Constructing Kinks and Sphalerons in One Dimension

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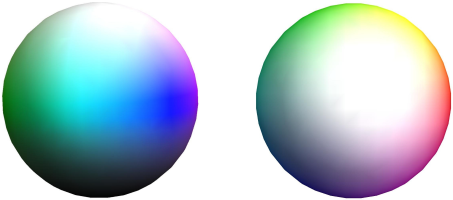
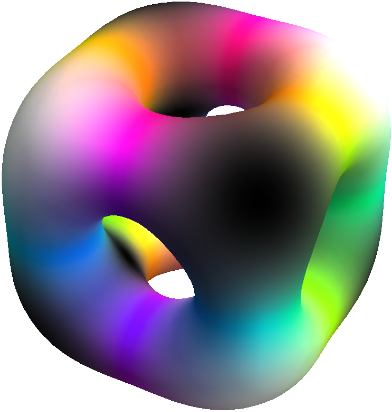
Topological Solitons

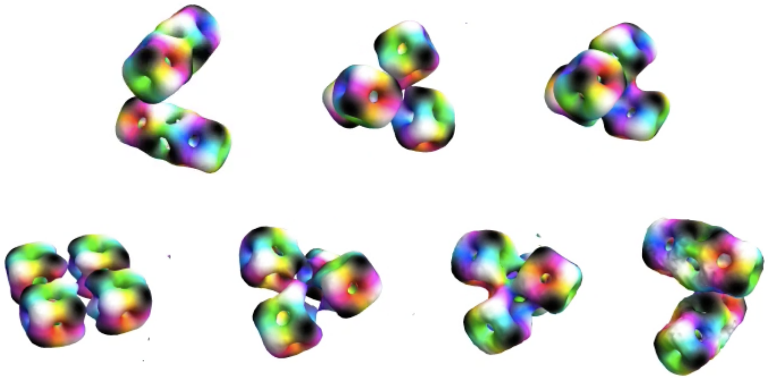
- Topological Solitons are classical solutions of equations of field theory – usually time-independent, localised, smooth and stable, with finite energy. They model a type of particle.
- Their stability relies on nonlinearity – multiple vacua, symmetry and spontaneous symmetry breaking, nonlinearity of fundamental field.
- Soliton character depends on the spatial dimension.
- 1d solitons are kinks – become domain walls in 3d.
  2d solitons are σ-model lumps, magnetic flux vortices, and magnetic Skyrmions – become string-like in 3d.
  3d solitons are magnetic monopoles, and baryonic Skyrmions.
- BPS (Bogomolny–Prasad–Sommerfield) solitons satisfy first-order field equations, as well as second-order E–L equations.



Baryon number $B = 1$ Skyrmion (two orientations). They attract and partly merge into larger-B Skyrmions.



$B = 4$ Skyrmion



$B = 16$ Skyrmion dynamics – via tetrahedron, square, and dual tetrahedron

Sphalerons

- Sphaleron is an unstable analogue of a soliton – a finite-energy saddle-point of field theory. Usually has a single mode of instability.
- Sphalerons depend on topological structure of field configuration landscape. A non-contractible path: Vacuum $\to$ Sphaleron $\to$ Vacuum has sphaleron as “mountain pass”.
- Sphalerons in 1d resemble kink-antikink pairs. Pair can annihilate or split up.
- Standard Electroweak theory (Gauge + Higgs Fields) has sphaleron but no monopole. Vacuum to vacuum tunnelling, via sphaleron, changes baryon and lepton numbers.

Scalar Field Theory in 1d

- Scalar field $\phi(x, t)$. Potential $U(\phi)$ provides nonlinearity, e.g. multiple minima (vacua).
- Lagrangian:

$$L = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - U(\phi)\right\} dx\,.$$

- Classical Euler–Lagrange equation:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{1}{2}\frac{dU}{d\phi} = 0\,.$$

Time-independent Fields

- Energy:

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + U(\phi)\right\} dx\,.$$

- E–L equation reduces to

$$\frac{d^2\phi}{dx^2} = \frac{1}{2}\frac{dU}{d\phi}\,,$$

with first integral

$$\left(\frac{d\phi}{dx}\right)^2 = U(\phi) + C\,.$$

- Absorb constant C into $U(\phi)$ and take square root.

Constructing Solutions

- Obtain autonomous first-order ODE as field equation:

$$\frac{d\phi}{dx} = \sqrt{U(\phi)} .$$

- Formal solution is

$$x = \int \frac{d\phi}{\sqrt{U(\phi)}} ,$$

  generally, a contour integral on the branched double-cover of the ϕ-plane (Semi-BPS case).
- If U only has zeros of even order (e.g. double zeros), then U has explicit square root. Integral then on ϕ-plane (BPS case) and simpler.

Canonical Example – ϕ^4 Kink

- Double-well ϕ^4-theory potential

$$U(\phi) = (1-\phi^2)^2 = (1-\phi)^2(1+\phi)^2\,,$$

has double zeros at ± 1. $\sqrt{U}$ has no branch points.
Solution of field equation,

$$x = \int \frac{d\phi}{1-\phi^2}\,.$$

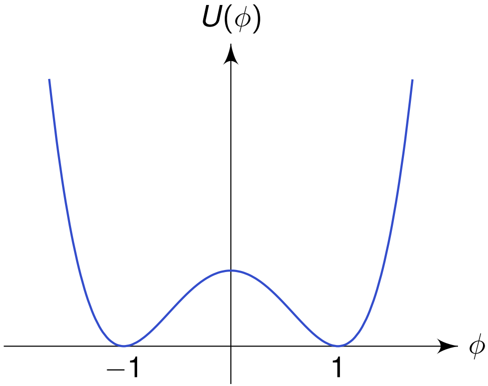
- Method of partial fractions gives

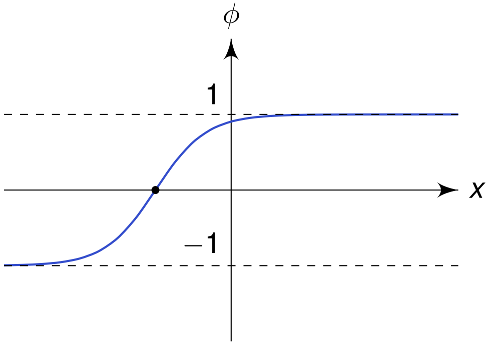
$$x = \frac{1}{2}\int \left\{ \frac{1}{1-\phi} + \frac{1}{1+\phi} \right\} d\phi = \frac{1}{2}\log\left(\frac{1+\phi}{1-\phi}\right)\,.$$

Invert to obtain explicit BPS kink solution

$$\phi(x) = \frac{e^{2x}-1}{e^{2x}+1} = \tanh x\,,$$

connecting $\phi_- = -1$ to $\phi_+ = 1$.


$U(\phi)$
ϕ
-1
1



ϕ
1
-1
x

A Mechanical Analogue

- Solution $\phi(x)$ is analogous to frictionless point-particle motion $y(t)$ in a potential $V(y)$.
- Conservation of energy for point-particle:

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + V(y) = \mathcal{E}\,.$$

- Set $\mathcal{E} = 0$. Analogue of $U(\phi)$ is $-2V(y)$.
- Kink profile $\phi(x)$ is equivalent to particle motion $y(t)$ in upside-down potential. Particle takes infinite time to roll between stationary points of potential.

Simplest Potential with Quartic Zeros

- For potential

$$U(\phi) = (1 - \phi^2)^4 \, ,$$

field equation has formal solution

$$x = \int \frac{d\phi}{(1 - \phi^2)^2} \, .$$

- Using partial fractions,

$$\begin{aligned} x &= \frac{1}{4} \int \left\{ \frac{1}{(1-\phi)^2} + \frac{1}{1-\phi} + \frac{1}{(1+\phi)^2} + \frac{1}{1+\phi} \right\} d\phi \\ &= \frac{1}{2} \frac{\phi}{1-\phi^2} + \frac{1}{4} \log\left(\frac{1+\phi}{1-\phi} \right) \, , \end{aligned}$$

an implicit BPS kink $\phi(x)$ connecting -1 to 1, with long-range, power-law tails.

An Algebraic Kink with Long-range Tails

- Rational potential

$$U(\phi) = \frac{(1-\phi^2)^4}{(1+\phi^2)^2}$$

again has quartic zeros at $\phi = \pm 1$, elsewhere $U > 0$.

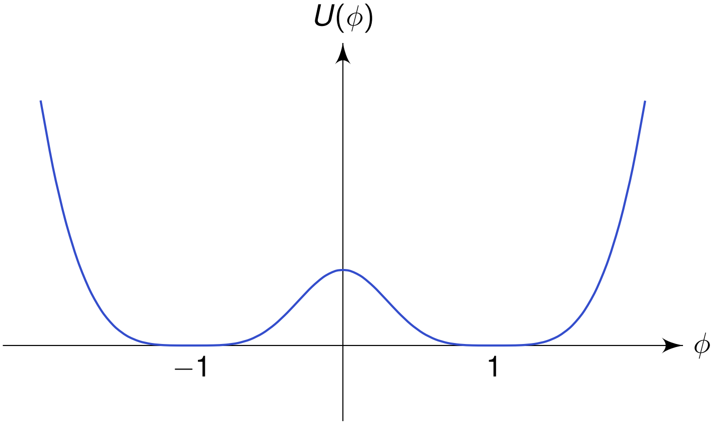
- Partial fraction solution

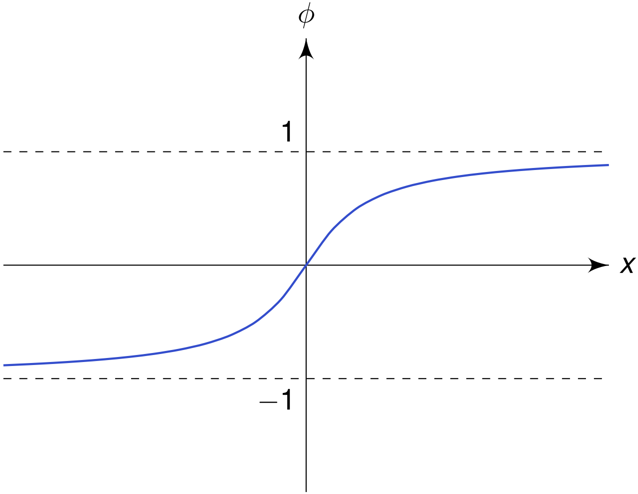
$$\begin{aligned} x &= \int \frac{1+\phi^2}{(1-\phi^2)^2}\, d\phi\,, \\ &= \frac{1}{2}\int \left\{ \frac{1}{(1-\phi)^2} + \frac{1}{(1+\phi)^2} \right\} d\phi \\ &= \frac{1}{2}\left\{ \frac{1}{1-\phi} - \frac{1}{1+\phi} \right\} = \frac{\phi}{1-\phi^2}\,, \end{aligned}$$

so

$$\phi(x) = \frac{\sqrt{1+4x^2}-1}{2x}\,,$$

an explicit BPS kink with long-range tails. Note, integrand has no simple pole terms.


$U(\phi)$
ϕ
-1
1



ϕ
1
x
−1

Semi-BPS Kink – U with Simple Zeros

- Christ–Lee potential

$$U(\phi) = \frac{1}{2}(1-\phi^2)^2(1+\phi^2)$$

  has double zeros at ± 1 and simple zeros at $\pm i$. Need to work on double-cover of ϕ-plane.

- Kink solution connecting -1 to 1 is

$$\begin{aligned} x &= \sqrt{2}\int \frac{d\phi}{(1-\phi^2)\sqrt{1+\phi^2}} \\ &= \frac{1}{2}\log\left(\frac{\sqrt{2(1+\phi^2)}+2\phi}{\sqrt{2(1+\phi^2)}-2\phi}\right) . \end{aligned}$$

  Invert to obtain explicit kink

$$\phi(x) = \frac{\sinh x}{\sqrt{\sinh^2 x + 2}} .$$

Bogomolny Argument

- Complete square in energy integral. Rederive first-order field equation, and obtain formula for kink energy:

$$\begin{aligned} E &= \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{d\phi}{dx}\right)^2 + U(\phi)\right\} dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{d\phi}{dx} - \sqrt{U(\phi)}\right)^2 + 2\sqrt{U(\phi)}\,\frac{d\phi}{dx}\right\} dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{d\phi}{dx} - \sqrt{U(\phi)}\right)^2 dx + \int_{\phi_-}^{\phi_+}\sqrt{U(\phi)}\,d\phi\,. \end{aligned}$$

Energy E is minimised provided

$$\frac{d\phi}{dx} = \sqrt{U(\phi)}\,.$$

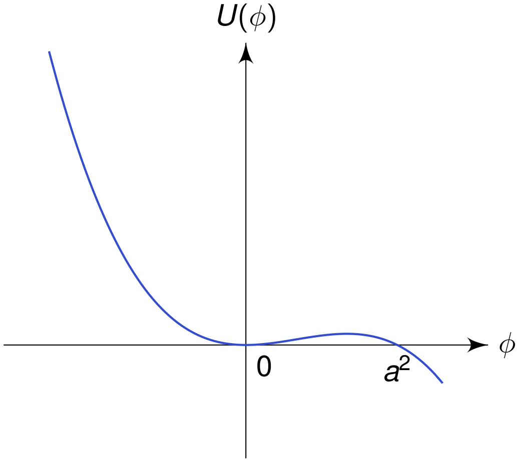
Then E "topological" – only depends on U and endpoints $\phi_{\pm}$.

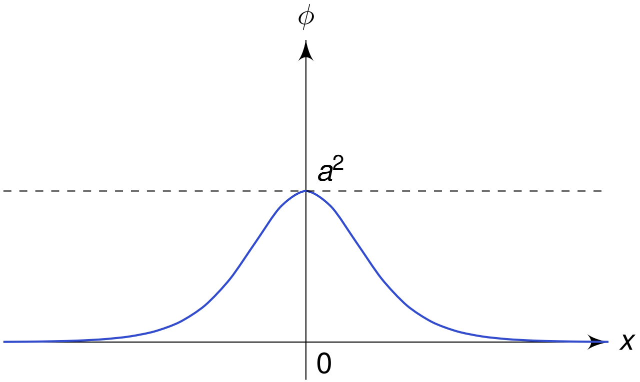
- Define
$$W(\phi) = \int \sqrt{U(\phi)}\, d\phi\,.$$
Then
$$E = W(\phi_+) - W(\phi_-)\,.$$
- Ex. 1. BPS ϕ^4-kink energy:
$$\sqrt{U(\phi)} = 1 - \phi^2 \quad \Longrightarrow \quad W(\phi) = \phi - \frac{1}{3}\phi^3$$
Kink energy $E = W(1) - W(-1) = \frac{4}{3}$.
- Ex. 2. Semi-BPS Christ–Lee kink energy:
$$\begin{aligned} W(\phi) &= \int (1-\phi^2)\sqrt{1+\phi^2}\, d\phi \\ &= \left(\frac{3}{8}\phi - \frac{1}{4}\phi^3\right)\sqrt{1+\phi^2} + \frac{5}{8}\sinh^{-1}\phi\,. \\ E &= W(1) - W(-1) = \frac{1}{4}\sqrt{2} + \frac{5}{4}\sinh^{-1} 1\,. \end{aligned}$$

Cubic Sphaleron

- Cubic potential $U(\phi) = \phi^2(1-\phi)$. Sphaleron solution
$$x = \int \frac{d\phi}{\phi\sqrt{1-\phi}} \,.$$
ϕ runs from double zero at $\phi = 0$ to simple zero at $\phi = 1$ and back to $\phi = 0$ on second sheet. Sphaleron is semi-BPS.
- Parametrisation $\phi = 1 - t^2$ gives
$$x = 2\int \frac{dt}{1-t^2} = 2\tanh^{-1} t \,.$$
so $t = \tanh\frac{x}{2}$, and
$$\phi(x) = 1 - \tanh^2\frac{x}{2} = \frac{1}{\cosh^2\frac{x}{2}} \,.$$

- Sphaleron energy given by Bogomolny formula. Find integral of $\sqrt{U}$ and integrate over complete ϕ-contour. t-integral is from -1 to 1. $E = \frac{8}{15} > 0$.
- Solitons and sphalerons have translation zero modes – energy unchanged. Sphaleron translation zero mode has a zero (a node). Therefore, sphaleron's second-variation operator has a nodeless eigenfunction with negative squared frequency – the unstable mode.


$U(\phi)$
ϕ
0
a^2



ϕ
a^2
x
0

Kink chains

- If $U(\phi)$ has adjacent simple zeros (with $U > 0$ between), then field equation has semi-BPS solution. x increases over finite range between zeros. x continues increasing on second sheet of $\sqrt{U}$, then on first sheet, etc.
- $\phi(x)$ is spatially periodic. It is a chain of alternating kinks and antikinks. Like a sphaleron, a kink chain is unstable.
- Simplest example has $U(\phi) = 1 - \phi^2$. Kink chain $\phi(x) = \sin x$ oscillates between -1 and 1 with period 2π. More realistic kink chain occurs for

$$U(\phi) = (1 - \phi^2)(1 - k^2\phi^2)$$

with $0 < k < 1$. Solution is Jacobi function $\phi(x) = \mathrm{sn}\, x$, oscillating between ± 1.

Trigonometric Potential $U = \cos^n(\phi)$

- For $n = 1$, U has simple zeros. Semi-BPS kink chain solution

$$x = \int \frac{d\phi}{\sqrt{\cos\phi}} \,.$$

Use $t = \tan\frac{\phi}{2}$ to convert to elliptic integral. Kink chain is Jacobi-type function.

- For $n = 2$, U has double zeros. BPS kink solution

$$x = \int \frac{d\phi}{\cos\phi} \,.$$

Invert to obtain (variant of) sine-Gordon kink

$$\tan\phi = \sinh x \,.$$

- Further solutions for other n.

Summary

- We have found a broad range of static solutions in 1d scalar field theories – kinks, sphalerons, kink chains – some stable, others unstable.
- Solution and its energy are

$$x = \int \frac{d\phi}{\sqrt{U(\phi)}} , \quad E = \int_{\phi_-}^{\phi_+} \sqrt{U(\phi)}\, d\phi \,.$$

- If integrals on ϕ-plane, then solution is BPS; if on double-cover then solution is semi-BPS, and calculations trickier.
- Static solutions are part of a dynamical story – moving kinks and sphalerons, forces, oscillatory shape modes, instabilities, quantization, physical applications.

References

- Manton and Sutcliffe: Topological Solitons (CUP).
- Shnir: Topological and Non-Topological Solitons in Scalar Field Theories (CUP).
- Manton: Integration theory for kinks and sphalerons in one dimension (arXiv, 2023).
- Many papers, by Skyrme, Lohe, Gonzáles, Bazeia, Khare, Gani, Romańczukiewicz, their collaborators, and others.