & \sigma_i & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad M = -\frac{i}{2}\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Therefore, the metric is given by STr rather than Tr ! [Crichigno-TM Yoshida]

Ex.4 Three-parameter γ-deformations of $T^{1,1}$

CYBE: Abelian classical r-matrix for $\mathfrak{su}(2)\times\mathfrak{su}(2)\times\mathfrak{u}(1)_R$ [TM-Yoshida Crichigno]

$$r_{\rm Abe}=\frac{1}{3}\hat{\gamma}_1 L_3\wedge M+\frac{1}{3}\hat{\gamma}_2 M\wedge K_3-\frac{1}{6}\hat{\gamma}_3 K_3\wedge L_3$$

Gravity: The resulting metric and B-field agree with [Lunin Maldacena] [Catal-Ozer 2005]

$$ds^2=G\Big[\frac{1}{6}\sum_{i=1,2}(G^{-1}d\theta_i^2+\sin^2\theta_i d\phi_i^2)+\frac{1}{9}(d\psi+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2)^2$$
$$+\frac{\sin^2\theta_1\sin^2\theta_2}{324}(\hat{\gamma}_3 d\psi+\hat{\gamma}_1 d\phi_1+\hat{\gamma}_2 d\phi_2)^2\Big]\,,$$

$$B_2=G\Big[\Big\{\hat{\gamma}_3\Big(\frac{\sin^2\theta_1\sin^2\theta_2}{36}+\frac{\cos^2\theta_1\sin^2\theta_2+\cos^2\theta_2\sin^2\theta_1}{54}\Big)$$
$$-\hat{\gamma}_2\frac{\cos\theta_2\sin^2\theta_1}{54}-\hat{\gamma}_1\frac{\cos\theta_1\sin^2\theta_2}{54}\Big\}d\phi_1\wedge d\phi_2$$
$$+\frac{(\hat{\gamma}_3\cos\theta_2-\hat{\gamma}_2)\sin^2\theta_1}{54}d\phi_1\wedge d\psi-\frac{(\hat{\gamma}_3\cos\theta_1-\hat{\gamma}_1)\sin^2\theta_2}{54}d\phi_2\wedge d\psi\Big]\,,$$

$$G^{-1}\equiv 1+\hat{\gamma}_3^2\Big(\frac{\sin^2\theta_1\sin^2\theta_2}{36}+\frac{\cos^2\theta_1\sin^2\theta_2+\cos^2\theta_2\sin^2\theta_1}{54}\Big)$$
$$+\hat{\gamma}_1^2\frac{\sin^2\theta_2}{54}+\hat{\gamma}_2^2\frac{\sin^2\theta_1}{54}-\hat{\gamma}_2\hat{\gamma}_3\frac{\sin^2\theta_1\cos\theta_2}{27}-\hat{\gamma}_3\hat{\gamma}_1\frac{\sin^2\theta_2\cos\theta_1}{27}\,.$$

Catalog of Gravity/CYBE correspondence

Typical constant skew-sym. R for $\mathfrak{gl}(4|4)$ and deformed $AdS_5 \times S^5$;

CYBE $\rightarrow$ Gravity

1. Trivial solution: $r = 0$ $\rightarrow$ Undeformed [Metsaev Tseytlin]
2. Jordanian: $r = E_{24} \wedge (E_{22} - E_{44})$ $\rightarrow$ Null Melvine twist [Dhokarh, Haque Hashimoto]
3. Abelian: $r = \gamma_3 h_1 \wedge h_2$ $\rightarrow$ γ_i-deformations of S^5 [Lunin Maldacena][Frolov 2005]
4. Abelian Jordanian: $r = p_1 \wedge p_2$ $\rightarrow$ Gravity duals for NC-YM[a] [Hashimoto Itzhaki][Maldacena Russo]

Non-integrable application for $AdS_5 \times T^{1,1}$;

5. Abelian: $r = \gamma_3 K_3 \wedge L_3$ $\rightarrow$ γ_i-deformations of $T^{1,1}$ [Lunin Maldacena][Catal-Ozer 2005]

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[a] The integrability is proved as byproduct!

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Summary

- proposed an integrable deformation of the $AdS_5 \times S^5$ superstring by the classical r-matrices

- The typical r-matrices reproduce the well-known deformed SUGRA b.g. such as LM, the gravity duals for NC-YM, ...

$\Rightarrow$ Observing many non-trivial examples, we conjecture that

Gravity/CYBE correspondence

The classical r-matrices could classify analytically the deformed supergravity backgrounds, which are partially described by coset spaces.

Open Problems

- Not only the NS-NS sector, but also the RR-sector should be confirmed.

Gravity/CYBE correspondence

The classical r-matrices could classify analytically the deformed supergravity backgrounds, which are partially described by coset spaces.

Schematically, given a parent theory x_0, we expect that there exist a map

$$\begin{aligned} f_0 : \text{“CYBE}_0\text{”} &\longrightarrow \text{“gravity}_0\text{”} \\ R &\longmapsto f_0(R) \end{aligned}$$

Q1: $f_0(R) \in$ “gravity$_0$”?

Q2: Is f_0 injective? *i.e.* $R_1 \neq R_2 \;\Rightarrow\; f_0(R_1) \neq f_0(R_2)$?

Q3: Is f_0 surjective? *i.e.* $f_0(\text{“CYBE}_0\text{”}) =$ “gravity$_0$”? How to find f_0^{-1}?

Q4: Is f_0 homomorphism? *i.e.* $f_0(R_1 \star R_2) = f_0(R_1) \cdot f_0(R_2)$?